On regular generators of $Z^2$-actions in exhaustive partitions

by

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Abstract. It is shown that for every totally ergodic $Z^2$-action with finite entropy there exists a regular generator in a given exhaustive partition and the set of regular generators is dense in the set of all generators.

1. Introduction. Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space, $\mathcal{M}$ the set of all measurable partitions of $X$ and $\mathcal{E}$ the subset of $\mathcal{M}$ consisting of partitions with finite entropy.

All relations between measurable partitions are to be taken mod 0.

Let $g$ be the metric on $\mathcal{E}$ defined by the formula

$$d(P, Q) = H(P) + H(Q) - P \cdot Q, \quad P, Q \in \mathcal{E}. $$

We denote by $e$ the measurable partition of $X$ into single points and by $\nu$ the measurable trivial partition whose only element is $X$.

Let $T$ be an automorphism of $(X, \mathcal{A}, \mu)$. For $P \in \mathcal{M}$ we define

$$P_T = \bigvee_{n=1}^{\infty} T^{-n} P, \quad P_T = \bigvee_{n=-\infty}^{\infty} T^n P. $$

If $P_T = e$ we say that $P$ is a generator of $(X, T)$.

A partition $\xi \in \mathcal{M}$ is said to be $T$-perfect if

$$T^{-1} \xi \leq \xi, \quad \xi_T = e, \quad \bigvee_{n=0}^{\infty} T^{-n} \xi = \pi(T) \quad \text{and} \quad h(\xi, T) = h(T)$$

where $\pi(T)$ and $h(T)$ denote the Pinski partition and the entropy of $T$ respectively.

Rokhlin and Sinai showed in [9] that for every automorphism $T$ there exists a $T$-perfect partition. If $T$ is aperiodic with $h(T) < \infty$ then for every generator $P$ of $(X, T)$ the partition $\xi = P \vee P^*_T$ is $T$-Perfect. Rokhlin [7] proved that if $h(T) < \infty$ and $\xi$ is $T$-Perfect then there exists a generator $P$ such that $\xi = P \vee P^*_T$, i.e. $\xi$ is a part of the process $(P, T)$.

Now, let $G$ be an abelian free group of rank 2 of automorphisms of $(X, \mathcal{A}, \mu)$. We denote by $h(G)$ the set of $\bigvee_{\Delta}^{\infty}$ all ordered pairs of independent generators of $G$. 


The quadruple \((X, \mathcal{A}, \mu, G)\) is said to be a two-dimensional dynamical system \((Z^2\text{-action})\) and is shortly denoted by \((X, G)\).

The entropy theory for such systems has been developed by Conze \([1]\), Katznelson and Weiss \([5]\), and the theory of invariant partitions by the author \([3]\).

Let \(Z^2\) denote the two-dimensional integers and \(\prec\) the lexicographical order in \(Z^2\).

We put \(\Pi = \{(i,j) \in Z^2; (i,j) < (0,0)\}\). Let \((T, S) \in h(G)\). For \(P \in \mathcal{P}\) we define

\[ P^G = \bigvee_{a \in \Pi} T^a P^S P, \quad P^G = \bigvee_{a \in \Pi} T^a S^a P. \]

A partition \(P \in \mathcal{P}\) is said to be a generator for \((X, G)\) if \(P^g = \emptyset\).

Now, let \(G\) be aperiodic and \(h(G) < \infty\). Following Conze \([1]\) we denote by \(P_G\) the set of all \(P \in \mathcal{P}\) with \(h(P, G) = h(G)\) and by \(P_G\) the set of all generators of \((X, G)\) with finite entropy. It is proved in \([1]\) that \(P_G \neq \emptyset\) and \(P_G\) is a dense subset of \(T_G\).

