4. The final norm. Let now
\[ \|p\| = \lim_{n \to \infty} |p|_{\infty} \]
for any polynomial \( p \). We have the following properties of the limit norm \( \| \cdot \| \):

**Proposition 5.**

(a) \( \|q_j - q\| \leq \epsilon_j \), \( j \geq 1 \).

(b) \( \|p_j\| \leq 2^{4j} \|p\| \).

(c) \( \|xp\| \leq 2 \|p\| \).

(d) The norm \( \| \cdot \| \) is hilbertian.

(e) For any \( n \geq 1 \) and any \( p \) with \( d^p \leq n \),
\[ \|p\| \geq \prod_{k \geq n} (1 - 4^{-k}) |p|_{n-1} \].

This last property ensures of course that the limit norm is nonzero. Therefore the completion of the polynomials for \( \| \cdot \| \) is a Hilbert space on which the multiplication by \( x \) is continuous. Every polynomial \( q \) with rational coefficients is hypercyclic. Indeed, let \( q' \neq 0 \) in \( H \), and let \( \epsilon > 0 \). We can find in the enumeration an integer \( j \) such that
\[ q_j = q, \quad \epsilon_j \leq \epsilon/2, \quad |q_j - q| \leq \epsilon/2. \]

Then \( \|q_j - q\| < \epsilon/2 \), and
\[ \|x^{N_j} q_j - q\| \leq \|x^{N_j} q_j - q\| + \|q_j - q\| < \epsilon, \]
which proves our claim.

**References**


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Some remarks on Triebel spaces

by

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Abstract. Some extensions of results in the recent monograph by Triebel [13] about Triebel spaces \( F^s_{p, \infty} \) are given. This concerns multiplication properties, dual spaces and some remarks on the spaces \( F^s_{p, \infty} \).

0. Introduction. Triebel spaces are a natural generalization of Sobolev–Hardy spaces. The characterization of these spaces by decompositions of Littlewood–Paley type provides a useful tool for the study of multiplication properties, dual spaces, etc.

The plan of this paper is as follows. Chapter 1 is used to fix the notation and to recall some results on Besov and Triebel spaces. In Chapter 2 multiplication properties of Triebel spaces are studied: multiplication by functions belonging to Hölder–Zygmund spaces, multiplication algebras and multiplication by the characteristic function of an interval.

Chapter 3 is devoted to some complementary results in the determination of dual spaces. The main result can be phrased as follows. Let us denote by \( \hat{F}^s_{p, \infty} \) the closure of the Schwartz space \( \mathcal{S}(\mathbb{R}^n) \) in \( F^s_{p, \infty} \). Then for \( 1 < p, q < \infty \) the dual of \( \hat{F}^s_{p, \infty} \) is isomorphic to \( F^t_{p', q'} \), \( 1/p + 1/q = 1/p + 1/q' = 1 \). Also some extensions to weighted spaces are given. The weight may belong to the Muckenhoupt class \( A_p \).

Finally, Chapter 4 contains some remarks on \( F^s_{p, \infty} \), \( 1 \leq q < \infty \). In particular, the trace problem is solved.

1. Besov and Triebel spaces. All functions and distributions are assumed to be defined on the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). \( \mathcal{S}(\mathbb{R}^n) \) is the Schwartz space of rapidly decreasing functions and \( \mathcal{S}(\mathbb{R}^n) \) its dual, the space of tempered distributions.

The Fourier transform is defined by
\[ \hat{f} (\xi) := \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x) dx, \quad f \in \mathcal{S}(\mathbb{R}^n), \]
and extended to \( \mathcal{S}(\mathbb{R}^n) \) by duality. The inverse Fourier transform is
\[ f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} \hat{f}(\xi) d\xi. \]
For suitable distributions $f$ and $g$, let us denote by $f * g$ the convolution of distributions.

Denote by $\phi(R^n)$ the set of all partitions $\{\phi_k\} \subset \mathcal{D}'(R^n)$ such that

1. $\text{supp } \phi_k \subset \{ \xi : |\xi| \leq 2^k \}$, $\text{supp } \phi_k \subset \{ \xi : 2^{k-1} \leq |\xi| \leq 2^k \}$ for $k = 1, 2, \ldots$,

2. $|\xi^\alpha \phi_k(\xi)| \leq C_k 2^{-k|\alpha|}$

for all multi-indices $\alpha$ and $k$.

3. $\sum_{k=0}^{\infty} \phi_k(\xi) = 1$.

$\phi(R^n)$ is not empty; see Triebel [13], Remark 2.3.1.1.

For $0 < p < \infty$, $0 < q \leq \infty$ and $s \in R$ we define the Triebel space $F^s_{pq}$ to be the set of all $f \in \mathcal{D}'(R^n)$ such that

4. $\|f\|_{F^s_{pq}} := \|2^{ks} \phi_k * f\|_{L^p} < \infty$

(modification for $q = \infty$).

