Complemented kernels of partial differential operators
in weighted spaces of (generalized) functions

by

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Abstract. It is shown that in a large class of weighted function spaces, every partial
differential equation with constant coefficients may be solved by means of a linear and
continuous operator.

Introduction. The existence of a (continuous linear) solution operator for
continuous linear equations has been studied by several authors ([8]–[13],
[15], [17], [18]). D. Vogt ([18]) showed that hypoelliptic partial differential
operators (with constant coefficients) have no right inverses in \( C^\infty(\mathbb{R}^d) \), thus
improving a classical result of Grothendieck for elliptic operators. Ideals in
weighted spaces of entire functions were studied in [10]–[13]. In [8] it was
proved that a large class of hypoelliptic operators have no right inverses in
the classical weighted spaces \( (W, \rho) \) and \( \delta = (M) \) (introduced by Gelfand–
Shilov and Palamodov, resp.).

In this paper, a complementary result will be proved: We will show that
a suitable choice of the system of weight functions will lead to so-called
general splitting spaces, i.e. every system of partial differential equations with
constant coefficients will have a (continuous linear) right inverse in these
spaces. We will consider spaces of (ultra)distributions and (ultra)differentiable
functions determined by the weight systems \( \{ \exp(W(x) + nV(x)) \mid n \in \mathbb{N} \} \) (and
\( \{ \exp(W(x) - nV(x)) \mid n \in \mathbb{N} \} \), resp.), where
\( W(x) = \sum_{i < N} W_i(l(x)_i) \) and \( V(x) = \sum_{i < N} V_i(l(x)_i) \) and \( \mathcal{W}, \mathcal{V} \in C^\infty(\mathbb{R}) \).

Let \( w_i := (W_i)^\delta \) and \( \eta = (\eta)^\delta \) and \( \delta = 1 \) or \( \delta = -1 \), according to the choice
of the sign of \( nV \) in the above weight system. We mainly need the following conditions
on the weight functions:
1. \( w_i + \delta \eta \) is strictly increasing and unbounded for any \( n \in \mathbb{N} \).
2. \( 2w_i \circ V_i^{-1}(t) \leq w_j \circ V_j^{-1}(Ct) \) for large \( t \).
3. \( \eta = o(w) \) and \( t = O(V(t)) \).
4. \( w_i(t) \leq \exp(CV_i(t)) \) for large \( t \).

The last condition is used for weighted spaces of distributions and \( C^\infty \)
families. A stronger assumption is needed when ultradistributions and
ultradifferentiable functions are treated (see (1.5) and Remark 1.3).
Simple examples satisfying these conditions are \( W(t) = t^r \) and \( V(t) = t^s \) for \( r > s \), or \( W(t) = \exp(t^r) \) and \( V(t) = \exp(y(t)^r) \) for \( s_t < a \), or \( a_t = a \) and \( y_t < y \).

The conditions are in a sense stable for taking compositions (see the remark after 3.3). Notice that the usual condition that the weighted space should be stable for shifts is not needed. This would imply that the weights are bounded by \( \exp(Ct) \), while no a priori bound is implied by the conditions of this paper. In fact, the whole space of distributions of finite order is filled with weighted spaces satisfying our conditions.

The paper is divided into three parts: We first show that the weighted spaces of (test) functions are isomorphic to power series spaces of infinite type. This extends the corresponding results in [16] and proves the linear topological invariants (DN) and (Q) (see [15]) for these spaces. The second section contains a Paley-Wiener Theorem and an existence theorem for suitable (pluri)subharmonic functions. In the third section, the Fundamental Principle of Ehrenpreis and the general splitting theorem of D. Vogt (Theorem 7.1 in [15]) are used to prove the final result.

Partial differential operators in weighted Gevrey spaces of ultradifferentiable functions or ultradistributions of Roumieu type are considered in a forthcoming paper ([9]). These spaces correspond to power series spaces of infinite type and one has to use the notion of graded spaces, tame linear maps and tame splitting theorems, as there is no general splitting theorem for continuous exact sequences of power series spaces of finite type ([19]).

1. Sequence space representations. The general splitting theorem of D. Vogt (Theorem 7.1 in [15]) is based on certain linear topological invariants (DN) and (Q), see [15], which characterize the (nuclear) subspaces and quotients of \( p \). We will prove these invariants in this section for the spaces of (test) functions of this paper. More precisely, we will show that the spaces are isomorphic to power series spaces of infinite type. We will need essentially weaker conditions than those normally used in the literature (see [16], § 3).

We will only consider functions defined on the real line, leaving the case of several variables to the reader (see Section 3).

1. Definition. Let \( W, V \in C^1(\mathbb{R}) \) and let \( V \) be positive.
(a) \( C^\infty(W, V) := \{ f \in C^\infty(\mathbb{R}) \mid p_k(f) := \sup_{x \in \mathbb{R}} \| f \|_{H^k} < \infty \} \),
(b) Let \( M_j \) be a sequence of positive numbers satisfying \( M_0 = 1 \) and the conditions (M.1), (M.2) and (M.3) of Komatsu ([5]). Then
\[
C_{[M_j]}(W, V) := \{ f \in C^\infty(\mathbb{R}) \mid p_k(f) := \sup_{x \in \mathbb{R}} \| f \|_{H^k} < \infty, 1 \leq k \}
\]

for any \( k \in \mathbb{N} \).

\( C^\infty(W, V) \) will sometimes be used as a common notation for \( C^\infty(W, V) \) and \( C_{[M_j]}(W, V) \) (similarly for \( P_k \) and \( R_k \)).

\( C^\infty(W, V) \) is a \( K \) \( \text{[M_j]} \)-space (see [2]). In Section 3, partial differential operators in \( C^\infty(W, V) \), and in \( C^\infty(-W, V) \) will be studied for positive and even functions \( W \) and \( V \).

We may suppose that
\[
V(t) \geq 1, \quad |t| + |w| \leq L^2(R),
\]
where \( v := W^r \) and \( w := W^t \). Indeed, we may consider \( V(t) + 1 \) (or \( V(t) + 2 + \cos t \)) instead of \( V \) (if \( |t| \in L^2(R) \)), and \( C^\infty(W, V) \) is not changed.

With \( U(t) := \int_0^t (v(\tau) + |v(\tau)|) \, d\tau \) and \( y \neq 0 \), let \( \tilde{y} \) solve the equation
\[
U(y) = U(\tilde{y}) + \text{sgn}(y) V(y).
\]
Then \( \text{sgn}(\tilde{y}) = \text{sgn}(y) \) and \( \epsilon(\tilde{y}) := |\tilde{y} - y| \) is strictly positive.

Let \( 0 \leq \phi \in D(\mathbb{R}) \) be fixed. Let \( \varphi_\epsilon \) be the convolution of a characteristic function with \( \varphi \cdot \epsilon(t) \) \( \epsilon(t) \), where \( t \in \text{sgn}(y) \), \( |\tilde{y}| = I_k \) or \( t \in I_k \), for some \( \epsilon \in I_k \), or let \( \varphi_\epsilon \) be the sum of two such functions. Moreover, we suppose that
\[
\text{supp} \varphi_\epsilon \subset I_k.
\]

Cut-off functions of the \( \varphi_\epsilon \)-type and Lemma 1.2 below will frequently be used in this paper (see 1.5, 2.2 and 2.3).

The Young conjugate \( F^* \) of a convex function \( F \) is defined by
\[
F^*(y) := \sup_{x} \{ xy - F(x) \}.
\]

Let \( M(t) \) be the function associated with \( (M_j) \) in the sense of Komatsu ([5]), i.e.
\[
M(t) = \ln(\sup_j \| M_j \|) \quad \text{for } t \in \mathbb{C}.
\]

Let \( F \) or \( ^\vee \) be the Fourier transform:
\[
f(t) = \int f(t) e^{-i\omega t} \, dt.
\]

1.2. Lemma. Let \( W \) and \( V \) satisfy (1.1). Let \( \varphi_\epsilon \) and \( \zeta \) be defined as above.
(1a) Let
\[
(1.4) \quad \|w + |\omega|\| \leq \exp(D, V(t)) \quad \text{for any } t.
\]

Then for any \( k \in \mathbb{N} \) there is \( C_k > 0 \) such that for any \( f \in C^\infty(\mathbb{R}) \) and any \( y \neq 0 \)
\[
\sup_{x \in \mathbb{R}} \| f \varphi_{\epsilon}(\cdot x) \| (1 + |x|) \leq C_k \sup_{x \in \mathbb{R}} \| f \varphi_{\epsilon}(x) \| e^{2\pi |x|} \int_{I_k} e^{2\pi i \xi x} \, d\xi.
\]

(1b) Let \( g_t \) and \( h_t \) be increasing sequences such that \( (1/g_t)_{t \in \mathbb{R}} \) and
\[
m_j := M_j/M_{j-1} \geq h_j g_j. \text{ Let } G_j := \prod_{i=1}^{j} g_i \text{ and } H_j := \prod_{i=1}^{j} h_i \text{ and let } H \text{ be the }
function associated to $(H_j)$. Suppose that

\[(\|w|+|v|)(t) \leq D_2 H^{-1}(D_1 V(t)) V(t) \quad \text{for any } t.\]

Then there is $(\tilde{G}_j)_{j \in \mathbb{N}}$ satisfying (M.1) and (M.3') such that for any $k \in \mathbb{N}$ there are $C_i > 0$ and $k_i \in \mathbb{N}$ such that for any $f \in C_{\tilde{G}_j}(R)$ and any $y \neq 0$

\[
|f(x)\|_{\infty} \leq C_1 \sup_{x \in \mathbb{R}} |f(x)| \leq C_1 \sup_{x \in \mathbb{R}} \|f(x)\|_{\infty} \leq C_1 \sup_{x \in \mathbb{R}} \|f(x)\|_{\infty} \leq C_1 \sup_{x \in \mathbb{R}} \|f(x)\|_{\infty}
\]

if $\varphi$ is defined by $\varphi \in D_{\tilde{G}_j}(R)$.