In \([3]\) the following two-dimensional analogue of the notion of perfect partition mentioned above is introduced. A partition \(\xi_g \in \mathcal{P}\) is said to be \((T, S)\text{-exhaustive} if

\begin{itemize}
  \item[(i)] \(T^a S^b \xi \leqslant \xi\) for \((a, b) \in \Pi,\)
  \item[(ii)] \(\xi_0 = x,\)
  \item[(iii)] \(\sum_{n=0}^{\infty} S^{-n} \xi = T^{-1} \xi,\)
\end{itemize}

If \(\xi\) also satisfies

\begin{itemize}
  \item[(iv)] \(T^a S^b \xi = \pi(G),\)
  \item[(v)] \(h(G) = h(\xi_0) = h(\xi S^{-1} \xi),\)
\end{itemize}

where \(\pi(G)\) and \(h(G)\) mean the Pinsker partition and the entropy of \(G\) respectively, then it is called \((T, S)\text{-perfect.}\)

It is clear that conditions (i) and (iv) are equivalent to the following:

\begin{itemize}
  \item[(i)] \(S^{-1} \xi \leqslant \xi,\)
  \item[(ii)] \(T^{-1} \xi \leqslant \xi,\)
  \item[(iv)] \(\sum_{n=0}^{\infty} T^{-n} \xi = \pi(G).\)
\end{itemize}

It is shown in \([3]\) that for every \((T, S) \in h(G)\) there exists a \((T, S)\text{-perfect partition.}\)

For \(P \in P_G\) we define \(P^G = P \vee P^G\). In \([4]\) we investigated the following question: is the partition \(P^G\) \((T, S)\text{-perfect} for any \(P \in B_G\)? As we have seen above, the analogue of this question for single automorphisms has the positive answer. Our question is equivalent to the following: is the equality

\[\sum_{n=0}^{\infty} (S^{-n} P^G \vee (P^G)P) = (P^G)P\]
satisfied for any \(P \in B_G\)? It turned out \([4]\) that in general the answer to this question is negative. A generator satisfying the above equality was called in \([4]\) \((T, S)\text{-regular.}\) We denote the set of all \((T, S)\text{-regular generators of } (X, G)\) by \(B_{T,S}\).

It is worth noting that Weiss \([11]\) considered a more general problem in probability theory.

Using the relative version of the Kolmogorov zero-one law one can show that the zero time partition in the two-dimensional Bernoulli dynamical system is regular with respect to the pair of the shifts. A more general example is given in \([4]\).

In this paper, using relative versions of some results of the ergodic theory of single automorphisms, we show that if \(G\) is totally ergodic then for any \((T, S) \in h(G)\) and for any \((T, S)\text{-exhaustive partition } \xi\) there exists \(P \in B_{T,S}\) with \(P \leqslant \xi\). Moreover, the set \(B_{T,S}\) is dense in \(B_G\). It appears that by the use of regular generators it is possible to characterize the groups with zero entropy in a manner similar to that for single automorphisms.

I am grateful to J. P. Thouvenot for suggesting the possibility of using relative generator theorems for a solution of the question stated above.

2. Some results of the relative ergodic theory. In the sequel we denote by \(Z\) the set of integers and by \(N\) the set of positive integers.

Let \(T\) be an automorphism of \((X, \mathcal{A}, \mu)\) and let \(\sigma \in \mathcal{P}\) be such that \(T \sigma = \sigma\). For \(P \in \mathcal{P}\) we put

\[h(P, T | \sigma) = H(P | P \sigma) \land \sigma.\]

We define the \(e\text{-relative entropy}\) \(h(T | \sigma)\) of \(T\) by the formula

\[h(T | \sigma) = \sup h(P, T | \sigma)\]

where the supremum is taken over all \(P \in \mathcal{P}\).

It is clear that \(h(T | \sigma) \leqslant h(T)\). There is a simple formula connecting \(h(T | \sigma)\) with \(h(T)\).

Proposition 1. \(h(T) = h(T | \sigma) + h(T | \sigma)\)

where \(T_{\sigma}\) denotes the factor automorphism of \(T\) on \(X/\sigma\).