Similarly for $0 < p < \infty$, $0 < q \leq \infty$ and $s \in R$ we define the Besov space $B^s_{pq}$ to be the set of all $f \in \mathcal{D}'(R^n)$ such that

5. $\|f\|_{B^s_{pq}} := \left( \sum_{k=0}^{\infty} 2^{ks} \|\phi_k * f\|_{L^p} \right)^{1/s} < \infty$

(modification for $q = \infty$).

These spaces are independent of the choice of the partition $\{\phi_k\} \subset \phi(R^n)$. Elementary properties are:

6. $B^s_{p_0,q_0} \subset F^s_{p_0,q_0} \subset B^s_{p_1,q_1}$

7. $F^s_{p_0,q_0} \subset B^s_{p_1,q_1}$

8. $F^s_{p_0,q_0} \subset F^s_{p_1,q_1}$

if $q_1 \leq q_2$ and $s_1 = s_2$, or $s_1 > s_2$.

An essential tool in Triebel [13] is the Peetre maximal function. It has the disadvantage of not giving optimal results. Therefore we will not use it. Note, however, that there exists a new maximal technique, which avoids this drawback in many situations; see Marshall [8], Chapter 4.

In this paper we use the following results.

**Lemma 1.** Let $f_k \in \mathcal{D}'(R^n)$ and suppose that for some constants $0 < c_1 < c_2$

$$\text{supp } f_k \subset \{ \xi : |\xi| \leq c_2 \}, \quad \text{supp } g_k \subset \{ \xi : c_1 2^k \leq |\xi| \leq c_2 2^k \}$$

for $k = 1, 2, \ldots$. Then for $0 < p, q \leq \infty$ and $s \in R$

$$\sum_{k=0}^{\infty} \|f_k\|_{L^p}^p \leq C \|f\|_{L^p}^p$$

$$\sum_{k=0}^{\infty} \|g_k\|_{L^p}^p \leq C \|g\|_{L^p}^p$$

The lemma follows immediately using the Nikol’skii representation; see Triebel [13], Theorem 2.5.2.

**Lemma 2.** Let $f_k \in \mathcal{D}'(R^n)$ be distributions such that for some constant $0 < c_1$

$$\text{supp } g_k \subset \{ \xi : |\xi| \leq c_0 \}$$

Then for any real number $s$ with $s > n \max \{1, 1/p - 1\}$

$$\sum_{k=0}^{\infty} \|f_k\|_{B^s_{p,q}} \leq C \|f\|_{B^s_{p,q}}$$

**Proof.** Let $\{\phi_0, \phi_1\} \subset \phi(R^n)$. Then for some $l = l(\xi) \in N$ one has

$$\phi_j(\xi) \left( \sum_{k=0}^{\infty} f_k \right) = \sum_{k=0}^{\infty} \hat{\phi}_j(\xi) \hat{f}_k(\xi)$$

Now for any integer $k$ satisfying $k \geq j - l$ the following inequality holds with $p_1 := \min \{1, p\}$:

$$\|\hat{\phi}_j * f_k\|_{L^p} \leq C_{j,k} \|\hat{\phi}_j\|_{L^p} \|f_k\|_{L^p} \leq C_{j,k} \|\hat{\phi}_j\|_{L^p} \|f_k\|_{L^p}$$

(see Triebel [13], 1.5.33). Hence the lemma follows by summation. 

**Note.** There is an analogous statement for weighted and unweighted Triebel spaces (see Marshall [8], Lemma 1.4 and 4.2).

We will need the following general Sobolev embedding theorem.

**Lemma 3.** (a) If $0 < p < q < \infty$ and $0 < r \leq \infty$ then

$$F^{s+1/p-1/\theta} \subset L^p$$

(b) If $0 < p < q \leq \infty$ then

$$F^{s+1/p-1/\theta} \subset B^r_{p,q}$$

(c) If $0 < p < q < \infty$ and $0 < r \leq \infty$ then

$$B^r_{p,q} \subset F^{s+1/p-1/\theta}.$$
Proof. For (a) and (b) see Jawerth [7]. The proof of (c) is similar to that of (b). If $p < q_0 < q < q_1 < \infty$ we get from (a)
\[ F^{i,p}_{\mathcal{L}}(f_{p-1}^{i-1}) \subseteq F^{i,q_1}_{\mathcal{L}} . \]
Using real interpolation (see Bergh and Löfström [2]) it follows that
\[ F^{k,q_1}_\mathcal{L} \times F^{j,q_0}_\mathcal{L} \subseteq (F^{i,q_1}_\mathcal{L})^{k,q_1}_\mathcal{L} \]
where $1/q = (1 - \theta)/q_0 + \theta/q_1$. Now $f \rightarrow \|f^*\|_q$ is a quasilinear operator. Hence by the Marcinkiewicz interpolation theorem (see Bergh and Löfström [2])
\[ (F^{i,q_1}_\mathcal{L})^{k,q_1}_\mathcal{L} \subseteq F^{j,q_0}_\mathcal{L} \]
which yields part (c). \[ \qed \]

Below we shall extend part (c) of the lemma to the case $q = \infty$ (see Corollary 4).

