Basic sequences and smooth norms in Banach spaces

by

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Abstract. This paper is concerned with the existence of basic sequences with small basis constants. We also study smoothness properties of norms.

Preliminaries. Let $E$ be a Banach space. The characteristic of a subspace $X$ of the dual $E^*$ is the number

$$r(X) = \inf_{\sigma \in \sigma(X)} \sup_{f \in E^* \setminus \sigma(X)} \frac{|f(x)|}{\|f\| \|x\|}.$$ 

We refer the reader to [9], [16] for the basic facts on this notion. Obviously $0 \leq r(X) \leq 1$.

We define

$$\chi(E) = \sup \{r(X) : X \not\subseteq E^*\}.$$ 

For a nonreflexive Banach space $E$, one has $\frac{1}{2} \leq \chi(E) \leq 1$. We notice that if $E$ is a Banach space with a basis $(x_n)$ and $X$ is the subspace of $E^*$ spanned by the coefficient functionals then we have $r(X) > K(x_n)^{-1} > 0$ (where $K(x_n)$ is the basis constant of $(x_n)$).

We get the following

Lemma 0.1. If $(x_n)$ is a basis of $E$ with basis constant strictly less than $\chi(E)^{-1}$ then $(x_n)$ is shrinking.

We will show that a kind of converse of this lemma is valid: namely, if $E$ is any Banach space, there exists, for any $\varepsilon > 0$, a nonshrinking basic sequence $(x_n)$ in $E$ with $K(x_n) \leq \chi(E)^{-1} + \varepsilon$. In this direction, A. Pelczyński has shown the existence of a nonshrinking basic sequence in a nonreflexive Banach space [14]. We also mention the result of M. Zippin: if $E$ is a Banach space with a basis, then $E$ is reflexive if and only if each basis of $E$ is shrinking [17]. In both results, there are no estimates of the basis constants.

We end the first chapter by some applications, in particular we get an extension of a result of [7].

In the second chapter, we introduce a class of Banach spaces satisfying smoothness conditions and we get $\chi(E) < 1$ for any Banach space $E$ in this class (Theorem 2.3). As a consequence, if $X$ and $Y$ are superreflexive Banach
spaces then $\chi(K(X, Y))$ is strictly less than 1, where $K(X, Y)$ is the space of compact operators from $X$ to $Y$. We also present some questions and in particular a renaming problem.

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Notation. If $E$ is a Banach space, the closed unit ball of $E$ is denoted by $B(E)$, the unit sphere by $S(E)$. The dual of $E$ is $E^*$. The basic constant of a basic sequence $(x_n)$ is denoted by $K(x_n)$. For a set $A$, conv $(A)$ is the norm-closed convex hull of $A$ and $[A]$ is the span of $A$. $K(X, Y)$ denotes the space of compact operators from $X$ to $Y$ (with the operator norm). For the definitions and basic facts on basic sequences the reader should consult [1].

1. Construction of basic sequences with a given constant. The main result of this chapter is the following:

**Theorem 1.1.** Let $E$ be a nonreflexive Banach space. For every $\epsilon > 0$, there exists a nonshrinking basic sequence $(x_n)$ such that $K(x_n) \leq \chi(E)^{-1} + \epsilon$ and there exists a nonboundedly complete basic sequence $(y_n)$ such that $K(y_n) \leq 1 + \epsilon$.

Let us first note that there is no restriction on $\chi(E)$. Hence any nonreflexive Banach space contains a nonshrinking basic sequence $(x_n)$ with $K(x_n) \leq 2 + \epsilon$; any nonreflexive dual space contains a nonshrinking basic sequence $(y_n)$ with $K(y_n) \leq 1 + \epsilon$. The proof of the theorem does not use the methods of A. Pełczyński [14] or M. Zippin [17] but the local reflexivity principle [12].