Proof. Let \(P_x, Q_x \in \mathcal{P}, x \in N\) be such that \(P_x \wedge \sigma\) and \(Q_x \wedge \sigma\). From the Pinsker formula and simple properties of the conditional entropy easily follow the inequalities:

\[h(P_x \vee Q_x, T | \sigma) \geqslant h(P_x, T) + h(Q_x, T | \sigma),\]

\[h(Q_x, T | \sigma) \leqslant h(P_x, T) + h(Q_x, T | \sigma) \vee (P_x | \sigma).\]

Applying to both the inequalities the well-known limit properties of entropy we obtain the desired equality.

We shall use in the sequel the following result given in \([8]\) (lemma 10.2).
LEMMA 1. For all \( P, Q \in \mathcal{M} \) such that \( P \geq Q \) and \( h(P | Q) < \infty \) there exists \( R \in \mathfrak{Z} \) with \( P = Q \lor R \) and \( h(R) < h(P | Q) + \frac{\delta}{2} \).

The main tool to obtain our main result is a relative version of the well-known Rokhlin generator theorem (cf. [6]). Since the proof runs in a similar way to that of Rokhlin we give only a sketch of it below.

For \( n \in \mathbb{N} \), \( B \in \mathcal{A} \) and a partition \( P = (P_i, i \in \mathbb{N}) \) we define the following partitions:

\[
P^\tau = \sum_{k=1}^{n-1} T^k P, \quad P \cap B = (P_i \cap B, i \in \mathbb{N}).
\]

Let \( \sigma, \tau \in \mathcal{A} \) be such that \( T_\sigma = \sigma, T_\tau = \tau, \tau \geq \sigma \).

LEMMA 2. If \( T \) is aperiodic with \( h(T | \sigma) < \infty \) then for all \( P, Q \in \mathfrak{Z} \) and \( \delta > 0 \) there exists a partition \( R \in \mathcal{Z} \) such that \( R \geq P^\tau \) and

\[
H(R | Q T^\delta | \sigma) \leq h(T | \sigma) - h(Q, T | \sigma) + \delta.
\]

Sketch of proof. Let \( \delta > 0 \) be arbitrary and \( n \in \mathbb{N} \) be such that

\[
\frac{\delta}{2n-1} H((P \lor Q)^n | \sigma) - h(P \lor Q, T | \sigma) < \frac{\delta}{3}.
\]

We choose \( \lambda > 0 \) satisfying the condition

\[
H(P \cap B) < \lambda
\]

for \( \mu(B) < \lambda. \)

The Rokhlin tower theorem implies there exists a set \( C \in \mathcal{A} \) such that the sets \( C, TC, \ldots, T^{n-1} C \) are pairwise disjoint and \( \mu(D) < \lambda \) where \( D = X \setminus (C \cup TC \cup \ldots \cup T^{n-1} C). \)

There exists \( 0 \leq k \leq n-1 \) with

\[
H(P^k \cap T^k C | TQ^\delta | \sigma) \leq h(T | \sigma) - h(Q, T | \sigma) + \frac{\delta}{6}.
\]

The partition \( R = P^k \cap T^k C \lor P \lor D \) satisfies the desired properties:

\[
R \geq P^\tau \quad \text{and} \quad H(R | Q T^\delta | \sigma) \leq H(P^k \cap T^k C | TQ^\delta | \sigma) + H(P \lor D)
\]

\[
\leq h(T | \sigma) - h(Q, T | \sigma) + \delta.
\]

RELATIVE GENERATOR THEOREM. If \( \tau \in \mathcal{A} \) is such that the factor automorphism \( T_\tau \) is aperiodic with \( h(T_\tau | \sigma) < \infty \) then there exists \( R \in \mathfrak{Z} \) such that \( P \geq \tau \lor P \lor \sigma = \tau \). Moreover, the set of \( P \in \mathfrak{Z} \) with \( P \leq \tau \lor P \lor \sigma = \tau \) is dense in the set of \( P \in \mathfrak{Z} \), \( P \leq \tau \lor P \).\( h(P, T | \sigma) = h(T_\sigma). \)