**Proof.** I. We may choose increasing unbounded sequences $\tilde{h}_j$ and $\tilde{g}_j$ such that $(1/\tilde{g}_j) \in C_i$ and $m_j \geq \tilde{h}_j \geq \tilde{g}_j$ and $h_j = \circ(\tilde{h}_j)$. So $(\tilde{G}_j)$ satisfies (M.1) and (M.3'). With $h(t) := \max |h_j|$, we get (see (3.11) in [6])

\[
H(\tilde{h}_j) = \frac{\tilde{h}_j(\lambda)}{\lambda} d\lambda = \int_0^{\tilde{h}_j(\lambda)} \frac{\tilde{h}_j(\lambda)}{\lambda} d\lambda + C_1
\]

\[
= \tilde{H}(j) + C_1 \geq \frac{1}{24} \tilde{H}(j) \quad \text{for large } i.
\]

We let $c < 1/(24 D_2)$ get for large $V(t)$

\[
D_1 H^{-1}(D_1 V(t)) V(t) \leq \frac{1}{24} \tilde{H}^{-1}(24 D_1 V(t)) V(t),
\]

and (1.5) implies that

\[
(1.5^*) \quad (\|w|+|v|)(t) \leq \frac{1}{4 A} \tilde{H}^{-1}(D_3 V(t)) V(t) \quad \text{for any } t \quad \text{and } D_3 = D_3(A).
\]

II. For $x \neq 0$ and $\eta \in I_x$ we get

\[
|V(\eta) - V(x)| \leq \text{sgn}(x) \int \text{sgn}(\eta) (U(\overline{x}) - U(x)) = \frac{1}{2} V(x),
\]

(1.7)

\[
\frac{1}{2} V(x) \leq V(\eta) \leq \frac{1}{2} V(x).
\]

By the mean value theorem, this implies

\[
\frac{1}{2} \leq \frac{V(t)}{t} \quad \text{for any } t \quad \text{and } D_3 = D_3(A).
\]

III. Let $t \in I_j \cup I_j$ for some $\zeta \in I_j$. Then

\[
(1.9) \quad \sup_j \exp [\tilde{H}(A(t)) + V(t)] \leq \exp [\tilde{H}(A(t)) + V(t)]
\]

\[
\exp [D_2 V(t)] \leq \exp [3D_2 V(t)] \leq \exp [9D_2 V(t)]
\]

where we have used (1.8), (1.5*) and (1.7). (1.8) and (1.9) imply

\[
(1.10) \quad \sup_j \exp [\tilde{H}(A(t)) + V(t)] \leq \exp [\tilde{H}(A(t)) + V(t)]
\]

\[
\exp [D_2 V(t)] \leq \exp [3D_2 V(t)] \leq \exp [9D_2 V(t)]
\]

(1.2)(b) is now an easy consequence of (1.3) and (1.10).

IV. 1.2(a) may be proved as above, taking the estimates with finite $i$. Indeed, (1.8), (1.1) and (1.7) give the following estimate:

\[
(1.11) \quad \epsilon(t) \leq C_i \exp \{\epsilon(t) \} \quad \text{for some } C_i \text{ and } (\epsilon(t)) \leq \exp (C_i V(t)) \text{ in } C_i V(t).
\]

(1.4) and (1.5) do not restrict the global growth of the weight functions, as does the assumption

\[
\sup \{\epsilon(t) \} \leq \exp (C_i V(t)) \quad \text{for some } C_i \text{ and } (\epsilon(t)) \leq \exp (C_i V(t))
\]

Indeed, for positive functions $W$ and $V$ this implies that

\[
W(t) + V(t) \leq C_i \exp (C_i t) \quad \text{for some } C_i > 0.
\]

(1.5) may often be obtained in an explicit form:

1.3. **Remark.** (a) Let $(M_j)$ satisfy (M.2) (see [6]) and let $m_j(\ln j) \rightarrow \infty$ for some $\alpha > 1$. Then (1.5) follows from

\[
(1.12) \quad (\|w|+|v|)(t) \leq C_j \exp (A(t) + 1), \quad 0 < \alpha \leq 1, \quad (\epsilon(t)) \leq \exp (C_i V(t))
\]

\[
\quad \text{for some } C_i > 0.
\]

**Proof.** (a) Let $h_j := m_j(\ln j) \in (1, \beta) \leq \beta < \alpha$. Then $h_j$ is strictly increasing, $H_j$ satisfies (M.1), (M.2) and

\[
\lim_{j \rightarrow \infty} H_j(t) \leq \lim_{j \rightarrow \infty} H_j(0) = \infty.
\]

From Lemma 1.4(c) in [14] we get for large $t$

\[
(\|w|+|v|)(t) \leq h_{C_2 V(t)} \leq C_j H^{-1}(C_j V(t)) + V(t).
\]
As \( (1/g_{i}) = (1/(j \ln j)^{p}) \in \mathcal{L}^{1} \), the proof is complete.

(b) Let \( h_{j} = \exp(\tilde{A}(j-1)^{p}) \) with \( \tilde{A} < A \) and \( g_{j} : = \exp(\tilde{A}(j-1)^{p}) \).
Then \( m_{j} \gg h_{j} g_{j} \) and \( (1/g_{j}) \in \mathcal{L}^{1} \). By (3.11) in [6] we get
\[
H(t) - H(1) = \int_{0}^{1} h_{j}(\lambda) \, d\lambda = \int_{0}^{1} h_{j}(\lambda) \, d\lambda \leq \int_{0}^{1} ((\lambda/\tilde{A})^{1/p} + 1) \, d\lambda \\
\leq C_{1} (\ln \lambda)^{p/\alpha + 10}\beta.
\]
(1.12) now implies (1.5).

The Gevrey sequence \( M_{j} = (j)^{p}, \alpha > 1 \), is a special case of 1.3(a). (1.5) holds in this case if
\[
(\mid w \mid + \mid t \mid)(t) \leq C_{1} V(t)^{p}/(\ln V(t))^{\beta} \text{ for some } \beta > 1.
\]
The sequence \( M_{j} = \exp(j^{p}) \) is maximal in \( (M_{j}) \) satisfies (M.2)' So the minimal restriction given by (1.5) is (by 1.3(b))
\[
(\mid w \mid + \mid t \mid)(t) \leq \exp(C_{1} V(t)^{p/2}) \text{ for large } t,
\]
while only (1.4) is needed for \( C^{\infty} \)-functions.

We now inductively define a partition of the real line by using (1.2), assuming that (1.1) holds. Let \( x_{0} : = 0, x_{+1} : = \pm 1 \) and let
\[
x_{t+1/10} = \frac{1}{2} (x_{t} + x_{t}), x_{t} = x_{t} + \frac{1}{2} \varepsilon (r) \text{ sgn}(r),
\]
where \( x_{t} \) solves the equation
\[
U(x_{t}) = U(x_{t}) + \frac{1}{2} \varepsilon (t) \text{ sgn}(t).
\]
A solution \( x_{t} \) may be chosen by (1.1) (\( x_{t} \)) is strictly increasing (by (1.1)) and (1.13) \( (x_{t}) \) is unbounded from above and below.

Otherwise, e.g. \( x_{t} = \lim_{x_{t} \rightarrow -\infty} x_{t} \) would exist and \( \lim_{x_{t} \rightarrow -\infty} x_{t} = \lim_{x_{t} \rightarrow +\infty} x_{t} = x_{t} \).
Hence \( U(x_{t}) = \lim x_{t} U(x_{t}) = U(x_{t}) + \frac{1}{2} \varepsilon (t) \) and \( U(x_{t}) = 0 \), contradicting (1.1).

The following definition depends on \( W \) and \( V \) through the definition of \( x_{t} \).
Let \( \varepsilon (0) \) be chosen by (1.1) (\( x_{t} \) is strictly increasing (by (1.1)) and (1.13) \( (x_{t}) \) is unbounded from above and below.

1.5. PROPOSITION. Let \( W \) and \( V \) satisfy (1.1) and let
\[
\exp(-CV(t)) \in \mathcal{L}^{1} \text{ for some } C > 0.
\]
Then \( C^{\infty}(W, V) \) and \( A \) contain each other as complemented subspaces if \( W \) and \( V \) satisfy (1.4) (for \( C^{\infty}(W, V) \) or (1.5) (for \( C^{\infty}(W, V) \)).