**Proof of the theorem.** Observe first that $\chi(E)$ may also be computed by the following formula:

$$
\chi(E)^{-1} = \inf_{\pi \in \mathbb{R}^*} \|\pi\|: \pi \text{ is the projection of } E \oplus Rx^* \text{ onto } E \text{ with kernel } Rx^*.
$$

Let $\epsilon$ be any positive number. There is $x^*_n \in E^*, \|x^*_n\| = 1$ and $\|x_n\| \leq \chi(E)^{-1}(1 + \epsilon)$, where $\pi$ is the projection of $E \oplus Rx^*$ onto $E$ with kernel $Rx^*$. We will construct a sequence $(x_n)$ in $E$ such that for any increasing sequence of integers $(n_i)$:

(a) The sequence $(x^*_n, x^*_n - x_{n_i})$ is basic with basis constant near to 1.
(b) $(x_{n_i} - x_{n_i-1})$ is a basic sequence with basis constant near to $\chi(E)^{-1}$.
(c) $(x_{n_i})$ is a nonshrinking basic sequence with basis constant near to 1.

Choose $(\epsilon_n)_{n \geq 0}$ so that $0 < \epsilon_n < 1$ for all $n \geq 0$ and $\prod_{n=0}^{\infty} (1 - \epsilon_n) \geq 1 - \epsilon_0$ for any increasing sequence of integers $(i)$. There is $f_i \in S(E^*)$ such that $x^*_n(f_i) \geq 1 - \epsilon_i$. Let $F_i$ denote the 1-dimensional subspace of $E^{**}$ spanned by $x^*_n$. By the local reflexivity principle [12], there is a one-to-one operator $T_i: F_i \to E$ so that:

1. $\|T_i\| \leq 1 + \epsilon_i$.
2. $f_i(T_i x^*_n) = x^*_n(f_i)$.

We put $x_i = T_i x^*_n$.

Let $F_2$ denote the 2-dimensional subspace of $E^{**}$ spanned by $x^*_n$ and $x^*_n - x_i$. There is a set $Z_2 = \{e_1, \ldots, e_{2N(2)}\} \subset S(F_2)$ which forms an $e_2/2$-net for $S(F_2)$. Pick $e^*_1, \ldots, e^*_{2N(2)} \in S(E^*)$ so that $e^*_i(e_i) > 1 - e_2/2$, and put

$$
Z^*_2 = \{e^*_1, \ldots, e^*_{2N(2)}\}.
$$

By the local reflexivity principle, there is a one-to-one operator $T_2: F_2 \to E$ such that:

1. $\|T_2\| \leq 1 + \epsilon_0$.
2. $f_i(T_2 x^*_n) = x^*_n(f_i), \quad \forall x^*_n \in F_2$.
3. $e^*_i(T_2 x^*_n) = x^*_n(e^*_i), \quad \forall e^*_i \in Z^*_2$.
4. $T_2 x_i = x_i$.

We put $x_2 = T_2 x^*_n$.

Notice that for any scalar $a$, we get

$$
\|x^*_n + a(x^*_n - x_i)\| \geq 1 - \epsilon_i.
$$

Indeed,

$$
\|x^*_n + a(x^*_n - x_i)\| \geq |f_i(x^*_n + a(x^*_n - x_i))|}

\geq |f_i(x^*_n)| - |a| |f_i(x^*_n - x_i)| \geq 1 - \epsilon_i.
$$

We repeat the above procedure. Inductively, we find for all $n \geq 2$

$$
F_n = \{x^*_n, x^*_n - x_{n+1}, \ldots, x^*_n - x_{n-1}\},
$$

$$
Z_n = \{e_1, \ldots, e_{2N(n)}\} \text{ an } e_2/2\text{-net for } S(F_n),
$$

$$
Z^*_n = \{e^*_1, \ldots, e^*_{2N(n)}\} \subset S(E^*) \text{ such that } e^*_i(e_i) > 1 - e_2/2,
$$

and a 1-1 operator $T_n: F_n \to E$ with:

1. $\|T_n\| \leq 1 + \epsilon_0$.
2. $f_i(T_n x^*_n) = x^*_n(f_i), \quad \forall x^*_n \in F_n$.
3. $e^*_i(T_n x^*_n) = x^*_n(e^*_i), \quad \forall e^*_i \in Z^*_n$.
4. $T_n x_i = x_i, \quad \forall x_i \in F_n \cap E$.

We put $x_n = T_n x^*_n$.