Sketch of proof. We may suppose \( \tau = \sigma \). Let \( \delta > 0 \) be arbitrary and \( Q \in \mathfrak{Z} \) be such that

\[
h(T | \sigma) - h(Q, T | \sigma) < \frac{\delta^2}{2}.
\]

We take a sequence \( (Q_n) \) of partitions in \( \mathfrak{Z} \) with \( Q_0 = Q \) and \( Q_n \geq \sigma \) and

\[
h(T | \sigma) - h(Q_n, T | \sigma) < \frac{\delta^2}{2n+1}, \quad n \in \mathbb{N} \cup \{0\}.
\]

Using Lemma 2 we may choose a sequence \( (R_n) \) in \( \mathfrak{Z} \) with \( R_n \geq Q_n \)

\[
H(R_n | (Q_{n-1})_T \lor \sigma) = H((Q_{n-1})_T \lor R_n \lor \sigma | (Q_{n-1})_T \lor \sigma) < \frac{\delta^2}{2n+1}, \quad n \in \mathbb{N}.
\]

Now Lemma 1 implies there exists a sequence \( (P_n) \) in \( \mathfrak{Z} \) such that

\[
(Q_{n-1})_T \lor R_n \lor \sigma = (Q_{n-1})_T \lor P_n \lor \sigma, \quad H(P_n) < \frac{\delta^2}{2n+1}, \quad n \in \mathbb{N}.
\]

This equality gives

\[
(Q \lor \bigvee_{k=1}^{n} P_k)_T \lor \sigma \geq (Q \lor \sigma)_T, \quad n \in \mathbb{N}.
\]

Therefore putting \( P = Q \lor \bigvee_{k=1}^{\infty} P_k \) we have \( P_T \lor \sigma = \sigma \),

\[
H(P) \leq h(Q) + \sum_{k=1}^{\infty} h(P_k) < h(Q) + \delta < \infty
\]

and \( g(P, Q) < \delta \), which completes the proof.

We denote by \( \pi(T, \sigma) \) the join of all \( P \in \mathfrak{Z} \) with \( P \leq \sigma \), \( h(P, T | \sigma) = 0 \) and call it the \( \sigma \)-relative Pinsker partition of \( T \). We shall write \( \pi(T | \sigma) \) instead of \( \pi(T, \sigma) \).

The concept of \( \sigma \)-relative Pinsker partition was introduced in [2] in the case \( \sigma = \varepsilon \) and called there the Pinsker closure of \( \sigma \).

It is clear that \( \pi(T, \varepsilon) \geq \sigma \). If \( \pi(T, \tau | \sigma) = \sigma \) then we say that \( T \) is a K-automorphism relative to \( \sigma \). Let us remark that \( T \) is a K-automorphism relative to \( \sigma \) if and only if \( \tau \subseteq \mathcal{A} \) with \( T_\tau = \zeta, \zeta \geq \sigma, h(T_\tau | \sigma) = 0 \) we have \( \zeta = \sigma \). Thouvenot also defined (cf. [10]) a concept of relative K-automorphism. Using Proposition 1 and the above remark one can easily check that in the case \( h(T) < \infty \) both concepts coincide.

Some properties of relative Pinsker partitions:

(a) \( T_\sigma = T_\tau \in \mathcal{A} \).

(b) \( h(T_\sigma \circ \varepsilon | \sigma) = 0 \).

(c) If \( S \) is an automorphism of \( (X, \mathcal{A}, \mu) \) commuting with \( T \) then

\[
S \pi(T, \sigma | \tau) = \pi(T, S \sigma | S \varepsilon).
\]

(d) If \( \tau_1, \sigma_1 \in \mathcal{A} \), \( T_{\tau_1} = \tau_1, T_{\sigma_1} = \sigma_1 \), \( i = 1, 2, \sigma_1 \leq \tau_1 \leq \tau_2, \sigma_1 \leq \sigma_2 \leq \tau_2 \) then

\[
\pi(T, \tau_1 | \sigma_1) \leq \pi(T, \tau_2 | \sigma_2).
\]
(e) $T$ is a $K$-automorphism relative to $\pi(T, \tau|\sigma)$.

