(p, q)-Convexity in quasi-Banach lattices and applications

by

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Dedicated to Prof. Luis Vigil

Abstract. We define (p, q)-convexity in quasi-Banach lattices for p, q > 0 and study the values r, s > 0 for which (p, q)-convexity implies (r, s)-convexity, showing the difference between this situation and the Banach case.

Finally, we apply our results to a problem of Turpin on the existence of tensor p-norms.

0. Introduction. In the recent development of the theory of Banach lattices the concepts of p-convexity and p-concavity play a very important role (see Lindenstrauss-Tzafriri [9]). They were first defined by Krivine [8] as “type ≥ p” and “type ≤ p”. Maurey [11] introduced the more general notions of “type ≥ (p, q)” and “type ≤ (p, q)” as follows: given a Banach lattice X, we say that X is of “type ≥ (p, q)” or (p, q)-convex, 1 ≤ q ≤ p < ∞, if there is some constant C such that for all finite sequences x₁,..., xₙ of elements of X we have

\[ \left\| \frac{1}{n} \sum_{i=1}^{n} |x_i|^p \right\|^{1/p} \leq C \left( \frac{1}{n} \sum_{i=1}^{n} |x_i|^q \right)^1/q. \]

((p, q)-concavity) is dually defined. Maurey himself showed ([11], especially pp. 11, 12 and 17) that for Banach lattices (p, q)-convexity (resp. (p, q)-concavity) adds nothing to p-convexity (resp. p-concavity), a (p, q)-convex Banach lattice also being r-convex for every r < q.

We shall show that the situation is entirely different when one considers quasi-Banach lattices (for the definition, see Kalton [6]). In this case (p, q)-convexity, now defined for p ≥ q > 0, cannot be reduced in general to r-convexity for any r ≥ 0.

Observe that on the left side of the inequality (*) the expression \( \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \) is used. This element in X is defined by means of a "homogeneous functional calculus", i.e., by proving that for every positive
integer $n$ and $x_1, \ldots, x_n$ in $X$ there is a unique continuous lattice homomorphism from $\mathcal{H}_n$ into $X$, where $\mathcal{H}_n$ is the Banach lattice of $1$-homogeneous continuous functions $h: \mathbb{R}^n \to \mathbb{R}$ normed by

$$\|h\|_{\mathcal{H}_n} = \sup \{ \|h(t_1, \ldots, t_n)\| : \text{max}(\|t_1\|, \ldots, \|t_n\|) = 1 \}$$

(Krivine [8], Lindenstrauss-Tzafriri [9]), such that it maps the coordinate projections $(t_1, \ldots, t_n) \in \mathbb{R}^n$ into $x_i \in X$ (1 \leq i \leq n). We denote this homomorphism by $\mathcal{T}_{x_1, \ldots, x_n}$ and the image of a function $h$ in it by $h(x_1, \ldots, x_n)

This construction also works for $p$-Banach lattices, 0 < p < 1 (see Popa [12]) and can be extended even to certain classes of vector lattices without any topology, for example to uniformly complete vector lattices (see Cuartero-Triana [2] for details).

Note. The uniqueness of each $\mathcal{T}_{x_1, \ldots, x_n}$ avoids possible ambiguities and allows in many cases to manage the expressions $h(x_1, \ldots, x_n)$ like the functions $h(t_1, \ldots, t_n)$, for example, if $f$ and $g$ are homogeneous continuous functions on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively, satisfying

$$f(t_1, t_2, \ldots, t_n) = g(t_1, t_1, t_2, \ldots, t_n)$$

for all $(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, then $f(x_1, x_2, \ldots, x_n) = g(x_1, x_1, x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \in X$.