(a) The same computation as before gives $\|x^*_n + a(x^*_n - x_i)\| \geq 1 - \epsilon_i$, for all $a > 1$. 


We now claim that for any \( y \in S(F_n), n \geq 2, \) any scalar \( a, \) and any \( p \geq 0, \)
\[
\|y + a(x^n - x_{n+p})\| \geq 1 - \varepsilon_a.
\]
Indeed, there exists \( e_0 \in Z_n \) such that \( \|y - e_0\| \leq \varepsilon_d/2 \) and \( e_0 \in Z_n \geq 1 - \varepsilon_d/2. \)
We get
\[
\|y + a(x^n - x_{n+p})\| \geq \|e_0^* (y + a(x^n - x_{n+p}))\| \\
\geq \|e_0^* (a e_0) - |e_0^* (e_0 - y) - |a|| e_0(x^n - x_{n+p})\| \\
\geq 1 - \varepsilon_d/2 - \|e_0 - y\| \geq 1 - \varepsilon_a.
\]
For any scalars \( a_1, \ldots, a_{n+p} \) and for any increasing sequence of integers \((i),\)
we get
\[
\sum_{i=1}^{n+p} a_i (x^n - x_{n+i}) = \sum_{i=1}^{n+p-1} a_i (x^n - x_i) + a_{n+p} (x^n - x_{n+p}) \\
\geq (1 - \varepsilon_a) \sum_{i=1}^{n+p-1} a_i (x^n - x_i) \\
\geq \prod_{i=1}^{n+p-1} (1 - \varepsilon_a) \sum_{i=1}^{n} a_i (x^n - x_i) \\
\geq (1 - \varepsilon_a) \sum_{i=1}^{n} a_i (x^n - x_i).
\]
This concludes the proof of (a).

(b) Let \((i)\) be any increasing sequence of integers and let \( a_1, \ldots, a_n \) be any scalars. Then
\[
\|\sum_{i=1}^{n} a_i x_i\| = \|(\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} a_i x^n)^+\| \\
\leq \|\|\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} a_i x^n\|\| \\
\leq \chi(E)^{-1} (1 + \varepsilon) \|\sum_{i=1}^{n} a_i x^n + \sum_{i=1}^{n} a_i (x^n - x_i)^+\| \\
\leq \chi(E)^{-1} (1 + \varepsilon) \|\sum_{i=1}^{n} a_i x^n + \sum_{i=1}^{n} a_i (x^n - x_i)^+\| \\
\leq \chi(E)^{-1} (1 + \varepsilon) \|\sum_{i=1}^{n} a_i x_i\|.
\]
This proves that the sequence \((x_i)\) is basic with
\[
K(x_i) \leq (1 + \varepsilon) (1 - \varepsilon_a)^{-1} \chi(E)^{-1}.
\]
Moreover, since for all \( n, f_1(x_i) = x_i^n, \) \( f_1(x_i) \geq 1 - \varepsilon_2, \) the sequence \((x_i)\) is nonshrinking.

Hence, in particular, we have proved that there exists a nonshrinking basic sequence \((x_\lambda)\) with basis constant near to \( \chi(E)^{-1}. \)

(c) We will suppose \( \|T_n\| \leq 1 \) and \( \|T_n^{-1}\| \leq 1 + \varepsilon_0. \) Then
\[
\|\sum_{i=1}^{n} a_i (x_i - x_{n+i})\| = \|a_1 (x_1 - x_2) + \ldots + a_n (x_{n-1} - x_n)\| \\
\leq \|a_1 (x_1 - x_2) + \ldots + a_n (x_{n-1} - x_n)\|. 
\]
Since \( x_{n-1} - x_n = x_{n-1} - x_n - x^n \) and \( x_{n-1} - x^n \) is a basic sequence, we have
\[
\|\sum_{i=1}^{n} a_i (x_i - x_{n+i})\| \\
\leq (1 - \varepsilon_0)^{-1} \|a_1 (x_1 - x_2) + \ldots + a_{n-1} (x_{n-1} - x_n) + a_n (x_n - x^n)\| \\
+ \|a_1 (x_1 - x_2) - x^n\| + \|a_2 (x_2 - x_3) + \ldots + a_{n-1} (x_{n-1} - x_n) - x^n\| + \ldots + \|a_{n-1} (x_{n-1} - x_n) + \ldots + a_n (x_n - x^n) - x^n\| \\
\leq (1 - \varepsilon_0)^{-1} \|T_{n-1}\| \|T_{n-1}\| \|T_{n-1}\| \|T_{n-1}\| \|T_{n-1}\| \|T_{n-1}\| \|T_{n-1}\| \|T_{n-1}\| \|T_{n-1}\| \|T_{n-1}\| \\
\leq (1 - \varepsilon_0)^{-1} (1 + \varepsilon_0) \|\sum_{i=1}^{n} a_i (x_i - x_{n+i})\|.
\]
and so \((x_i - x_{n+i})\) is a basic sequence with basis constant near to 1.