2. Multiplication properties of Triebel spaces. For abbreviation set $f_j := \hat{q}_j \ast f$ if $f \in \mathcal{S}^r(\mathbb{R}^n)$. If $f$ and $g$ belong to an appropriate Besov or Triebel space we make the following decomposition:
\[ h = \sum_{j=0}^{\infty} \sum_{k=0}^{j-1} g_j f_{j-k} + \sum_{j=0}^{\infty} \sum_{k=0}^{j-2} g_k f_j + \sum_{j=0}^{\infty} \sum_{k=0}^{j-2} f_j g_k \]
\[ = h_1 + h_2 + h_3. \]

If each of these sums converges in $\mathcal{S}^r(\mathbb{R}^n)$, we call $h$ the product of $f$ and $g$. The convergence is usually shown by estimating $h$ in a suitable $F^{i,q_1}_\mathcal{L}$ quasinorm. However, we shall not stress this point here, we only have to consider the following estimates.

Let $g \in L^p$. Since the spectrum of $\sum_{j=0}^{\infty} g_j f_j$ is contained in the annulus $|z| \sim 2^j$, Lemma 1 yields
\[ \|h_1\|_{F^{i,q_1}_\mathcal{L}} \leq C \|\sum_{k=0}^{\infty} g_k f_j\|_{L^p(\mathbb{R}^n)} \leq C \|g\|_{L^p} \|f\|_{F^{i,q_1}_\mathcal{L}} \]
for $0 < p < \infty$, $0 < q_1 < \infty$ and $s \in \mathbb{R}$.

Hence $h_1$ is well defined for $g \in L^\infty$ and arbitrary $f \in F^{i,q_1}_\mathcal{L}$. Since we will assume that $g \in L^\infty$, we only have to estimate $h_2$ and $h_3$.

Theorem 1. Let $g \in B^{s,n}$, $r > 0$. Then for $0 < p < \infty$, $0 < q < \infty$ and
\[ n(\max \{1,1/p\} - 1) - r < s < r \]
the following estimate holds:
\[ \|g \ast f\|_{F^{i,q_1}_\mathcal{L}} \leq C \|g\|_{B^{s,n}} \|f\|_{F^{i,q_1}_\mathcal{L}}^r. \]

Proof. The estimate for $h_1$ follows from (10).

Now observe that the spectrum of $\sum_{j=0}^{\infty} g_j f_j$ is contained in the ball $|z| \leq 2^{j+1}$. Hence Lemma 2 yields
\[ \|h_2\|_{F^{i,q_1}_\mathcal{L}} \leq C \|g\|_{B^{s,n}} \|f\|_{F^{i,q_1}_\mathcal{L}} \leq C \sup_{j \leq n} \|\sum_{k=0}^{j} g_k f_j\|_{L^p(\mathbb{R}^n)} \]
\[ \leq C \|g\|_{B^{s,n}} \|f\|_{F^{i,q_1}_\mathcal{L}} \]
provided $s > n(\max \{1,1/p\} - 1) - r$.

Finally, the spectrum of $\sum_{j=0}^{\infty} f_j g_j$ is contained in the annulus $|z| \sim 2^j$.

Therefore by Lemma 1 if $s < r$
\[ \|h_3\|_{F^{i,q_1}_\mathcal{L}} \leq C \|\sum_{j=0}^{\infty} f_j g_j\|_{L^p(\mathbb{R}^n)} \leq C \|g\|_{B^{s,n}} \|f\|_{F^{i,q_1}_\mathcal{L}} \]
\[ \leq C \|g\|_{B^{s,n}} \|f\|_{F^{i,q_1}_\mathcal{L}}^r. \]

Now the theorem follows. \[ \qed \]

This theorem improves Corollary 2.8.2 in Triebel [13]. For its generalization to weighted Triebel spaces see Marshall [8]. Chapter 4. There pseudodifferential estimates can also be found. Another improvement concerns the case $s = r$ and $q = \infty$; see Marshall [8], Chapter 11. There it is shown that for these values of the parameters the theorem remains true provided that $g \in F^{i,q_1}_\mathcal{L}$ (for the definition of $F^{i,q_1}_\mathcal{L}$ see Chapter 3 below).

Moreover, one has the following

Theorem 2. Let $0 < p < q < \infty$, $0 < q < \infty$ and
\[ s > \frac{n(1,1/p - 1/r)}{2}. \]

Then for $s > n/q$ or $s = n/q$ and $0 < q < 1$ the following estimate holds:
\[ \|g \ast f\|_{F^{i,q_1}_\mathcal{L}} \leq C \|g\|_{B^{s,n}} \|f\|_{F^{i,q_1}_\mathcal{L}}. \]

Before we prove the theorem, let us state its immediate consequence.

Corollary 1. If $s > n/p$ or $s = n/p$ and $0 < p < 1$, then for $0 < r < \infty$, $F^{i,q_1}_\mathcal{L}$ is a multiplication algebra. \[ \qed \]

Proof of the theorem. Note that by Lemma 3(b) we have $g \in L^\infty$. Hence in view of (10) it remains to provide the necessary estimates for $h_2$ and $h_3$.