Proof. (i) For \( r \neq 0 \) let \( t 

(1.15)
\[
\begin{align*}
W(t) + V(t) - W(x_{t}) - V(x_{t}) & \leq \varepsilon x_{t}(t) \left( U(x_{t}) - U(x_{t}) \right) = \frac{1}{2} V(x_{t}).
\end{align*}
\]
This implies by (1.7)
\[
\begin{align*}
W(t) + n V(t) & \leq W(x_{t}) + 2n V(x_{t}) \quad \text{for } t \in I.
\end{align*}
\]
\[
W(x_{t}) + n V(x_{t}) \leq W(t) + 2n V(t)
\]
(ii) Let \( 0 \leq R \in D(R) \) be such that \( R(t) dt = 1 \) and \( \text{supp } R \leq [x | x | < 1/64] \).
For \( r \neq 0 \) let \( \tilde{R}_{r} : = \chi_{k} \varepsilon (r) \text{ sgn}(r), \) and \( \tilde{R}_{r} : = \varepsilon (r) + \varepsilon (r) \text{ sgn}(r) \).
For \( r \neq 0 \) let \( \tilde{R}_{r} = \tilde{R}_{r} \text{ supp } R \). Then \( \tilde{R}_{r} \text{ supp } R \) is a resolution of \( \text{identity subordinate to } [t \mid t \in Z] \) \( t_{0} : = (x_{t} - x_{t}), \phi \) is a function as considered in (1.2) (for \( | r | > 2 \), \( y = y_{x_{t} - x_{t}}, \) and \( z = x_{t} \)). This also holds for
\[
\psi : = \chi_{k} \varepsilon (r) \text{ sgn}(r)
\]
with \( \tilde{R}_{r} : = (\text{supp } R, t \leq y_{x_{t} - x_{t}}, \text{ and } t = x_{t}) \to [z : z \in Z] \), \( t = x_{t} \).

1.4. DEFINITION.
(a) \( A_{\omega} := \{ (c_{\omega}) \in C^{\infty}} \mathbb{R}^{2} | \)
\[
q_{\omega}(c_{\omega}) := \sum_{\omega} \left| c_{\omega} \right| (1 + \mid x_{t} \mid \text{ sgn}(r)) ^{p} \exp(k V(x_{t})) < \infty \text{ for any } k \in N.
\]
(b) \( A_{\omega} := \{ (c_{\omega}) \in C^{\infty} \mathbb{R}^{2} | \)
\[
q_{\omega}(c_{\omega}) := \sum_{\omega} \left| c_{\omega} \right| \exp(M(k/\varepsilon (r)) + k V(x_{t})) < \infty \text{ for any } k \in N.
\]
We will sometimes use \( A_{\omega} \) as a common notation.
where \( (f \phi) - := \sum_{x \in \mathbb{Z}} S_{8/15}(f \phi)_x \). We have used the fact that
\[
(1.17) \quad \text{supp } (f \phi)_y = 1 \quad \text{for all } y \in \mathbb{Z},
\]
and the fact that \( \text{supp } (f \phi)_y \subseteq \{ x \in \mathbb{Z} : x \neq x_2 \} \) and \( \epsilon(0) = x_2 \).

So \( x_2 \circ x_2 \) is the identity on \( C_{8/15}(R) \).

(iv) For \( g \in D(R) \), we have \( (S_{-2}(f \phi)) (t) = \hat{g}(t) e^{it \phi/2} \). Lemma 2.2(a) now shows the following estimate for \( f \in C_{8/15}(W, V) \):
\[
(1.18) \quad \sum_{x \in \mathbb{Z}} (g(t))^2 \leq C \left( 1 + \sum_{\| x \|_{8/15} < 1} \right) e^{-C_{8/15} \| x \|_{8/15}^2}
\]
by (1.15) and (1.11), if we choose \( C_2 \) so large that
\[
(1.19) \quad M(g) - M(g) \geq \frac{B}{M(g)} \log(t/0) \quad \text{for any } y \geq 0 \text{ and } t \geq 1
\]
(see Prop. 3.4 in [6]). For \( s = 0 \) we use (1.9) and (1.18):
(a) $C^\infty(W,V)$ is isomorphic to $(\psi)$ if
\[(\omega + |\pi|)(t) \equiv \exp\{D_1(V(t) + 1)\}\] for any $t$.

(b) $C^\infty_{Mb}(W,V)$ is isomorphic to a nuclear and stable power series space $A_\infty$ if
\[(\omega + |\pi|)(t) \equiv D_2 H^{-1}(D_1(V(t) + 1))\{V(t) + 1\}\] for any $t$, and
\[
m(2t) \leq Cm(t)\] for large $t$.

Here $H$ is chosen as in (1.5) and $m(t) := \max \{1, \frac{|\pi|}{\mu m_1^{\infty}}\}$, where $\mu h(t)$ is the $L$-measure of $(x \in V(x) \leq t)$. If $V(t)$ and $V(-t)$ are nondecreasing for $t > 0$, then $C^\infty_{Mb}(W,V)$ is isomorphic to
\[
\hat{A}_{Mb} := \{(c_{\alpha}) \in C^{\infty}(\beta) : \sum_{\alpha} |c_{\alpha}| e^{\mu|\alpha| + \mu V(\beta)} < \infty \text{ for any } k \in N\}.
\]

Proof. We may assume that (1.1), (1.4), (1.5) and (1.14) hold (see (1.1)). So $A_\infty$ is defined.

(a) $(\psi) \equiv C^\infty_{Mb}(\psi)$ for suitable $C_\infty$ by (1.18). So $e^{\psi(x)}(\psi)$ tends to $\infty$ for $|\psi| \to \infty$, and $A_\infty$ is isomorphic to a power series space $A_\infty$ of infinite type (via an increasing rearrangement of $\{C_\infty(x) = \psi(1 + |\pi|)/(x, r) \in Z \times Z\}$). Also, $A_\infty$ is nuclear by the Grothendieck–Pietsch criterion. So $A_\infty$ is isomorphic to a complemented subspace of $(\psi) (15)$, Th. 1.5. On the other hand, $A_\infty$ contains $(\psi)$ as a complemented subspace via the projection $(c_{\alpha}) 
\rightarrow (\sum_{\alpha} c_{\alpha})$. So $A_\infty$ is isomorphic to $(\psi)$ by (1.17), and $C^\infty_{Mb}(W,V)$ is isomorphic to $(\psi)$ by (1.5) and (1.17) again.

(b) As $m$ is increasing, we get from (1.20) and (3.11) in [6]
\[
M(kt) - M(t) = \int_{[t]} m(z) d\lambda \mathbf{1}_{\lambda} \mathbf{1}_{km(kt) - C_4 m(t)}
\] for $t \geq 1$.
\[
M(kt) - M(t) \gg \text{ln} km(kt) 
\]
So $A_{Mb}$ is isomorphic to
\[
\{(c_{\alpha}) \sum_{\alpha} |c_{\alpha}| e^{\mu|\alpha| + 4\mu|V(\beta)} + 2\mu V(\beta)} < \infty \text{ for any } k \in N\}.
\]

Again by (1.20), $A_{Mb}$ is isomorphic to
\[
\hat{A}_{Mb} := \{(c_{\alpha}) \sum_{\alpha} |c_{\alpha}| e^{\mu|\alpha| + 2\mu V(\beta)} < \infty \text{ for any } k \in N\}
\]
with $f(s, 0) := m(|s| + 1)$ and $f(s, \pm 1) := m(|s| + 1)/(x, s \pm 1) - x, s \pm 1)$ for $r > 0$, since $x(s) = z((x, s + 1) - x, s + 1)$, and since by (1.9)
\[
f(0, r) = m(2e(\pi)) \equiv M(2e(\pi) + C_4 \equiv \hat{A}(2e(\pi) + C_4 \leq C_4 V(x)).
\]

(ii) Let $\beta$, be an increasing arrangement of $(f(s, r) : (s, r) \in Z \times Z)$ and let
\[
\begin{align*}
\hat{K}_f(t) := & (s, r) \in Z \times Z: f(s, r) \leq t, \\
\hat{K}_r(t) := & (s, r) \in Z \times Z: f(s, r) \leq t.
\end{align*}
\]
Then
\[
\begin{align*}
\min \{t \mid \hat{K}_f(t) \geq n \} & \leq \beta_n \leq \min \{t \mid \hat{K}_r(t) \geq n\}, \\
\hat{K}_f(t) & \leq 2m_{\beta+1} + 2 \{s, r \in (Z \times Z) : f(s, r) \leq t, s \geq \mu(h(t))m_{\beta+1} - 1\} \\
& \leq 2m_{\beta+2} + 1 \mu(h(t)) \leq 2m_{\beta+2} + 1 \mu(h(t))
\end{align*}
\]
since by (1.7), $\hat{f} := \hat{f}(n) \{x, x \in (\beta_n \in \infty)\}$ is contained in $(x \leq \mu(k) \in \infty)$ if $X(s) \leq t$. Next,
\[
\begin{align*}
\hat{K}_f(t) & \geq \{(s, r) \in Z \times Z : f(s, r) \leq t, s \geq \mu(h(t))m_{\beta+1} - 1\} \\
& \geq \{(s, r) \in Z \times Z : f(s, r) \leq t, s \geq \mu(h(t))m_{\beta+2} - 1\} \\
& \geq 1/2 \mu(h(t))m_{\beta+2}
\end{align*}
\]
Since $X(s) \leq t/4$ if $X(s) \leq t/8$ for some $x \in Z$, we have proved that
\[
\hat{K}_f(t) \leq 1/4 \mu(h(t))m_{\beta+2}
\]
(in particular, $\beta_n$ increases to $\infty$).