We are ready to finish the proof of the second assertion of the theorem. We first assume that \( E \) contains an isomorphic copy of \( l_1. \) Then \( E \) contains almost isometric copies of \( l_1 [11]. \) Let \((e_0)\) be the usual basis of \( l_1. \) Consider the sequence \((x_\lambda) = (e_1, e_2, e_3, \ldots, e_2, e_1, \ldots). \) Then \((x_\lambda)\) is a non-boundedly complete basis of \( l_1\) with basis constant equal to 1.

Suppose now that \( E \) does not contain a copy of \( l_1. \) There is no loss of generality in assuming that \( E \) is separable. Then our sequence \((x_\lambda)\) has a subsequence \((x_\mu)\) which is \( \sigma(E^n, E^n)\)-convergent. But \((x_\mu)\) does not converge to a point of \( E, \) since 0 is the only possible limit point of a basic sequence, but for all \( n, f_1(x_\mu) \geq 1 - \varepsilon_1. \)

We have
\[
\sum_{i=1}^{n} (x_i - x_{n+i}) = -x_{n+1} + x_n.
\]
Therefore \( \sup_{\|x\|} \|T^*_{\lambda}(x_n - x_{n-1})\| \) is finite, but the series \( \sum_{n=1}^\infty (x_n - x_{n-1}) \) does not converge in \( E \). Moreover, \( (x_n - x_{n-1}) \) is basic with basis constant near to 1.

Let us mention some applications.

We define \( \chi(E) \) for a Banach space \( E \) by

\[
\chi(E) = \inf \{ \chi(F)^{-1} : F \subseteq E \}.
\]

\( \chi(E) \) can be considered as the "shrinking basic sequences index" because of the following corollary:

**Corollary 1.2.** Let \( E \) be a Banach space. Then

\[
\chi(E) = \sup \{ \lambda : \text{every basic sequence } (x_n) \text{ with } K(x_n) < \lambda \text{ is shrinking} \}.
\]

**Corollary 1.3.** If \( E \) is a separable Banach space with \( \chi(E) > 1 \), then \( E^* \) is separable.

**Proof.** Since \( E \) is separable, there exists a \( \sigma(E^*, E) \)-dense sequence \( (x_n) \) in \( B(E^*) \). So \( r[x_n] = 1 \) and \( [x_n] = E^* \).

The above corollary should be compared with the following result: if \( E \) is a separable space such that \( \{x_n\} \) is separable for every basic sequence \( (x_n) \) then \( E^* \) is separable [10].

**Remark.** Let \( E \) be a Banach space which is an \( M \)-ideal of its dual. Then \( \chi(E) = 1/2 \) [7]. This class is stable under subspaces. Hence \( \chi(E) = 2 \).

We now give an isomorphic version of the following result of G. Godefroy and P. Saabhar [7]. Let \( E \) be a Banach space such that \( E^* \) contains no proper 1-norming subspace and let \( (T_n) \) be a sequence of contractions on \( E \) such that for every \( x \in E \), \( \lim_{n \to \infty} \|T_n x - x\| \to 0 \). Then we have \( \lim_{n \to \infty} \|T_n^* x - x^*\| = 0 \) for every \( x^* \in E^* \).