Estimate for $h_2$. If $p = q$ then $g \in L^\infty$ and hence by (10)
\[ \|h_2\|_{F^{i,q_1}_\mathcal{L}} \leq C \|g\|_{L^\infty} \|f\|_{F^{i,q_1}_\mathcal{L}}. \]

If $p < q$ let $1/p = 1/p_1 + 1/q$. Then by Hölder's inequality and Lemma 3(a) we obtain
\[ \|h_2\|_{F^{i,q_1}_\mathcal{L}} \leq C \|g\|_{L^\infty} \|f\|_{F^{i,q_1}_\mathcal{L}}. \]
Estimate for $h_2$. Let $q = s - n/q$. Note that $s > \frac{1}{2} n(1/p + 1/q - 1)$ and
$s \geq n/q$ imply
\begin{equation}
(11)
0 > n(\max \{1, 1/p\} - 1).
\end{equation}

If $s > n/q$ then $g \in B_{r,p}^{\infty}$ and Lemma 2 yields
\[\|h_2\|_{r,p} \leq C \|h_2\|_{p^{\infty}, r} \leq C \|g\|_{r,q} \|f\|_{s,p}.\]

If $s = n/q$ choose $p_1 < p < p_2$ such that $1/p_1 = 1/p_2 + 1/q$ and $n/q > n(1/p_1 - 1)$. Because of (11) this is possible. Then by Lemmas 3(c) and 2
\[\|h_2\|_{s,p} \leq C \|h_2\|_{p^{\infty}, r_0} \leq C \sum_{|\alpha| = 3} \|g_{|\alpha|}^{(1+1/p_1-1)}(x_1)\|_{r_0, L^p}\]
\[\leq C \|g\|_{r,q} \|f\|_{s,p} \|f\|_{r_0, L^p}.\]

Hence Lemma 3(b) yields
\[\|h_2\|_{s,p} \leq C \|g\|_{r,q} \|f\|_{s,p} \|f\|_{r_0, L^p}.
\]

This completes the proof of the theorem. \hfill \blacksquare

Let us also mention the following result.

**Theorem 3.** Let $q < p$ if $0 < p \leq 2$ and $q < p/(p-1)$ if $2 \leq p < \infty$. Then for $0 < r \leq \infty$ and
\[n(1/p - 1) < s < n/p\]
one has the following estimate:
\[\|f \|_{r,p} \leq C \|g\|_{r,q} \|f\|_{s,p}.\]

**Proof.** The estimate for $h_1$ follows from (10). For the estimation of $h_2$, choose $p_1 < p < p_2$ such that $1/p_1 = 1/p_2 + 1/q$ and $s < n(1/p_1 - 1/p_2)$. Then by Lemma 3 and Hölder's inequality
\[\|h_2\|_{r,p} \leq C \|h_2\|_{p^{\infty}, r_0} \leq C \sum_{|\alpha| = 3} \|g_{|\alpha|}^{(1+1/p_1-1)}(x_1)\|_{r_0, L^p}\]
\[\leq C \|g\|_{r,q} \|f\|_{s,p} \|f\|_{r_0, L^p}.\]

Next choose $p_1 < p < p_2$ such that $1/p_1 = 1/q + 1/p_2$ and
\[s+n\left(\frac{1}{p_1} - \frac{1}{p}\right) > n(\max \{1, 1/p_1\} - 1).\]

For $q = p$ if $0 < p \leq 2$, resp. $q = p/(p-1)$ if $2 \leq p < \infty$, this choice is possible since in either case $s+n(1/p_1 - 1/p) > n(1/p_1 - 1)$. Then by Lemmas 2 and 3 we get
\[\|h_2\|_{r,p} \leq C \|h_2\|_{p^{\infty}, r_0} \leq C \|g\|_{r,q} \|f\|_{s,p}.\]

This proves the theorem. \hfill \blacksquare

This theorem is the counterpart for Triebel spaces of Remark 2.6.4.5 in Triebel [12]. Let us give two applications.

**First, Proposition 3.4.12 in Triebel [13] can be improved as follows.**

**Corollary 2.** Let $\alpha \in C^\infty$ be a function such that $|\partial^\alpha a(x)| \leq C_\alpha$ for all multi-indices $\alpha$. Further, let $\varphi \in C^\infty_0(R^n)$ be supported in the unit ball and set
$\varphi_n(x) := \varphi((x-x_0)/r)$. Then for $0 < p < \infty$, $0 < q < \infty$ and
\[n(1/p - 1) < s < n/p\]
there exists a constant $C > 0$ such that for $0 < t \leq 1$
\[\|a(x) - a(x_0)\|_{L^p_0} \leq Ct \|f\|_{s,p}.\]

**Proof.** It is shown in the aforementioned proposition of Triebel that for $0 < r \leq \infty$
\[\|a(x) - a(x_0)\|_{L^p_0} \leq C.\]

Since obviously
\[\|a(x) - a(x_0)\|_{\infty} \leq C_\varphi,
\]
the corollary follows from Theorem 3. \hfill \blacksquare

Denote by $\kappa_{x_0,h}$ the characteristic function of a bounded or unbounded interval $[a, b] \subset R^n$.