(f) If $\sigma \in \mathcal{M}$, $T = \sigma$ and $\xi \in \mathcal{M}$ is such that $\sigma \leq T^{-1} \xi \leq \xi$ then

$$\bigwedge_{n=0}^{m} T^{-n} \xi \geq \pi(T, \tau|\sigma).$$

(g) If $\sigma \in \mathcal{M}$, $T = \sigma$ and for every $P \in \mathcal{Y}$

$$\bigwedge_{n=0}^{m}(T^{-n} P_{T} \vee \sigma) = \pi(T, P_{T} \vee \sigma|\sigma).$$

(h) If $\tau_{n} \in \mathcal{M}, n \in \mathbb{N}$, are such that $T \tau_{n} = \tau_{n-1}, \sigma \leq \tau_{n} \leq \tau_{n+1}$ and $T_{\tau_{n}}$ is a $K$-automorphism relative to $\sigma$, $n \in \mathbb{N}$, then $T_{\tau_{n}}$ is a $K$-automorphism relative to $\sigma$, where $\tau = \bigwedge_{n=1}^{\infty} \tau_{n}$.

Proof. Properties (a)-(d) are easy consequences of the definition. The proofs of (f) and (h) are similar to the proofs of Theorems 12.1 and 13.4 of [9] respectively and we omit them.

To prove (e) let us suppose $P \in \mathcal{Y}, P \leq \xi$ and

$$h(P, T) \pi(T, \tau|\sigma) = 0.$$

Let $Q_{n} \in \mathcal{Y}, n \geq 1$, and $Q_{n} \geq \pi(T, \tau|\sigma)$.

Using the relative version of the Pinsker formula (cf. [1]) and simple properties of the conditional entropy we have

$$h(P \cup Q_{n}, T|\sigma) = h(P, T|\sigma) + H(Q_{n} \cup P_{T} \vee \sigma)$$

$$= H(P) P_{T} \vee (Q_{n} \cup P_{T} \vee \sigma), \quad n \in \mathbb{N}.$$

Therefore the choice of $P$ and $Q_{n}$ implies

$$h(P, T|\sigma) = h(P) P_{T} \vee (Q_{n} \cup P_{T} \vee \sigma), \quad n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$, we have

$$h(P, T|\sigma) = h(P) P_{T} \vee \pi(T, \tau|\sigma) = 0,$$

i.e. $P \leq \pi(T, \tau|\sigma)$ and (e) is proved.

In order to check (g) let us observe that the inequality

$$\bigwedge_{n=0}^{m}(T^{-n} P_{T} \vee \sigma) \geq \pi(T, P_{T} \vee \sigma|\sigma)$$

is an easy consequence of (f). To prove the converse inequality we take $Q \in \mathcal{Y}$ and $Q \leq \bigwedge_{n=0}^{m}(T^{-n} P_{T} \vee \sigma)$. Hence $Q \leq P_{T} \vee \sigma$ and

$$H(Q) Q_{T} \vee T^{-n} P_{T} \vee \sigma = 0, \quad n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$ we obtain $H(Q) Q_{T} \vee \sigma = 0$. This means that $Q \leq \pi(T, P_{T} \vee \sigma|\sigma)$ which proves (g).

3. Existence of regular generators. Let $(X, G)$ be a two-dimensional dynamical system and let $(T, S) \in b(G)$. In order to prove our result we shall need the following.

Lemma 2. If $\xi \in \mathcal{M}$ is $(T, S)$-exhaustive then $S$ is a $K$-automorphism relative to $T^{-1} \xi$, $k \in \mathbb{Z}$.