We will replace the metric assumption (the \( T_n \) are contractions) by an algebraic one (the \( T_n \) commute). Our next result is

**Corollary 1.4.** Let \( E \) be a separable Banach space such that \( (2x > 1, \forall (x_n) \text{ basic sequence: } K(x_n) < x \Rightarrow (x_n) \text{ shrinking}) \). Then \( \forall X \subseteq E, \forall T_n : X \to X \) with

\[
\text{rank}(T) < \infty, \quad \sup \|T\| < \alpha,
\]

(\( * \))

\[
T_0 = T_n T_0, \quad \forall k, n, \quad \|T_n x - x\| \to 0, \quad \forall x \in X,
\]

we have \( \|T_n^* x^* - x^*\| \to 0, \quad \forall x^* \in X^* \).

**Proof.** Let \( (T_{\lambda})_{\lambda \geq 1} \) be a sequence of operators on \( X \) satisfying \( (\ast) \). Then

\[
\lim_{\lambda \to \infty} \|T_{\lambda} x^* - x^*\| = 0 \text{ for every } x^* \text{ in the norm-closed subspace } \Gamma \text{ of } X^* \text{ generated by } \bigcup_{n=1}^\infty T_n^* (X^*) \text{ [16]}. \]

But it is easy to check that \( r(\Gamma) = (\sup \|T_n\|)^{-1} \) and thus \( r(\Gamma) > x^{-1} \), and this implies that \( \Gamma = X^* \).

Let us note that under the same assumptions on \( E \), the projections associated with a basic sequence satisfy the conclusion of Corollary 1.4. We get here the result \( (\lim_{\lambda \to \infty} \|T_{\lambda} x^* - x^*\| = 0 \text{ for every } x^* \in X^*) \text{ for any sequence } T_{\lambda} \text{ satisfying } (\ast) \).

**2. A smoothness condition. Examples.** Let us introduce a condition of uniform smoothness which is close to the one given in [4], [5]. We use the same terminology. Let \( E \) be a Banach space. \( \mathcal{D}(E) \) is the set of points of the unit sphere where the norm is Fréchet-smooth; for every \( x \in \mathcal{D}(E) \), we denote by \( f_x \) the differential of this norm at \( x \).

**Definition 2.1.** We say that \( E \) is almost uniformly smooth (a.u.s.) if there exists a family \( (A_{\lambda})_{\lambda < 1} \) of subsets of \( \mathcal{D}(E) \) such that:

(a) \( \forall \lambda \in ]0, 1], \exists \delta(\lambda) > 0, \exists \varepsilon > 0 \):

\[
y \in B(E^*), \quad x \in A_\lambda, \quad \|y - f_x\| \geq 1 - \delta(\lambda) \Rightarrow \|y - f_x\| \leq \varepsilon.
\]

(b) \( \forall \lambda \in B(E^*) = \text{conv } \{ f_x : x \in A_\lambda \} + \varepsilon B(E^*) \).

**Examples.** \( c_0(\Gamma) \) is a.u.s. for any \( \Gamma \). Every superreflexive Banach space is a.u.s. for every equivalent norm [5]. A nontrivial class of new examples is provided by our next result:

**Proposition 2.2.** If \( X, Y \) are superreflexive Banach spaces then \( K(X, Y) \) is a.u.s.

**Proof.** The proof follows the method of [4], [7]; we only give the main ideas. The following facts are known. Let \( x \in S(X) \) (resp. \( y \in S(Y) \)) and assume that \( x \) (resp. \( y \)) is strongly exposed in \( B(X) \) (resp. \( B(Y) \)) by \( f_x \in X^* \) (resp. \( f_y \in Y^* \)). Then \( x \otimes y \) is strongly exposed in \( B(X \otimes Y) \) by \( f_x \otimes f_y \) [7]. Moreover, if there is a uniformity in the strong exposition of \( x \) and \( y \) then there is also a uniformity in the strong exposition of \( x \otimes y \) [4]. Since a superreflexive Banach space is a.u.s., the unit ball of \( X \) and the unit ball of \( Y \) are of type (b) from Definition 2.1. Using the technique of [7], we conclude that the unit ball of \( X^* \otimes Y \) is also of type (b).