**Corollary 3.** For $0 < p < \infty$, $0 < q \leq \infty$ and $1/p - 1 < s < 1/p$
\[\|\kappa_{x_0,h}\|_{L^p_0} \leq C \|f\|_{s,p}.\]

**Proof.** It is shown in Proposition 2.8.4, Triebel [13], that $\kappa_{x_0,h} = \rho_1 + \rho_2$ with $\rho_1 \in B_{r_0}^{\infty}$, $0 < r < \infty$, and $|\rho_2(x)| \leq C_\varphi$ for $k = 0, 1, 2, \ldots$. Again the corollary follows from the theorem.

Note that this corollary solves the extension problem for $F_0^\infty(R^n)$. For details see Triebel [13], Chapter 2.9.

**3. Dual spaces.** Let $1 \leq p, q < \infty$ and $\{\varphi_j\} \in \Phi(R^n)$ be a partition. Denote by $L_{p, q}^\infty$ the space of all $f \in \Phi(R^n)$ represented by
\[f = \sum_{j=0}^{\infty} \varphi_j \ast f_j.\]
such that

\[ \|f\|_{M_0^*} := \inf \{ \|2^j f_j\|_{L^q(\mathbb{R}^n)} : j \in \mathbb{N} \} < \infty \]

where the infimum is taken over all representations (12). These spaces have been introduced in Triebel [12] for the study of the duals of Triebel spaces.

For \( 1 \leq p, q < \infty \) the Schwartz space \( S' (\mathbb{R}^n) \) is dense in \( L_{p_0}^q \), and if \( 1 < p, q < \infty \) then \( L_{p_0}^q \approx F_{p_0}^q \) (see Triebel [12], Proposition 2.5.1 and Triebel [13], Proposition 2.3.4.1).

Actually the last statement is true for \( 1 \leq q \leq \infty \).

**Proposition 1.** If \( 1 < p < \infty, 1 \leq q \leq \infty \) and \( s \in \mathbb{R} \) then \( L_{p_0}^q \approx F_{p_0}^q \).

**Proof.** The inclusion \( F_{p_0}^q \subset L_{p_0}^q \) is obvious. For the other direction we use the Hardy–Littlewood maximal function

\[ Mf(x) = \sup_{r > 0} \frac{1}{mB(x, r)} \int |f(y)| dy. \]

One has (see Stein [10], Theorem 3.2.2)

\[ \|\varphi_j * f_j(x)\| \leq CM_j f(x). \]

Then the boundedness of the maximal function on \( L^p(M) \) for \( 1 < p < \infty \) and \( 1 < q < \infty \) (see Fefferman and Stein [5]) implies the assertion for \( 1 < q < \infty \). If \( q = 1 \) we use

**Lemma 4.** For \( 1 \leq p < \infty \) and \( 1/p + 1/\gamma' = 1 \) we have

\[ L^p(\mathbb{R}^n) \subset L^{p'}(\mathbb{R}^n). \]

Here \( c_0 \) is the Banach space of all sequences converging to zero. For a proof see Edwards [5], Theorems 8.18.2 and 8.20.3.

**Proof of the proposition, the case \( q = 1 \)**. Let \( f = \sum a_j \varphi_j * f_j \) be such that \( \{2^j \varphi_j \} \in L^q(\mathbb{R}^n) \). We show that

\[ \{2^j \varphi_j * \sum_j \varphi_j * f_j\} \in L^q(\mathbb{R}^n). \]

By the lemma this implies \( f \in F_{p_0}^q \).

Since for \( \{a_k\} \in L^q(\mathbb{R}^n) \)

\[ \langle \varphi_j * \sum_k \varphi_k * f_k, a_k \rangle = \langle f_j, \sum_{k \in \mathbb{N}} \varphi_j * \varphi_k * a_k \rangle, \]

\[ \| \sum_{k \in \mathbb{N}} \varphi_j * \varphi_k * a_k(x) \| \leq C \sum_{k \in \mathbb{N}} M a_k(x), \]

we get

\[ \| \{2^j \varphi_j * \sum_j \varphi_j * f_j\}, \{a_k\} \| \leq C \|\{2^j \varphi_j\}\|_{L^q(\mathbb{R}^n)} \|\{M a_k\}\|_{L^q(\mathbb{R}^n)} \]

\[ \leq C \|f\|_{2^j M_0^*} \|\{a_k\}\|_{L^q(\mathbb{R}^n)}. \]

This yields the conclusion. ■

It turns out that the right choice for \( F_{p_0}^q, 1 \leq q \leq \infty \), is

\[ F_{p_0}^q = L_{p_0}^q. \]

In particular, \( F_{p_0}^\infty = B_{p_0}^\infty \). Denote by \( \hat{F}_{p_0}^q \) the closure of the Schwartz space \( S' (\mathbb{R}^n) \) in \( F_{p_0}^q \). Now the main result in this chapter is

**Theorem 4.** If \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \) then

\[ \hat{F}_{p_0}^q \approx F_{p_0}^q \]

where \( 1/p + 1/q = 1/\gamma + 1/q' = 1 \).