(iii) $A_{Mb}$ is nuclear by the Grothendieck–Pietsch criterion, since
\[
\begin{align*}
\sum_{\alpha} e^{-c_{4m(\beta|\alpha|) + 2\mu V(\beta)}} & \equiv \sum_{\alpha} \sum_{\alpha} e^{-c_{4m(\beta|\alpha|) + 2\mu V(\beta)}} < \infty
\end{align*}
\]
by (3.12) in [6], (1.19) and (1.18).

(iv) $\hat{A}_{Mb} \times A_{Mb}$ is isomorphic to $\hat{A}_{Mb}$ via the mapping $(c_{\alpha}, d_{\alpha}) 
\rightarrow (a_{\alpha})$, where $a_{2s+1} := c_{\alpha}$ and $a_{2s+2} := d_{1}$. So the assumptions of (1.7) are satisfied and $C_{Mb}(W, V)$ is isomorphic to $\hat{A}_{Mb}$. As $\hat{A}_{Mb}$ is stable, we have
\[
\hat{K}_f(t) \leq 1/4 \mu(h(t))m_{\beta+2}
\]
for any $C \in N$ and some $C_1(C)$.

So (1.22) shows that $A_{Mb}$ is isomorphic to $\hat{A}_{Mb}(\alpha)$.

(v) To prove the last statement in (1.6), we may assume that $V(\beta) := V_{\beta} \gamma$ and $V(\beta) := V(\beta)$, are strictly increasing to $\infty$ (by adding $\tan^{-1}(\beta)$). Then
\[
h(t) = V(\beta) + V^{-1}(t)
\]
for large $t$.

Let $\beta_n$ be an increasing arrangement of $(g(s, r) := m(s) + V(\beta)) : (s, r) \in Z \times Z$ and let $\hat{K}_g$ and $\hat{K}_r$ be defined as above. Then
\[
\hat{K}_g(t) \leq 2m_{\beta+1} + 1 \mu(h(t)) m_{\beta+1} - 1\}
\]
\[
\hat{K}_r(t) \geq \{(s, r) \in Z \times Z : f(s, r) \leq t, s \geq \mu(h(t))m_{\beta+2} - 1\} \\
\geq 1/2 \mu(h(t))m_{\beta+2} + 1 \mu(h(t))m_{\beta+2}.
\]

So (1.22) shows that $A_{Mb}$ is isomorphic to $\hat{A}_{Mb}(\alpha)$. 


As also $\alpha_n$ is stable, this shows as above that $\alpha_n$ and $\gamma_n$ are equivalent, i.e.
\[ \Lambda_n(t) \approx \Lambda_n(t) = C_{ \mathcal{M} }^n(W, V). \]

This completes the proof of the theorem.

Theorem 1.6(a) extends Theorem 3.4 of [16], where the weighted space is assumed to be shift stable (see the remark after 1.2). Sequence space representations for weighted spaces of ultradifferentiable functions seem to appear here for the first time. $\mathcal{A}_{m_j}$ is the sequence space, which is also obtained in the shift invariant case.

The isomorphism class of $C^n(W, V)$ is independent of $W$ and $V$, while the isomorphism type of $C^n(W, V)$ is independent of $W$, but depends on $V$.

The proof of 1.6(b) implies a sequence space representation for periodic ultradifferentiable functions, which was also given in [16]. Lemma 8 (with a longer proof), (1.20) holds if
\[ 2m_i \leq m_j \quad \text{for some } C \in \mathcal{N}, \]
which is satisfied if (M.1) and (M.3) hold ([16]), or if $M_j(l)$ satisfies (M.1), since then $m_j/I$ is increasing and
\[ 2m_j = 2(m_j/l) = 2(m_i/l) = m_i. \]

2. The Fourier transform on $C^n(W, V)$ and $C_n(W, V)$. We will check the assumptions of the Fundamental Principle of Ehrenpreis (resp. the Division and Extension Theorem, see [33] in this section for $C_n(W, V)$ and $C^n(W, V)$, where $W, V \in C^n(W, V)$ are even functions larger than 1. Again, $w := W^*$ and $e := V^*$.

Let $F^{*}$ be the Young conjugate of $F$ ($F^{*}(x) := \sup_{\{e \leq F(t)\}}$). For $k \geq 0$ let $M_k := k \ln(1 + |x|^2)$, $\Lambda_k := M_k(\varepsilon)$, and $M_{-k} := -M_k$, where $M_k$ is the common notation for $M_k$ and $M_{-k}$.

2.1. Definition. Let $L_k(z) := (W + kV^*)(\ln z) - M_k(z)$ for $k \in \mathbb{R}$.
(a) $\mathbb{H}^a_{k} := \{ f \in \mathcal{F}( \mathcal{C} ) \ | \ [f, \mathcal{F}( \mathcal{C} )] \leq \| f \| \chi^a_n < \infty \}$ for any $k \in \mathbb{N}$.
(b) $\mathbb{H}^a := \{ f \in \mathcal{F}( \mathcal{C} ) \ | \ [f, \mathcal{F}( \mathcal{C} )] \leq \| f \| \chi_n < \infty \}$ for any $k \in \mathbb{N}$.

The spaces carry their natural projective (and inductive, resp.) topologies.

2.2. Paley-Wiener Theorem. For $\delta = 1$ (in (a)) or $\delta = -1$ (in (b)) suppose that
\[ \begin{align*}
(2.1) & \quad (w + \delta x)(x) \text{ is increasing to } \infty \text{ for any } k \in \mathbb{N} \text{ and large } t, \\
(2.1') & \quad V(i) \to \infty \quad \text{for } t \to \infty. 
\end{align*} \]

(a) The Fourier transform $\mathcal{F}$ (or $\hat{\cdot}$) is an isomorphism of $C^n(W, V)$ onto $\mathcal{H}^a$.

(b) If $W, V$ (and $M_j(l)$) satisfy (1.4) (resp. (1.5)), then $\mathcal{F}$ is an isomorphism of $C^n(W, V)$ onto $\mathcal{H}^a$.

Proof. (a) $C^n(W, V)$ is contained in $\mathcal{H}^a(R)$. We may use the Fourier inversion formula and shift the path of integration into the complex plane (by (M.2)) to show that $\mathcal{F}$ is surjective.

(b(i)) For $T \in C^n(W, V)$, there is $k \in \mathbb{N}$ such that
\[ \| \mathcal{F}(T) \| \chi^a_n = \| \mathcal{F}(T, e^{-i\alpha}) \| \chi^a_n \leq C_R(e^{-i\alpha}) \quad \text{for any } \alpha \in \mathbb{C}. \]

This shows (by (2.1)) that $\mathcal{F}$ is (a continuous) mapping in $C^n(W, V)$ into $\mathcal{H}^a$.

(ii) Let $C^d$ denote the $d$-periodical functions in $C^n(W, V)$. The Fourier series expansion of $f \in C^d$ converges in $C^n(W, V)$ (by (2.1) and (M.2)). So if $\mathcal{F}(T) = 0$ for $T \in C^n(W, V)$, then $T$ vanishes on $C^d$ for $d > 0$, and $T = 0$, since $C^d$ is dense in $C^n(W, V)$. So $\mathcal{F}$ is injective.

(iii) For $g \in \mathcal{H}^a$ we define a linear functional $T$ on $D(R)$ by
\[ \langle T, \psi \rangle := \langle 2\pi \rangle^{-1} \int g(x) \psi(x) dx. \]

Then $T$ is defined by (M.2), since
\[ |\psi(z)| \leq C e^{\chi^a_n |z|} \quad \text{for some } C > 0 \text{ and any } k > 0, \]
if $\psi \in D(R)$.

(iv) If $T$ is continuous on $D(R)$ for the topology induced by $C^n(W, V)$, then $T$ may be uniquely extended to $\mathcal{F}(C^n(W, V))$, since $D(R)$ is dense in $C^n(W, V)$. So $\mathcal{F}(\hat{T})$ is defined and is an entire function.

Choose $\varphi \in D(R)$ such that $\varphi = 1$ near 0. Then $\tilde{T} := \varphi(-i\alpha) \hat{T}$ in $C^n(W, V)$, and $\mathcal{F}(\varphi \tilde{T}) \to \mathcal{F}(\hat{T})$ uniformly on compact sets by (b(i)). On the other hand,
\[ \langle \mathcal{F}(\varphi \tilde{T}), \psi \rangle = \langle \varphi \tilde{T}, \varphi e^{-i\alpha} \hat{T} \psi \rangle = \langle \varphi \tilde{T} \hat{T}, \varphi \hat{T} \psi \rangle = \langle \varphi \tilde{T} \hat{T}, \varphi \hat{T} \psi \rangle \]
by (M.2) and the theorem of dominated convergence.

So $\mathcal{F}$ is surjective if $T$ (as defined in (2.2)) is continuous on $D(R)$ for the topology of $C^n(W, V)$.