Of course, the proposition is not true in general for reflexive Banach spaces. If \( X = \bigoplus_{\lambda \geq 1} \mathbb{T}_\lambda \) and \( Y = R \) then \( K(X, Y) = \bigoplus_{\lambda \geq 1} \mathbb{T}_\lambda \) and \( K(X, Y) \) is not a.u.s. [4]. J. Partington has shown that every Banach space may be equivalently renormed to have property \( \beta \) [13], see also [15]. For a reflexive Banach space it is easy to see that property \( \beta \) implies almost uniform smoothness. Therefore if \( X \) and \( Y \) are reflexive Banach spaces then \( K(X, Y) \) admits an equivalent almost uniformly smooth norm.
We use the technique of [4] in the proof of the main result of this chapter.

**Theorem 2.3.** If $E$ is a.u.s. then $\chi(E)$ is strictly less than 1.

**Proof.** Suppose $\chi(E) = 1$. There is a sequence $(y_n)_{n=1}^{\infty}$ in $(E^*)^*$, $E$ and a sequence of projections $(x_n)$ such that $x_n$ is the projection of $E \oplus B_{\infty}$ onto $E$ with kernel $R_{x_n}$ and $1 < \|x_n\| < 1 + 1/n^2$. Therefore
\[
\|x\| < (1 + 1/n^2) \|x + y\| \quad \text{for any } y \in E.
\]
By using a technique similar to that in [4] we show that $\ker x \cap B(E^*)$ is "almost" $\sigma(E^*, E)$-dense in $B(E^*)$. More precisely, there is an integer $N$ such that for every $n \geq N$ and every $x \in S(E)$, one has
\[
\ker x \cap B(E^*) = \{ x \in B(E^*): \|x\| < 1 - 1/n \}.
\]
The idea is the following. Suppose this is not true. Let $N$ be an integer such that there are $n > N$ and $x \in S(E)$ satisfying
\[
\ker x \cap B(E^*) = \{ x \in B(E^*): \|x\| < 1 - 1/n \}.
\]
Then $u = (n+1/n) x \in E$ it can be shown as in [4] that $\|u\| < (1 + 1/n^2) \|u - y\|$, a contradiction with (I).

Therefore
\[
\exists N, \forall n > N, \forall x \in A_n, \exists y_0(x) \in \ker x \cap B(E^*): \|y_0(x)\| > 1 - 1/n.
\]

Let $M \geq \max \{ N, 1/\delta(x) \}$; then for $n > M$ we get $\|y_0(x) - f_n\| \leq \varepsilon$. Now,
\[
\|y_0(x) - y_n(x)\| \leq \|y_0(x) - f_n\| + \|f_n - y_n(x)\|.
\]
Since $y_0(x) \in \ker x$, one has
\[
\forall \varepsilon > 0, \exists M, \forall n > M: \sup_{x \in A_n} \|y_0(x) - f_n\| \leq \varepsilon.
\]
By Definition 2.1(b), this implies $\|y_n\| \to 0$ as $n \to \infty$, a contradiction. ■

**Corollary 2.4.** If $X$ and $Y$ are superreflexive Banach spaces then $\chi(K(X, Y))$ is strictly less than 1.

We now mention two questions.

**Question 2.5.** Let $X$, $Y$ be superreflexive Banach spaces. Does there exist $\alpha > 1$ such that if $(x_n)$ is a basic sequence in $K(X, Y)$ with $K(x_n) < \alpha$ then $(x_n)$ is shrinking? Of course, Theorem 2.3 does not imply the existence of $\alpha$ since the property $\chi(E) < 1$ is not hereditary (L. V. Gidun and A. M. Plichko have constructed [6] an example of a Banach space $X$ which contains a norm-one complemented hyperplane $Y$ with $\chi(Y) = 1$, $\chi(X) = 2/3$).

The second question concerns a renorming problem:

**Question 2.6.** Is it possible to renorm a Banach space $E$ with separable dual so that $\chi(E) < 1$? (see [2]). The answer is affirmative whenever $E$ is a quasi-reflexive Banach space: if $E$ is a quasi-reflexive Banach space then there exists on $E$ an equivalent norm such that $\chi(E) = 1/2$ [8].

**References**