Proof. Let $k \in \mathbb{N}$ and $P, Q \in \mathcal{Y}$ satisfy the following conditions:

$$P \leq T^{k} S^{k}, \quad Q \leq \pi(S, T^{k} \xi), \quad k \in \mathbb{N}.$$

Hence

$$\pi(S, T^{k} \xi), \quad k \in \mathbb{N}$$

(1) $Q \leq T^{k} S^{k}, \quad h(Q, S)T^{-1} \xi) = 0$.

Let $m \in \mathbb{N}$ be arbitrary. The relative Pinsker formula implies

$$h(P \cup Q, S^{m} \cup T^{-1} \xi) = h(P, S^{m} \cup T^{-1} \xi) + H(Q) Q_{S} \cup P_{S} \vee T^{-1} \xi)$$

$$= h(Q, S^{m} \cup T^{-1} \xi) + H(P) P_{S} \vee Q_{S} \vee T^{-1} \xi)$$

Hence by (1) we have

$$h(P, S^{m} \cup T^{-1} \xi) = H(P) P_{S} \vee Q_{S} \vee T^{-1} \xi).$$

Therefore

$$h(P, \pi(S, T^{k} \xi), \pi(T, T^{-1} \xi)) \geq H(P) P_{S} \vee Q_{S} \vee T^{-1} \xi)$$

Taking the limit in (2) as $m \to \infty$, using (iii) we obtain

$$\pi(S, T^{k} \xi) \leq \pi(S, T^{k} \xi), \quad k \in \mathbb{N}.$$

Since $P$ runs through a dense subset of the set $\{R \in \mathcal{Y}; R \leq T^{k} \xi\}$, we conclude that (3) is valid for any $P \in \mathcal{Y}$ with $P \leq T^{k} \xi$. Assuming $P = Q$ we get $Q \leq T^{k} \xi$ and so

$$\pi(S, T^{k} \xi) \leq T^{k} \xi, \quad k \in \mathbb{N}.$$

Therefore

$$\pi(S, T^{k} \xi) \leq \pi(S, T^{k} \xi), \quad k \in \mathbb{N}$$

and thus

$$\pi(S, T^{k} \xi) = T^{-1} \xi, \quad k \in \mathbb{N}.$$

Now (ii) and (h) give $\pi(S, T^{-1} \xi) = T^{-1} \xi$, and so by (c) we obtain the result.

Now, let $G$ be aperiodic with $h(G) \leq \infty$.

Corollary 1. A generator $P \in B_{T, \xi}$ iff $S$ is a $K$-automorphism relative to $(P_{\xi})_{T}$. 

Proof. If \( P \in B_{T, S} \), then the partition \( \xi = P \vee P_0^* \) is \((T, S)\)-exhaustive and so, by Lemma 2, \( S \) is a \( K \)-automorphism relative to \( T^{-1} \xi = (P_0 S \vee P_0) \).

Now, let \( S \) be a \( K \)-automorphism relative to \( (P_0 S \vee P_0) \). From this and (g) it follows that

\[
\forall_{S} \exists_{P_0} (S \vee (P_0 S \vee P_0)) \leq \pi(S \vee (P_0 S \vee P_0)) = (P_0 S \vee P_0),
\]

i.e. \( P \in B_{T, S} \).

From Corollary 1 and (e) we obtain at once

**Corollary 2.** If \( P \in B_0 \) and \( Q \in \mathcal{F} \) is such that \( (Q_0 S \vee P_0) = \pi(S \vee (P_0 S \vee P_0)) \) then \( Q \in B_{T, S} \).

In the theorem below we prove that for a wide class of groups \( G \) and for any generator \( P \) of \((X, G)\) such a generator \( Q \) exists.

**Definition.** The group \( G \) is said to be totally ergodic if every automorphism \( \varphi \in G \) different from the identity transformation of \( X \) is ergodic.

**Theorem.** If \( G \) is totally ergodic with \( h(G) < \infty \), \( P \in B_0 \) and \((T, S) \in b(G)\) then for every \( \varepsilon > 0 \) there exists \( Q \in B_{T, S} \) such that \( P \leq Q \) and \( q(P, Q) < \varepsilon \).