(iv) We choose $\varphi$, as in part (ii) of the proof of Proposition 1.5. Then there are $k_i$ such that for any $\psi \in D(R)$ and any $|\alpha| \geq 1$
\[ \langle T, \hat{\psi} \rangle \leq \sum_{\gamma} \langle T, \hat{\psi}_\gamma \rangle \leq (2\pi)^{-1} \int \hat{f}(x+i\gamma) (\varphi \varphi^*) (x+i\eta) \, dx \]
\[ \leq C_1 \sum_{\gamma} e^{-4(\varphi \varphi^*)} \int \exp(M_\gamma (x+i\eta) - M_\eta (x+i\eta)) \, dx \]
\[ \times \sup_{\gamma} \int \exp(M_\gamma (x+i\eta)) (\varphi \varphi^*) (x+i\eta) \, dx \]
\[ \leq C_2 \sup_{\eta, x} e^{-4(\varphi \varphi^*)} \int \exp(M_\gamma (x+i\eta)) (\varphi \varphi^*) (x+i\eta) \, dx \]
by (M.2), Lemma 1.2 and (1.15). By (1.18), the last sum is finite. We now choose
\[ y_\eta = -\text{sgn}(\eta)(w-k\eta) (x_\eta) \quad \text{for large } |\eta|, \]
\[ y_\eta = 1 \quad \text{otherwise.} \]
This implies
\[ \sup_{\eta} \int \exp(M_\gamma (x+i\eta)) (\varphi \varphi^*) (x+i\eta) \, dx \leq -\int (W-k\gamma)(x_\eta) + C_3 \]
\[ \leq -\int (W-k\gamma)(x_\eta) + C_4. \]
So \( T \) is continuous for the topology of \( C_0^\infty(-W, V) \) and the proof is complete.

2.3. Theorem. Let \( L^* \) be defined as in 2.1. Let \( W \) and \( V \) satisfy (2.1) and (1.4) (resp. (1.5)) and let
\[ t = O(V (t)). \]
(a) For any \( k > 0 \) there are \( k', C > 0 \) (resp. for any \( k' > 0 \) there are \( k < 0 \) and \( C_1 > 0 \) such that
\[ \sup_{|\xi| \leq k_1} L^*_\xi (x + \xi) + 2 \ln (1 + |\xi|) \leq L_\xi (x) + C_1. \]
(b) \( L_\xi (x) \) is subharmonic (sh.) for \( k < 0 \).
(c) For any \( k > 0 \) there are \( k', C > 0 \) and sh. functions \( \Phi_k \) such that
\[ -C + L^*_\xi (x) \leq \Phi_k (x) \leq L_\xi (x). \]
Proof. (a) For large \( |k| \) and \( |\xi| \leq 1 \) we get by (2.4)
\[ (W+kV)^* (\text{Im } \xi + \text{Im } z) \leq C_1 + (W+kV - 1d)^* (\text{Im } z) \]
\[ \leq C_1 + (W+kV - C_0V)^* (\text{Im } z). \]
(a) is now trivial (by (1.19)).
(b) follows from [4], Section 1.6.
(c) The construction in some sense uses formula (1.2) twice: For fixed

\[ k > 0 \text{ and } k' > k \text{ (to be determined later) we define} \]
\[ (W+k'V)^* (y_\eta) = (W+(k+1)V)^* (y_{k+1}), \quad r \geq 1, \quad y_1 = 1. \]
y_\eta is strictly increasing to \( \infty \), since \( V \) is larger than 1 by assumption. Let
\[ x_0 := 0, \quad x_r := t(y_\eta) \quad \text{for } r \geq 1, \]
where
\[ (W+kV)^* (y_\eta) = y_{r-1}(t(y_{r-1})) - (W+kV)^* (t(y_r)). \]
Let \( 0 \leq \eta \in D(0, 1/4) \) and \( \int \eta(t) \, dt = 1 \). For \( r \geq 1 \) let
\[ \psi_k := (\eta(t))^{-k} \eta(t) \quad \text{with} \quad B_k := (x_0 + c(t)/4, x_0 + 3c(t)/4) \text{ and} \quad c(t) := x_r - x_r \text{ (see (1.2).} \]
Lemma 1.2, (1.8) and (1.15) show for large \( k' > k \) and \( r \geq 1 \)
\[ |\hat{\psi}_k (z) e^{V_0 z}| \leq C_1 \exp((C+1)V(X_0)) |X_0| e^{2V}|z| \]
\[ -W(x_0) - k'(WV)_0 (x_0) \exp(-V(x_0)) \]
\[ \leq C_3 \exp(W+kV)^* (\text{Im } z) \exp(-V(x_0)) \]
\[ \leq C_3 \exp(W+kV)^* (\text{Im } z) \exp(-V(x_0)). \]
(iii) Let
\[ |\text{Re } z| \leq 1/2, \quad y_{r+1} \geq |\text{Im } z| \geq y_r \quad \text{for } r \geq 1. \]
Then
\[ |\hat{\psi}_k (z)| \geq e^{V_0 z} e^{aV_0 |z|/2} \int \text{Re } e^{-i\pi |z|} \, dx \geq \frac{1}{2} \]
\[ \geq \epsilon(t) \cos(1/2) e^{V_0 z/2}, \]
\[ \int \eta(t) \, dt = 1 \]
by (2.8). The choice of \( x_r \) and (2.8) now imply for large \( r \)
\[ |\hat{\psi}_k (z)| \geq C_1 e^{(W+kV)^* (x_r)} \geq C_1 e^{(W+kV)^* (x_r)} \]
\[ \geq C_1 e^{(W+kV)^* (x_0)} \geq C_3 \epsilon(t) \exp(1/4) \]
\[ \text{by } (1.5). \]
(2.10) shows that this supremum locally is a maximum and that
\[ \Phi_k(z) \leq L^k_2(z) + C_k. \]
So \( \Phi_k \) is continuous and sh, since \( \psi_{\eta} \) is sh. ([4], Sect. 1.6).
If \( |\text{Im} z| \geq C \), then there are \( r \in \mathbb{N} \) and \( s \) such that for \( \eta = 1 \) or \( \eta = -1 \)
\[ y_{s+1} \geq \eta \text{Im} z \geq y_s, \quad |\eta \text{Re} z - s/2(r)| \leq 1/2z(z). \]
So we get by (2.9)
\[ \Phi_k(z) \geq C_k + (W'((k + 1) V') |\text{Im} z| - M_k^{2k}(z) \geq C_k + L_{2k}(z). \]
\( \Phi_k \) has the desired properties for large \( |\text{Im} z| \). Taking the maximum of \( \Phi_k \) and finitely many functions of the form
\[ \sup_{x \in \mathbb{R}} |\psi_{\eta}(x - iy)|, \quad \psi_{\eta}(x - iy), \quad \psi_{\eta}(x - iy), \quad \eta \in \mathbb{R}, \]
for suitable \( \gamma \), the resulting function \( \Phi_k \) shows 2.3(c) for \( k/4 \).


The structural results of the preceding sections may now be applied to prove the existence of continuous linear solution operators for systems of partial differential equations with constant coefficients. For the convenience of the reader, we first restate the conditions on the weight functions (now defined on \( \mathbb{R}^n \)) which were used so far, and which will generally be assumed in this section:

Let \( W(x) = \sum_{i+j} W_i(x) j \) and \( V(x) = \sum_{i+j} V_i(x) j \) and let \( (W_i, V_i) = 0 \) be continuous. For \( i \leq N \) and large \( t \) we assume that:
1. \( W_i \) and \( V_i \) are strictly increasing and \( t = O(V_i(t)) \).
2. \( W_i + \delta i \) is strictly increasing and unbounded for any \( k \in \mathbb{N} \). Here and below \( \delta \) may be \( 1 \) or \( -1 \).
3. For ultradifferential functions \( C^\infty_{\delta W_i} \) we assume that \( M_i = \prod_{i < k} M_{i,k} \),
where \( (M_{i}, M_{j}) \) satisfy (M.1), (M.2) and (M.3) (see [6]) and that
\[ m_i(t) \leq C (m_i(t)) + 1 \]
for \( m_i(t) = \max |f_j| M_{j,i} / M_{j,i-1,1} \leq t \).
The spaces \( C^\infty(\delta W_i, V_i) \) and \( \mathcal{H}^\infty \) are now defined as in 1.1 and 2.1.
The weight functions \( L_{k}^2 \) now have the form
\[ L_{k}^2(z) = (W + kV)^* |\text{Im} z| - M_k^*(z), \]
where
\[ M_k^*(z) = k \sum_{i < k} \ln(1 + |z|) \quad \text{for } C^\infty(\delta W_i, V_i), \]
\[ M_k^*(z) = \sum_{i \in \mathbb{N}} M_i(k |z|) \text{sgn}(k) \quad \text{for } C^\infty_{\delta W_i}(\delta W_i, V_i). \]
may be solved with \( f \in CB \), if \( f \in B_k \). So \( \varphi : (\mathcal{H}^*)' \to \mathcal{H}^*_\mathfrak{a} \) is surjective. \( \mathcal{H}^*_\mathfrak{a} \) is defined by a compact (injective) spectrum, since \( B_k \subseteq CB \), which is (relatively) compact in \( \mathcal{H}^*_\mathfrak{a} \) for some \( m \) by (3.1) and (3.2).