**Proof.** (a) First take \( 1 \leq p < \infty \).

Then for \( 1 < q < \infty \) one has \( F_{p_0}^q \approx L_{p_0}^q \) (see Triebel [12], Theorem 2.5.1). Now for \( q = \infty \) the mapping

\[ f \mapsto \{2^j \varphi_j * f_j\}: \hat{F}_{p_0}^q \to L^p(\mathbb{R}^n) \]

is an isometric embedding. Then by Lemma 4, Triebel’s proof yields \( \hat{F}_{p_0}^q \approx L_{p_0}^q \). Hence Proposition 1 and the definition of \( F_{p_0}^q \) imply the assertion for \( 1 \leq p < \infty \).

(b) Let \( p = \infty \). Here we show that the norm topology on \( \hat{F}_{p_0}^q \) is compatible with the duality \((\hat{F}_{p_0}^q, F_{p_0}^q)^*\), i.e. we show that the norm topology is finer than the weak topology \( \sigma(\hat{F}_{p_0}^q, F_{p_0}^q)^* \) and weaker than the Mackey topology \( \tau(\hat{F}_{p_0}^q, F_{p_0}^q)^* \). Then the Mackey topology (see Edwards [5], 8.3.3) yields \( \hat{F}_{p_0}^q \approx F_{p_0}^q \).

(c) The norm topology on \( \hat{F}_{p_0}^q \) is the topology of uniform convergence on the closed unit ball \( B \) of \( F_{p_0}^q \) (observe that \( F_{p_0}^q \approx \hat{F}_{p_0}^q \)). Consequently, the norm topology is finer than \( \sigma(\hat{F}_{p_0}^q, F_{p_0}^q)^* \). That it is weaker than \( \tau(\hat{F}_{p_0}^q, F_{p_0}^q)^* \) is a consequence of the following assertion:

\[ \hat{B} = \sigma(\hat{F}_{p_0}^q, F_{p_0}^q)^* \text{compact}. \]

**Proof.** (i) Let us first show that \( \hat{B} \) is \( \sigma(\hat{F}_{p_0}^q, \hat{F}_{p_0}^q)^* \)-sequentially compact. In fact, let \( f \in B \) be a sequence. Then \( f \to f \) in \( \mathcal{S}' (\mathbb{R}^n) \) for some subsequence. Now if \( \{\phi_k\} \in \mathcal{S}(\mathbb{R}^n) \) then for fixed \( i \)

\[ \varphi_i * f_k \to \varphi_i * f \text{ pointwise.} \]

Hence Fatou’s lemma yields

\[ \|f\|_{L^p_{1/\gamma}} \leq \liminf \|f_k\|_{L^p_{1/\gamma}} \leq 1 \]

and therefore \( f \in B \). Obviously we have \( f_k \to f \) in \( \sigma(\hat{F}_{p_0}^q, \hat{F}_{p_0}^q) \).

(ii) \( \hat{F}_{p_0}^q \) is separable because \( \mathcal{S}' (\mathbb{R}^n) \) is dense in it. Let \( (\varphi_i) \in \mathcal{S}' (\mathbb{R}^n) \) be dense in \( \hat{F}_{p_0}^q \). Then

\[ d(f, h) = \sum 2^{-j} \min \{1, |\langle f - h, \varphi_i \rangle|\} \]

defines a metric on \( B \), which induces a weaker topology than \( \sigma(\hat{F}_{p_0}^q, \hat{F}_{p_0}^q) \).
Now recall that any sequentially compact space with a weaker metrizable topology is actually compact, because both topologies are identical. Consequently, $B$ is $A_p$-compact.

This completes the proof of the theorem. ■

Remark. If $0 < p < 1$ then

$$F^p_{<\infty} \approx B^p_{\infty} = B^{p+1}_1,$$

This is a consequence of

$$B^p_{1,q} \subset F^p_{<\infty} \subset B^p_{1,q},$$

and the known duality theory for $B^p_{1,q}$; see Triebel [13], Chapter 2.11. The determination of the dual of $F^p_{<\infty}$ for $1 < p < \infty$ and $0 < q < 1$ remains open. It is known that

$$F_{1,q'} \approx F^p_{<\infty} \approx F^p_{1,q}.$$

The conjecture is of course $F^p_{1,q} \approx F^p_{1,q'}$. ■

Let us state some consequences of Theorem 4.