**Proof.** Let us suppose \( G \) is totally ergodic, \((T, S) \in b(G)\), \( P \in B_0 \) and \( \varepsilon > 0 \) is arbitrary.

By our assumption, the factor automorphism \( S_{(S_0 \cup P_0^*)^{-1}} \) is ergodic and since \( P \in B_0 \), the factor measure induced by \( \mu \) on \( X/\pi(S \vee (P_0 S \vee P_0)) \) is continuous. Therefore the above factor automorphism is aperiodic. Now property (b) implies

\[
h(T^{-1} P, S \vee (P_0 S \vee P_0)) = h(S_{(S_0 \cup P_0^*)^{-1}} \vee (P_0 S \vee P_0)) = 0.
\]

It follows from the Relative Generator Theorem that there exists \( R \in \mathcal{F} \) such that

\[
R \leq \pi(S \vee (P_0 S \vee P_0)) \quad \text{and} \quad q(T^{-1} P, R) < \varepsilon.
\]

Putting \( Q = P \vee TR \) we have

\[
P \leq Q, \quad (Q S \vee P_0) \leq \pi(S \vee (P_0 S \vee P_0)) \quad \text{and} \quad q(P, Q) \leq q(T^{-1} P, R) < \varepsilon.
\]

By Corollary 2 we see that \( Q \) satisfies all desired properties.

Since \( B_0 \) is a dense subset of \( T_0 \) the theorem above implies at once

**Corollary 1.** If \( G \) is totally ergodic with \( h(G) < \infty \) then for every \((T, S) \in b(G)\) the set \( B_{T, S} \) is dense in \( T_0 \).

Now, let \( \xi \) be \((T, S)\)-exhaustive.

**Corollary 2.** If \( G \) is totally ergodic with \( h(G) < \infty \) then for every \((T, S) \in b(G)\) there exists \( P \in B_{T, S} \) with \( P \leq \xi \).

**Proof.** Let \( Q \in B_0 \) with \( Q \leq \xi \). The existence of such a generator \( Q \) may be proved by the same method as that used by Rohlin in [7]. Let \( Q \in B_0 \) be

such that \( (Q_0 S \vee P_0) = \pi(S \vee (Q_0 S \vee P_0)) \). As we already know, \( Q \in B_{T, S} \). It follows from Lemma 2 that

\[
T^{-1} Q \leq (Q_0 S \vee P_0) \leq \pi(S \vee (Q_0 S \vee P_0)) = T^{-1} \xi.
\]

Hence \( P = T^{-1} Q \in B_{T, S} \) and \( P \leq \xi \).

In the ergodic theory of single automorphisms the following characterization of automorphisms with zero entropy is well known. Namely, an automorphism \( T \) has zero entropy if and only if \( P_T = 0 \).

However, it appears that it is possible to obtain a two-dimensional analogue of this result by the use of regular generators. First we define the concept of two-dimensional strong generator with respect to the lexicographical order.

**Definition.** A generator \( P \in \mathcal{F} \) is said to be a \((T, S)\)-strong generator of \((X, G)\) if

\[
\bigvee_{0 \leq n \leq \infty} T^n S^P P = 0.
\]

**Proposition 2.** A totally ergodic group \( G \) has zero entropy if every generator \( P \in B_{T, S} \) is \((T, S)\)-strong.

**Proof.** The sufficiency is obvious. Let us suppose \( h(G) = 0 \), \((T, S) \in b(G)\) and \( P \in B_{T, S} \). Since \( \text{if } P \neq 0 \text{ then } P = \bigvee_{0 \leq n \leq \infty} T^n S^P P \neq 0 \), \( P \) is a strong generator of \((X, G)\). Therefore \( P \in B_{T, S} \) and thus \( P \) is \((T, S)\)-strong.