\( \varphi : (\mathcal{H}^*)' \to \mathcal{H}^*_\mathfrak{a} \) is open by the open mapping theorem for (DFS)-spaces.

Finally, \( R(-z)(\mathcal{H}^*)' = \text{Ker } \varphi \cap (\mathcal{H}^*)' \) by the D/E-Theorem and Theorem 2.3(a)(b).

(a(i)) \( \varphi : (\mathcal{H}^*)' \to \mathcal{H}^*_\mathfrak{a} \) is continuous by Theorem 2.3(a).

For any \( k \in \mathbb{N} \) there is \( n \in \mathbb{N} \) such that

\[ R(-z)(\mathcal{H}^*)' \text{ is dense in } R(-z)(\mathcal{H}^*)' \text{ for the topology of } (\mathcal{H}^*)'. \]

As \( R(-z) \) is continuous by Th. 2.3(a) and \( \mathcal{H} \) is an isomorphism, we only have to show that \( C^\omega(W, V) \) is dense in \( C^\omega(W, V) \) for the topology of \( C^\omega_k(W, V) \), where \( C^\omega_k(W, V) := \{ f \in C^\omega_k(W, V) \mid \rho_k(f) < \infty \} \) (see 1.1). This is trivially proved using cut-off functions and convolution. By the D/E-Theorem and Th. 2.3(a)(c), \( R(-z)(\mathcal{H}^*)' \) is contained in \( \text{Ker } \varphi \cap (\mathcal{H}^*)' \), and \( \text{Ker } \varphi \cap (\mathcal{H}^*_\mathfrak{a})' \) is contained in \( R(-z)(\mathcal{H}^*)' \). So \( \text{Ker } \varphi \cap (\mathcal{H}^*)' \) is dense in \( \text{Ker } \varphi \cap (\mathcal{H}^*_\mathfrak{a})' \) for the topology of \( (\mathcal{H}^*)' \) for any \( k \in \mathbb{N} \) and some \( n(k) \in \mathbb{N} \).

Again by the D/E-Theorem and Th. 2.3(a)(c), for any \( k \in \mathbb{N} \) there are \( C \) and \( n \in \mathbb{N} \) such that the equation

\[ \varphi g = (f_0) \]

has a solution \( g \in B_k \) if \( (f_0) \in CB_k \). Hence (3.3) has a solution \( g \in (\mathcal{H}^*)' \cap B_k \) if \( (f_0) \in (CB_k) \) (by the classical Mittag-Leffler argument).

So \( \varphi : (\mathcal{H}^*)' \to \mathcal{H}^*_\mathfrak{a} \) is surjective and open.

(ii) \( \text{Ker } \varphi \cap (\mathcal{H}^*)' = R(-z)(\mathcal{H}^*)' \).

Proof. We have already noticed that \( \text{Ker } \varphi \cap (\mathcal{H}^*)' \) contains \( R(-z)(\mathcal{H}^*)' \). To prove the opposite inclusion, we have to show that

\[ 1R(-z) = \varphi g \]

is solvable with \( g \in (\mathcal{H}^*)' \) if \( g \neq 0 \) and \( f \in (\mathcal{H}^*)' \).

(a) If \( R(-z)f = 0 \) for some \( f \neq 0 \neq f \in C[z]' \), then there is an \( n \times m \) matrix \( Q \) of polynomials such that

\[ R(-z)f = 0 \neq f \in C[z]' \text{ iff } f = Q(-z)g \text{ for some } g \in C[z]' \].

Then \( \left\{ \rho_j \right\}_{j \leq s} \) is a Noetherian operator for \( Q(-z) \), where the \( R_j \) are the columns of \( R \). The solvability of (3.4) now follows as in (b)(i) by the D/E-Theorem and Th. 2.3(a)(c) (with \( \left\{ \rho_j \right\}_{j \leq s} \) instead of \( Q \) and \( Q(-z) \) instead of \( R(-z) \)).

(b) If \( R(-z)f = 0 \) for \( f \in C[z]' \) iff \( f = 0 \), then \( 1R(-z)f = 0 \) for \( f \in C[z]' \) iff \( f = 0 \). So the solutions \( g \in (\mathcal{H}^*)' \) of (3.4) (existing by the D/E-Theorem) are fact unique and therefore contained in \( (\mathcal{H}^*)' \).

Let \( P(D) \) be an \( r \times s \) system of partial differential operators with constant coefficients. Let \( N \) be the kernel of \( P(D) \) (in the hyperfunctions), say, and let \( Q(D) \) be the matrix of relations implied by \( P(D) \) (i.e. \( P(-z)f = 0 \) for \( f \in C[z]' \) iff \( f = Q(-z)g \) for some \( g \in C[z]' \)). \( Q(D) \) may be 0.

We may now prove the main result of this paper, concerning the following two sequences:

\[ \begin{align*}
(3.5) & \quad 0 \to N_p \cap C^\omega(-W, V)' \to C^\omega(-W, V)' \overset{P(D)}{\to} N_q \cap C^\omega(-W, V)' \\
(3.6) & \quad 0 \to N_p \cap C^\omega(W, V)' \to C^\omega(W, V)' \overset{P(D)}{\to} N_q \cap C^\omega(W, V)' 
\end{align*} \]

3.2. Theorem. Let \( W \) and \( V \) (and \( M \)) satisfy the general assumptions of this section and (1.4) (resp. (1.5)). Let moreover

\[ v_1 = 0(w_1), \]

\[ v_2 = 0(w_2). \]

Then the sequences (3.5) and (3.6) are exact and split, i.e. \( P(D) \) has continuous linear solution operators

\[ \begin{align*}
L_1 & : C^\omega(-W, V)' \cap N_q \to C^\omega(-W, V)' \\
L_2 & : C^\omega(W, V)' \cap N_q \to C^\omega(W, V)' 
\end{align*} \]

Proof. (a) The spaces in (3.5) (resp. (3.6)) are (FS)-spaces (resp. (DFS)-spaces). So the Paley–Wiener Theorem 2.2 implies that (3.5) and (3.6) are exact iff

\[ 1P(-z) : (\mathcal{H}^*)' \to (\mathcal{H}^*)' \]

is injective with closed range for \( \delta = 1 \) (resp. \( \delta = -1 \)). But \( 1P(-z) \) is closed by 3.1, being the kernel of the continuous mapping \( \varphi \). We have already noticed that \( 1P(-z) \) is injective if \( Q \) vanishes, and that \( 1P(D) \) is a Noetherian operator for \( Q(-z) \) if \( Q \neq 0 \). So the injectivity follows from 3.1 (applied to \( R = Q \)) in this case.

The exactness of (3.5) and (3.6) is thus proved.

(b) The splitting of (3.5) will be proved by the general splitting theorem of D. Vogt ([15], Th. 7.1). All spaces are nuclear by Theorem 1.6. We now have to check the linear topological invariants (DN) and (Q) (see [15]):

\[ C^\omega(-W, V)' \text{ and the subspace } C^\omega(-W, V)' \cap N_q \text{ have (DN), since } C^\omega(-W, V)' \text{ is isomorphic to a power series space of infinite type by Theorem 1.6. } C^\omega(-W, V)' \cap N_q \text{ is isomorphic to } \mathcal{H}^*_\mathfrak{a} \text{ by Th. 2.2 and 3.1(a).} \]

\( \mathcal{H}^*_\mathfrak{a} \) may be defined by the equivalent inductive spectrum (DFS)-spaces.
(3.9) \[ \mathcal{F}_p := \{ f \in \mathcal{F}(V)_p \mid \langle f, \xi \rangle \leq C \varepsilon^{(p+1)+k}\nu^{(p)} \}, \quad k \in \mathbb{N}, \]
where
\[ L_1(\xi) = (W - V)^* (\text{Im} \, \xi) \quad \text{for} \quad C^\infty(-W, V) \]
(resp. \[ L_2(\xi) = (W - V)^* (\text{Im} \, \xi) + M(\xi) \quad \text{for} \quad \mathcal{C}_{\mu}(W - V), \]
\[ L(p, \xi) = V_0 \circ (w_0 - V)^{-1} (\text{Im} \, \xi) + \ln(1 + |\xi|) \]
(resp. \[ L(p, \xi) = V_0 \circ (w_0 - V)^{-1} (\text{Im} \, \xi) + m_0(\xi). \]

**Proof.** (i) We have shown in the proof of 1.6 that for any \( k > 0 \) there is \( C > 0 \) such that (by (1.20))
\[ M(k) \leq M(t^0) + C \mu(t), \quad M(t^0) + k \mu(t) \leq M(e^{t^0}). \]

Here and below we have omitted the index \( i \). This shows the claim for the functions \( M_k(z) \).

(ii) \[ (W - V)^*(t) - (W - V)^*(t) \geq C_2 + \frac{(w - v)^{-1}}{C_2} \]
\[ \geq C_3 + (k - 1) \mu (w - v)^{-1}(t) \quad \text{for large } t, \]
\[ (W - V)^*(t) - (W - V)^*(t) \leq C_4 + (k - 1) (V_0 (w - k)^{-1}(t) \]
\[ \geq C_5 + C_0 (k - 1) (V_0 (w - v)^{-1}(t) \quad \text{for large } t, \]
where \( (w - v)^{-1} \) holds iff \( (w - v)^{-1} (C(t)) \geq (w - v)^{-1}(t) \) for large \( t \).