Corollary 4. For $0 < p < \infty$

$$B^p_{1,q} \approx F^p_{1,q} \approx F^p_{1,q'}.$$

Proof. We may suppose $1 < p < \infty$. But then the statement follows by duality from $F^p_{1,q} \approx B^p_{1,q}$, ■

We can also extend the Fourier multiplier result, Theorem 2.4.8 in Triebel [13], to the case $q = \infty$. Let $\varphi$, $\psi \in \mathcal{S}(R^n)$ be such that

$$\text{supp } \psi \subset \{x : |x| \leq 4\}, \quad \psi (x) = 1 \text{ if } |x| \leq 2,$$

$$\text{supp } \varphi \subset \{x : \frac{1}{2} \leq |x| \leq 2\}, \quad \varphi (x) = 1 \text{ if } \frac{1}{2} \leq |x| \leq 2.$$

For abbreviation set

$$m = \sup_{x \in R^n} |\psi(2^j x)|, \quad j = 1, 2, \ldots ,$$

$$\mathcal{C} = n(\text{max } \{1, 1/p\} - 1).$$

Then

$$\|\mathcal{C} \|_{L^p} \leq C \|m\|_{L^p} \|f\|_{L^p}.$$

Proof. For $1 < p < \infty$ the result follows by duality from the case $q = 1$. For $0 < p \leq 1$ one uses complex interpolation. For details see the aforementioned reference. ■

The preceding results can be extended to weighted Triebel spaces. Let

$$1 < p < \infty \text{ and } w \geq 0 \text{ be such that}$$

$$\sup_{Q} \frac{1}{w} \int_Q w dx \left( \int_Q w^{-1/(p-1)} dx \right)^{1/p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset R^n$. Then $w$ is said to satisfy Muckenhoupt's $A_p$-condition.

Let $w \in A_p := \bigcup_{p < \infty} A_p$. Define $F_{<\infty}^p (w)$ by

$$\|f\|_{L_p^p (w)} := \| (2^p \varphi) \ast f \|_{L_p^p (w)},$$

(modification for $q = \infty$).

Similarly define $L_{<\infty}^p (w)$ as the space of all $f \in \mathcal{D}' (R^n)$ which can be represented by $f = \sum \phi_i \ast f_i$ in such a way that

$$\|f\|_{L_{<\infty}^p (w)} := \inf \| (2^p \varphi) \ast f \|_{L_p^p (w)} < \infty.$$

Analogously to Proposition 1 one may prove

Proposition 2. If $1 < p < \infty$, $1 \leq q < \infty$ and $w \in A_p$, then

$$L_{<\infty}^p (w) \approx F_{<\infty}^p (w).$$

Here one has to use the weighted Hardy-Littlewood maximal theorem; see Andersen and John [1]. Now using the results of Bui Huei Quôc [4], we can show similarly to Theorem 4

Theorem 5. Let $1 < p < \infty$, $1 \leq q < \infty$, $w, w^{-1/(p-1)} \in A_p$, and $1/p + 1/p' = 1/q + 1/q' = 1$. Then

$$F_{<\infty}^p (w) \approx L_{<\infty}^p (w^{-1/(p-1)}) \approx L_{<\infty}^p (w^{-1/(p-1)}) \approx F_{<\infty}^p (w^{-1/(p-1)}).$$

Obviously, $F_{<\infty}^p (w)$ resp. $L_{<\infty}^p (w)$ denotes the closure of $\mathcal{S}'(R^n)$ in $F_{<\infty}^p (w)$ resp. $L_{<\infty}^p (w)$. Since $\mathcal{S}'(R^n)$ is dense in $F_{<\infty}^p (w)$ and $L_{<\infty}^p (w)$ for $p, q < \infty$, we obtain

Corollary 6. For $1 < p, q < \infty$ and $w, w^{-1/(p-1)} \in A_p$, the spaces $F_{<\infty}^p (w)$ and $L_{<\infty}^p (w)$ are reflexive. ■

Now let $w \in A_1$, i.e. let $Mw (x) \leq Cw (x)$ a.e. Define $b_{w_0} (w)$ by

$$\|f\|_{b_{w_0} (w)} := \sup \frac{1}{w_0} \int_{w_0} J (x) dx + \sup \frac{1}{w_0} \int_{w_0} j (y) dy < \infty.$$

Here we have set $w_0 = [w]_{A_1}$. Let $\psi \in C_0^\infty$ be a cut-off function such that $\psi (x) = 0$ for $|x| \leq 1/2$ and $\psi (x) = 1$ for $|x| > 1$. Let for $j = 1, \ldots , n$

$$r_j (D) f (x) := (2\pi)^{-n} \int e^{-ix \cdot \xi} \psi (\xi) \frac{1}{|\xi|^2} (\xi) d\xi$$

be the inhomogeneous Riesz transform.
The Hardy space $h^1(w)$ is defined by

\begin{equation}
\|f\|_{h^1(w)} := \|f\|_{L^1(w)} + \sum_{j=1}^{n} \|\tau_r(D_jf)\|_{L^1(w)} < \infty.
\end{equation}

Let $\text{cmo}(w)$ be the closure of $\mathcal{S}(\mathbb{R}^n)$ in $\text{bmo}(w)$. Since $h^1(w) \approx \text{bmo}(w)$ (Bui Huy Que [3]) and $h^1(w) \approx F^1_2(w)$ (Bui Huy Que [4]), the method described above yields

**Corollary 7.** For $w \in A_1$, the dual of $\text{cmo}(w)$ is isomorphic to $h^1(w)$. ■

The case $w \equiv 1$ of this corollary has been shown by Neri [9].