**Remark.** The conditions \( h(G) = 0 \), \( P \in B_0 \) do not imply that \( P \) is \((T, S)\)-strong.

**Example.** Let \((Y, S, \lambda)\) be a Lebesgue probability space and \( S_0 \) an aperiodic automorphism of \( Y \) with \( h(S_0) = 0 \). We denote by \((\mathbb{X}, \mathbb{A}, \mu)\) the product space

\[
\prod_{i=0}^{\infty} (Y_i, S_i, \lambda_i), \quad \text{where } Y_i = Y, \quad S_i = \mathcal{F}, \quad \lambda_i = \lambda.
\]

Let \( T, S \) be automorphisms of \((\mathbb{X}, \mathbb{A}, \mu)\) defined by the formulas

\[
(T x)(n) = x(n+1), \quad (S x)(n) = S_0 x(n), \quad n \in \mathbb{Z},
\]

and let \( G \) be the automorphism group generated by \( T \) and \( S \). It is clear that \( G \) is totally ergodic. It is shown in [7] that \( h(G) = h(S_0) = 0 \). Let \( \alpha = [A_0, A_1] \) be a generator of \((Y, S_0)\). The partition \( P = [C(0, A_0), C(0, A_1)] \) where \( C(0, A_i) = \{x \in X : x(0) \in A_i\}, \quad i = 1, 2, \) is a generator of \((X, G)\). We shall check that \( P \) is not \((T, S)\)-strong. Let us suppose \( P_T \neq 0 \). Since \( \mu \) is a product measure the partitions \( P \) and \( P_T \) are independent. Hence \( P \) and also \( \alpha \) are trivial partitions, which is impossible.

**Remark.** Let \( G \) be totally ergodic, \((T, S) \in b(G)\) and let \( \xi \) be a \((T, S)\)-perfect partition. It would be interesting to know whether \( \xi \) may be represented as the past of a certain two-dimensional process \((P, G)\), i.e. \( \xi \).
\( P \lor P_2, P \in B(G) \). This question has a positive answer if \( h(G) = 0 \), because in this case every perfect partition is the partition into points and it is sufficient to use Corollary 2 and Proposition 2. We have been unable to decide whether this question has a positive answer in the general case.

References


On drop property

by

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Abstract. Let \( (X, || ||) \) be a Banach space. We say that the norm \( || || \) has the drop property if for each closed set \( C \) disjoint with the closed unit ball \( B = \{ x : ||x|| \leq 1 \} \), there is a point \( x \in C \) such that \( \text{conv}(a \cup B) \cap C = \{ a \} \).

We say that a Banach space \( (X, || ||) \) has the drop property if there is a norm \( || || \) equivalent to the given one such that \( || || \) has the drop property.

In the paper it is shown that each superreflexive space has the drop property and each space \( X \) which has the drop property is reflexive.

Let \( (X, || ||) \) be a Banach space. Let \( B \) denote the unit ball in \( X \). By a drop induced by a point \( a \notin B \) we mean the set

\[ (1) \quad D(a, B) = \text{conv}(a, B). \]

Danef [3] proved the following

Theorem 1. (Drop theorem). Let \( C \) be a closed set such that

\[ (2) \quad \inf \{ ||x|| : x \in C \} = R > 1. \]

Then there is a point \( a \in C \) such that

\[ (3) \quad D(a, B) \cap C = \{ a \}. \]

The drop theorem was used in various situations (see [1], [2], [4], [5], [10]).

Recently Penot [9] discussed the relations between the drop theorem and Ekeland's variational principle [7].

It is a natural question to ask when we can replace in the drop theorem assumption (2) by the weaker assumption that \( C \) is disjoint with \( B \).

We shall say that the norm \( || || \) has the drop property if the drop theorem holds under this weaker assumption. If there is a norm \( || || \) equivalent to the norm \( || || \) and having the drop property, then we say that the space \( X \) has the drop property.

In this paper we shall show that the uniformly convex norms have the drop property and that the spaces \( X \) with the drop property are reflexive.

Let \( (X, || ||) \) be a Banach space. We recall that the space \( (X, || ||) \) is called uniformly convex if there is an increasing positive function \( \delta(u) \) defined