This is seen by (3.7) and (3.8):
\[ (w - v)^{-1} (C(t)) \geq (w - v)^{-1}(t) \geq (w - v)^{-1}(t). \]

The spectrum (3.9) is compact by 3.1(b). Hence it is regular and the unit balls \( B_p \) of \( \mathcal{F}_p \) are a fundamental system of (absolutely convex closed) bounded sets, whose polars are a basis of \( \mathcal{C}_p \)-neighbourhoods in \( C^\infty(-W, V)^* \cap N_p \). So this space has (Q) (see the proof of 3.2 in [15]) if for any \( p \in \mathbb{N} \) there is \( k \geq p \) such that for any \( i \in \mathbb{N} \) there are \( C, n \geq 1 \) such that \( \langle f, \xi \rangle \leq C \varepsilon^{(p)+k}\nu^{(p)} \).

This is easily shown. So \( C^\infty(-W, V)^* \cap N_p \) has (Q) and the sequence (3.5) is split by the splitting theorem 7.1 in [15].

(c) \( C^\infty(W, V)^* \) and the quotient \( C^\infty(W, V)^*/Q(-D)C^\infty(W, V)^* \) have (Q), since \( C^\infty(W, V)^* \) is isomorphic to \( \mathcal{F}_p \) by 2.2 and 3.1, where the norms in \( \mathcal{F}_p \) may be defined by
\[ \| f \|_p := \| f \|_{e^{t^0+1+\nu}} \]
with \( L \) and \( L \) defined as in (b) (with \( (W + V)^* \) and \( (w + V)^* \) instead of \( (W - V)^* \) and \( (w - V)^* \)). Indeed, this follows from (3.7) and (3.8) as above.

By using these new norms, (DN) is easily proved for \( \mathcal{F}_p \) (and hence for \( C^\infty(W, V)^*/P(-D)C^\infty(W, V)^* \).

So the dual sequence to (3.6) is split by the splitting theorem of D. Yog. and (3.6) also splits.

The conditions of Thm 3.2 will be illustrated by some simple examples:

3.3. **Examples.** (a) Let \( W(t) = t^0 (\ln t)^3 \) and \( V(t) = t^0 (\ln t)^3 \). Then the conditions of 3.2 are satisfied:

(i) for \( C^\infty(\delta W, V) \), if \( 1 < \beta_1 < \beta, 0 \leq \gamma_1 < \infty \), or if \( 1 < \beta_1 = \beta, 0 \leq \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < \gamma_5 < \gamma_6 < \gamma_7 \);

(ii) for \( C^\infty(\delta W, V), 1 < \alpha, 1 \leq \beta_1 \text{ and } (\beta_1 - 1)/\alpha \leq \beta_2 < \beta_3 < \gamma_4 < \infty \), or if \( 1 < \beta_1 = \beta_2 = \beta_3, (\gamma_4 + 1)/\gamma_5 < \gamma_6 < \gamma_7 < \gamma_8 \).

(b) Let \( W(t) = \exp(t^2) \) and \( V(t) = \exp(t^2) \) for \( 0 < \gamma_4 < \beta_1, \beta_2 < \gamma_5 < \gamma_4 < \gamma_5 < \gamma_6 < \gamma_7 \). Then the conditions of 3.2 are satisfied:

(i) for \( C^\infty(\delta W, V) \), if \( 0 < \beta_1 < \beta_2, \gamma_1, 0 < \gamma_1 < 1 \);

(ii) for \( C^\infty(\delta W, V), 1 < \alpha, 1 < \beta_1 = \beta_2, 1/\alpha < \gamma_1 < 1 \).

The functions in 3.3(a)(i) and (b)(i) also satisfy the conditions of 3.2 for \( C^\infty(\delta W, V) \) with \( M_1 = \exp(A^{(p)^+}), 0 < \alpha < 1 \).

Notice that the sequence representation in Th. 1.6 holds for an essentially larger class of the above functions.

The conditions needed in 3.2 for \( C^\infty(\delta W, V) \) (and for \( C^\infty(\mu)^{-1} \delta W, V) \) with \( M_1 = \exp(A^{(p)^+}), (1.12) \) are in a sense stable for compositions: If \( W \) and \( V \) satisfy these conditions, then they are also satisfied by \( W \circ F \) and \( V \circ F \) if \( f := F \) is nondecreasing, positive and
\[ f(t) \leq \exp(C_1 F(t)) \quad \text{for large } t \]
(3.10)
\[ (\text{resp. } f(t) \leq \exp(C_1 F(t)^{p\nu^{(p)}}) \quad \text{for large } t). \]

(3.10')

So all conditions follow for \( \delta = 1 \) from (3.10) (and (3.10')) for \( W = F, V = F^p, \beta < 1, \) if \( (F^p) \) is increasing and positive.

The conditions of 3.2 hold for \( C^\infty(\delta W, V) \) and \( W(t) = \exp(t^2), V(t) = \exp(t^2) \) with \( 1/\alpha < \gamma < 1 \) if \( W = W \) is strictly increasing and
(3.11) \( \hat{w}(t) \leq \exp(\mu \hat{W}(t)) \) for any \( \epsilon > 0 \) and large \( t \).

This essentially is (1.4) for \( W = V \) so the conditions seem to get weaker for faster growing weight functions.

Indeed, Theorem 3.2 may be shown for \( C^0(W, W) \) in this case, i.e. we may omit the \( \alpha \)-condition (3.7):

3.4. THEOREM. Let \( W(t) = \exp \hat{W}(t) \), where \( \hat{W} \) and \( \hat{w} = \hat{W} \) are strictly increasing from 0 to \( \infty \) on \( [0, \infty) \). Let

(3.12) \( \hat{w}(t) \leq \exp(\epsilon \exp(\hat{W}(t))) \) for any \( \epsilon > 0 \) and large \( t \)

for \( C^0(W, W) \).\(^\dagger\)

(3.13) \( \exp(\hat{w}(t)) \leq C_t H \hat{w}(t) \) for large \( t \)

for \( C_t W, W \), where \( H_t \) is chosen as in (1.5).

Then the sequence

\[ 0 \rightarrow N \rightarrow (C^0(W, W), W)^n \rightarrow (C^0(W, W), W)^n \rightarrow N \rightarrow 0 \]

is exact and split.

Proof. All the assumptions of 3.2 are valid except for (3.7). This was only used in 3.2 to prove (1.2b) for \( \hat{w} \). So we have to show the following: For any \( p \neq N \) there are \( k \geq p \) and \( C > 0 \) such that

\[ \| f \|_{C^0(W, W), W}^n \leq C \| f \|_{C^0(W, W), W}^n + M \| f \|_{C^0(W, W), W}^n \]

for \( f \in C_t W, W \), where \( M \) is chosen as in (1.5).

The assumptions on \( M_t \) as being as before, this reduces to the following statement: For any \( n \neq N \) there are \( k \geq p \) and \( C > 0 \) such that

(3.14) \( (M W)^n (t) + (k W)^n (t) \leq 2 (p W)^n (t) + C \).

Here again only one variable is considered. (3.14) holds if

\[ A := \{ \xi \in C^0(N) : \sum |\xi_j| e^{-\mu W^2}(\xi_j) < \infty \text{ for } j \in N \} \]

has (DN) \(^\dagger\)

(155), Prop. 5.1. \( A \) is nuclear by the Grothendieck–Pietrow criterion, since

(3.15) \( (n W)^n (t) - (n W)^n (t) \geq (n - n_1) W \circ e^{-\mu W^2}(t) + (n_2 - n_1) \ln(t/n) / C \).

The last estimate follows from (3.12). (3.12) is also implied by (3.13): \( \ln(t/A) B \ln k \leq M(k) \leq H(k) \) by (1.19) and hence \( \ln t = o(H(t)) \). (3.15) also implies that

\[ A^* := \{ \xi \} : \sum |\xi_j| e^{-\mu W^2}(\xi_j) < \infty \text{ for some } n \in N \}

and that \( A^* \) is isomorphic to

\[ \mathcal{H}_C := \{ f \in \mathcal{H}(C) : f(t) \leq ke^{\mu t} \text{ for some } k \in N \} \]

with \( U(t) = W(\ln t) = \exp(\hat{W}(\ln t)) \) for \( |t| \geq 1 \) and \( U(0) = 0 \) otherwise. Theorem 3.4 now follows from

3.5. PROPOSITION. Let \( \hat{W} \) be convex and increasing and \( \hat{W}(t) = 0 \) for \( t \leq 0 \). Let

(3.16) \( \hat{w}(t) \leq \exp(\epsilon \exp(\hat{W}(t))) \) for any \( \epsilon > 0 \) and large \( t \)

where \( \hat{w} = \hat{W} \). Then \( \mathcal{H}_C \) has (DN) for \( U(t) = \exp(\hat{W}(\ln t)) \).

Proof. Let \( B := \{ f \in \mathcal{H}(C) : f(t) \leq e^{\mu t} \} \). By Lemma 2.1 in [15], we have to show the existence of \( \mu \in N \) such that for any \( \mu \in N \) there are \( p, C \) with

(3.17) \( B := e^\mu B + C e^\mu B \) for \( \mu > 0 \).