**4. Remarks on $F^p_{aq}$.** By Theorem 4, many results for $F^p_{aq}$ extend to the case $p = \infty$. For example, one can make the following statement about Fourier multipliers. If $\alpha > n/2$ then

\begin{equation}
\|\hat{m}\|_{F^\infty_{aq}} \leq C\|m\|_{\mathcal{M}_2} \|\hat{f}\|_{F^\infty_{aq}}.
\end{equation}

This follows by duality from Theorem 2.4.8, Triebel [3], if $1 < q < \infty$ and from Corollary 5 if $q = 1$. More generally, let us consider pseudodifferential operators. Define $S^m(r, N)$ to consist of symbols $a(\cdot, \cdot)$ such that for all $|\xi| \leq N$

\begin{equation}
|\hat{a}(\alpha, \xi)| \leq C_1(1 + |\xi|)^{m-|\alpha|}, \quad |\hat{a}(\cdot, \xi)|_{\mathcal{S}_{aq}} \leq C_2(1 + |\xi|)^{m-|\alpha|}.
\end{equation}

We associate pseudodifferential operators to these symbols by

\begin{equation}
\text{Op}(a) f(x) := (2\pi)^{-n} \int e^{ix\xi} \hat{a}(\alpha, \xi) \hat{f} (\xi) \, d\xi
\end{equation}

for $f \in \mathcal{S}(\mathbb{R}^n)$.

Let us take $N > \frac{5n}{2}$ and $0 < \delta < 1$. Then for $1 \leq q < \infty$ we have

\begin{equation}
\text{Op}(a) : F^\infty_{aq} \rightarrow F^\infty_{aq}
\end{equation}

provided that $- (1 - \delta) r < s < r$ and

\begin{equation}
\text{Op}(a) : F^\infty_{aq} \rightarrow F^\infty_{aq}
\end{equation}

provided that $- r < s < (1 - \delta) r$. Here $\text{Op}(a)$ is the transpose of $\text{Op}(a)$ defined by $\langle \text{Op}(a)f, g \rangle = \langle f, \text{Op}(a^\top g) \rangle$ for $f, g \in \mathcal{S}(\mathbb{R}^n)$, (26) and (27) follow by duality from the results in Chapter 3 of Marschall [3]. Then (26) can be used to show that for $s > 0$ $F^s_{aq}$ is a multiplication algebra. More precisely,

\begin{equation}
\|g f\|_{F^s_{aq}} \leq C \|g\|_{L^\infty} \|f\|_{F^s_{aq}} + \|\tau_r(D_jf)\|_{L^1(w)}.
\end{equation}

The proof is identical to the one given for $q = 2$ in Marschall [3], Chapter 11.

One topic which cannot be treated by duality is the trace operator. Decompose $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ and $x = (x', x_n)$. On continuous functions the trace operator is defined by $\text{Tr}_v f(x') := f(x', 0)$.
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Drop property equals reflexivity

by

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Abstract. We prove that in a reflexive Banach space \((X, \| \cdot \|)\) property (H) of Radoz-Riesz (if \(\{x_n\}_{n=1}^{\infty}\) is a sequence of elements in \(X\) converging weakly to an element \(x\) in \(X\) such that \(\|x_n\| \to \|x\|\), then \(\{x_n\}_{n=1}^{\infty}\) is norm-convergent to \(x\) is equivalent to a geometric condition (the "drop property") introduced by Rolewicz: \(\| \cdot \|\) has the drop property if for every closed set \(S\) disjoint with \(B_X\) (the closed unit ball of \(X\)) there exists an element \(x \in S\) such that the "drop" defined by \(x\) (the convex hull of \(x\) and \(B_X\)) intersects \(S\) only at \(x\). We also prove that a Banach space is reflexive if and only if it has an equivalent norm with drop property.

§ 1. Introduction. Let \((X, \| \cdot \|)\) be a Banach space and \(B_X\) its closed unit ball. By the drop \(D(x, B_X)\) defined by an element \(x \in X\), \(x \notin B_X\), we shall mean the convex hull of the set \(\langle x \rangle \cup B_X, \text{conv}(\{x\} \cup B_X)\). In [4], Danell proved ("Drop Theorem") that, for any Banach space \((X, \| \cdot \|)\) and every closed set \(S \subseteq X\) at positive distance from \(B_X\), there exists a point \(x \in S\) such that \(D(x, B_X) \cap S = \{x\}\).

This result, as its author points out, allows to prove in a simple way certain theorems of Browder [2] and Zabreiko-Krasnosel'skii [17] which are very important in the theory of nonlinear operator equations. In [14], Rolewicz mentions a number of papers where the Danell result is used. Recently, Danell has discussed the relationship between his Drop Theorem and several other results [5].

Motivated by Danell's theorem, Rolewicz introduced in the aforesaid paper the notion of drop property for the norm in a Banach space: \(\| \cdot \|\) in \(X\) has the drop property if for every closed set \(S\) disjoint with \(B_X\) there exists an element \(x \in S\) such that \(D(x, B_X) \cap S = \{x\}\). He proved that if \(X\) is a uniformly convex Banach space then its norm has the drop property, and also

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