We set \( \psi(x) := (x - x_k + 1) g(x) \) and \( x_k = \ln(g(k)) \). Then

(3.18) \( \psi(x) \geq e^\mu \) for \( x \geq 0 \)

and \( \psi(x) \leq (k + 1 + p) e^\mu \) for \( x > 0 \), \( p = e^{\mu x} \)

(consider only the unique point where the derivatives of the left and right hand sides are equal). Moreover,

(3.19) \( \psi(x) \geq (k + 1) e^\mu \) for \( \hat{x}_k = x_k - k \leq x \leq x_k \).

Let \( \varphi(x) := \psi(\hat{W}(\ln x)) \). Then \( \varphi(x) \) is sh, since \( \varphi(\mu) = \psi(\hat{W}(x)) = \hat{W}(x) C \).

(157) and (158) give the following estimates:

(3.17) \( \varphi(x) \geq \mu^{\mu / \mu} \) for \( x > 0 \)

(3.18) \( \varphi(x) \leq (k + 1 + p) U(x) \) for \( p = e^{\mu x} \), \( x > 0 \),

(3.19) \( \varphi(x) \geq (k + 1) U(x) \) for \( \hat{x}_k = \hat{W}(\ln x) \),

where we suppose that \( \mu > 0 \) is so large that \( x_k = \ln(g(k)) > k \). Now,

\[ \delta_k := 1/\exp(\hat{W}(x_k) - \exp(\hat{W}(x_k))) \leq \hat{W}(x_k) / k \]

\[ \leq \exp(\hat{W}(\ln k)) \exp(\hat{W}(x_k)) \]
Let $f \in B_1$ and $H(z) := h(z)$. Then
\[ \hat{\gamma}(HF)(z) \leq C_4 e^{\epsilon_{1+2z/3}} \leq C_5 e^{\epsilon_{1+2z/3}}. \]
So $\hat{\gamma}(HF) e^{-\epsilon} \in L^2(C)$ and we may choose a solution $G$ of
\[ \hat{\gamma}G = -\hat{\gamma}(HF) \]
such that for some $C_9$ independent of $g$ (see Th. 4.4.2 in [4])
\[ \|G(z)^2 e^{-2\epsilon_{1+2z/3}}(1 + |z|^2)^{3/2} \| \leq C_9. \]
Set $f_1 := G + HF$ and $f_2 := (1 - H)f - G$. Then $f = f_1 + f_2$ and $f \in H(C)$. For large $\epsilon$, we get
\[ \begin{align*}
\left\| \left| f_1(z)^2 e^{-2\epsilon_{1+2z/3}} \right| dz \right\|^{1/2} &\leq C_4 \left( \left\| f_1(z)^2 \right\| e^{-2\epsilon_{1+2z/3}}(1 + |z|^2)^{3/2} \right)^{1/2} \leq C_5 e^{\epsilon}, \\
\left\| f_2(z)^2 e^{-2\epsilon_{1+2z/3}} \right\|^{1/2} &\leq C_0 \left( \left\| f_2(z)^2 \right\| e^{-2\epsilon_{1+2z/3}}(1 + |z|^2)^{3/2} \right)^{1/2} \leq C_7 e^{-\epsilon}. \end{align*} \]

The proof is completed by showing that the sup-norms may be estimated by the $L^2$-norms. For $f \in X(C)$ we have
\[ \|f\| \leq C \left( \int |f(z)|^2 dz \right)^{1/2} \text{ for any } R > 0. \]
For fixed $z$ choose $R_5$ such that $U(|z| + R_5) = 2U(|z|)$. Then
\[ 1/R_5 = u(z)/U(z) \leq C e^{\epsilon_{1+2z/3}} \leq C e^{\epsilon_{1+2z/3}} \]
by the mean value theorem and (3.12). This shows (for $|z| \geq 2$)
\[ \|f(z)\| \leq C \left( \int |f(z + \xi)|^2 dz \right)^{1/2} \leq C e^{\epsilon_{1+2z/3}} \left( \int |f(z)|^2 e^{-2\epsilon_{1+2z/3}} dz \right)^{1/2} \leq C e^{\epsilon_{1+2z/3}} \left( \int |f(z)|^2 e^{-2\epsilon_{1+2z/3}} dz \right)^{1/2} \text{ for } \xi \in N. \]
So the proposition is proved.

Property (DN) for $(\mathcal{F}_C)_0$ was characterized in terms of $U$ by R. Meise and B. A. Taylor ([12]), assuming, however, that $X_C$ is invariant for shifts, which gives a priori estimates for $W$.

If $\tilde{W} + \ln \tilde{w} = \tilde{w}$ is convex, then (3.14) may be seen directly: $w^{-1} \circ \exp$ is then concave and
\[ w^{-1}(t/p) - w^{-1}(t/p^2) \geq w^{-1}(t) - w^{-1}(t/p). \]
This shows (3.14) for $k = p^3$, $m = 1$ and $C = 0$.

Theorem 3.4 does not hold for stable weight functions $M$ (i.e. $M(2t) \leq CM(t)$). In this case, $C^\infty(M, M)$ is the space $W_{M^\infty}$, which was considered in [8]. A large class of hypoelliptic operators have no solution operators in $(W_{M^\infty})_0$. In particular, semilinear nonelliptic equations have no solution operators in $(W_{h^\infty})_0$ for $0 < h < 1$ (except for at most one value of $h$).

3.6. Remark. Let $G \in C^1(0, \infty)$ and let $g := G'$ be strictly increasing from 0 to $\infty$ on $[0, \infty)$. Then there is $W \in C^1(0, \infty)$ such that $w := W'$ is increasing to $\infty$ and
\[ G(t) \leq W(t) \leq G(t + C) + 1 \text{ for some } C > 0 \text{ and large } t, \]
\[ w(t) \leq W(t) \text{ for } \text{any } \epsilon > 1 \text{ and large } t. \]

Proof. Let $c_n := G^{-1}(n)$ for $n > 0$. Then $\gamma := c_n - c_{n-1}$ is strictly decreasing to 0 and $\sum c_n = \infty$. Let the graph of $v_0$ be defined by the line segments joining the points $(n, \Gamma_n)$ and $(0, 0)$ for $\Gamma_n := 1/(n(n(n+1)^3)) + \gamma_n$. Let $V$ be the inverse function of $V_1(t) := \int_0^t v_1(x) dx$. Then $v := V'$ is strictly increasing and for large $n$ we get for $t \in (n(n-1), n]$,
\[ G^{-1}(t) \leq c_n = \sum_{k=1}^n \gamma_k \leq \sum_{k=1}^n \int_0^t v_1(x) dx \leq V_1(t+1), \]
\[ G^{-1}(t) \geq c_{n-1} = \sum_{k=1}^{n-1} \gamma_k \geq \sum_{k=1}^{n-1} \int_0^t v_1(x) dx \leq \int_0^t C_1, \]
\[ \sum_{k=n}^{n-1} \nabla n(n) \frac{1}{n(n(n+1)^3)} \leq \infty. \]
So
\[ G(t) \leq W(t) := V(t + C) \leq G(t + C) + 1 \text{ for large } t. \]

Fix $\epsilon > 1$. Then we have for large $n$ and $t \in \{n, n+1\}$
\[ v_1(t) \geq \Gamma_n \geq 1/(n(n+1)(n(n+2)^3)) \geq \epsilon^{-4}, \]
\[ v(t) = V'(t) = \frac{1}{\Gamma_n} V_1'(t) \leq V_1'(t) = W'(t) \text{ for large } t. \]
So $W$ has the above-stated properties.

So (1.4) and (1.5) do not imply a priori bounds for the growth of the weights by the remarks after 3.3. Moreover, the whole space of distributions of finite order may be filled with weighted spaces satisfying the assumptions of this paper. However, there is no solution operator for hypoelliptic equations $P(D)$ in the space of distributions of finite order. Indeed, if the kernel of $P(D)$ were complemented in $D_1(R^k)$, then it would be complemented in $C^\infty(R^k)$, contrary to the result of Vogt ([18]).

Similarly, $(W_{(M^\infty)}_0$ is the union of the spaces $W_{(M^\infty)}_0$ which are general splitting spaces by 3.4. Nevertheless, $(W_{(M^\infty)}_0$, in general allows no linear continuous solution operator for partial differential equations ([8]).
If \( P \) is hypoelliptic, then \( N_P \cap C^\infty(-W, V)^p \) may coincide (topologically) with \( N_P \cap A^p \) for some continuously embedded space \( A \subset C^\infty(-W, V) \). Then \( N_P \cap A^p \) is complemented in \( A^p \) if it is complemented in \( C^\infty(-W, V)^p \). So Th. 3.2 may give solution operators in weighted spaces of ultradifferentiable functions assuming only (1.4) (and not (1.5)). Using this argument for \( C^\infty(W, V)^p \), one may get solution operators in spaces which are not tractable directly by the methods of this paper, since they fail to be power series spaces of infinite type (for example \( \| f \in C^\infty(R^n) \|_p^{\mu,\nu} = \int_0^1 e^{-([\nu+\mu]t)} \| f(t) \|_p \| < \infty \text{ for some } \mu, \nu \in \mathbb{N} \)).

References