In fact, if $S: \mathcal{H}_{n+1} \to \mathcal{H}_n$ is defined by

$$(Sh)(t_1, \ldots, t_n) = h(t_1, t_1, t_2, \ldots, t_n),$$

then $T = \mathcal{T}_{x_1, x_2, \ldots, x_n} \circ S$ is a lattice homomorphism which maps the coordinate projections into $x_1, x_2, \ldots, x_n$ and so $T = \mathcal{T}_{x_1, x_2, \ldots, x_n}$ in a similar way, for each permutation $\sigma$ of the indices 1, 2, n we take $S_{\sigma}: \mathcal{H}_n \to \mathcal{H}_{n}$ such that

$$(S_{\sigma}h)(t_1, \ldots, t_n) = h(x_{\sigma_1}, \ldots, x_{\sigma_n})$$

we see that $(S_{\sigma}h)(x_1, \ldots, x_n) = h(x_{\sigma_1}, \ldots, x_{\sigma_n})$, and so on. This result will be used later in § 2.

This paper is an improved version of the first chapter in Triana [13]. Not too surprisingly for us, Kalton [6] has been independently working on $p$-convexity in quasi-Banach lattices but his results, though interacting with ours, go in a different direction.

In § 1 we define $(p, q)$-convexity in quasi-Banach lattices $L$ and by means of the $s$-convexification of $L$ we give several answers to the question: For what values $r, s > 0$ the $(p, q)$-convexity of $L$ implies its $(r, s)$-convexity? Taking account of an example of Kalton [6], the more general result (Proposition 1.3) cannot be improved. With supplementary conditions of $q$-concavity, we can obtain better results (Proposition 1.6).

What about the $s$-convexifications of Banach lattices? As such $s$-convexifications are $s$-convex quasi-Banach lattices, in order to be sure that we have something essentially different from Banach lattices we must find quasi-Banach lattices not $s$-convex for any $s > 0$. We describe here an example; some others of a different kind can be seen in Kalton [6]. Thus we have an important classification of quasi-Banach lattices into two nonvoid groups:

1. The $s$-convexifications of Banach lattices, i.e., the $s$-convex quasi-Banach lattices for some $s > 0$, for which Kalton has given a nice intrinsic characterization, the $L$-convexity (see [6]);
2. The non-$L$-convex quasi-Banach lattices, those for which there is no $s > 0$ such that their $(1/s)$-convexification is Banach (equivalently, which are not $s$-convex for any $s > 0$).

In § 2 we apply our results on $(p, q)$-convexity to tensor products of $p$-Banach spaces. Turpin [14], solving a problem which goes back to Waelbroeck, proved that if $E$ is a $p$-normed space and $F$ is a $q$-normed space, then a tensor $r$-norm may be given in $E \otimes F$ with $r = pq/(p + q - pq)$ (recall that a tensor $r$-norm is an $r$-norm in $E \otimes F$ such that the canonical bilinear map $E \times F \to E \otimes F$ is continuous). We obtain in the general case the same value for $r$ as Turpin does and we can improve it under additional conditions on one of the spaces $E, F$. Moreover, the examples of non-$L$-convex quasi-Banach lattices suggest that the value $r = pq/(p + q - pq)$ is best possible. After the elaboration of this paper and using very different ideas, Kalton [7] has been able to prove this in § 1. $(p, q)$-convexity in quasi-Banach lattices. Let $L$ be a quasi-Banach lattice, i.e., a complete quasi-normed space $(L, \| \cdot \|)$ where $L$ is a vector lattice and $\| \cdot \|$ is a lattice quasi-norm, i.e., a map $\| \cdot \|: L \to \mathbb{R}$ such that

$$\|x\| > 0 \quad \text{if } x \in L \setminus \{0\},$$

$$\|\alpha x\| = |\alpha| \|x\| \quad \text{if } \alpha \in \mathbb{R}, x \in L,$$

$$\|x + y\| \leq M (\|x\| + \|y\|) \quad \text{if } x, y \in L,$$

for some constant $M$ independent of $x$ and $y$ (the best constant $M$ is called the multiplier of the quasi-norm) and

$$\|x\| \leq \|y\| \quad \text{whenever } |x| \leq |y| \text{ in } L$$

$L$ is said to be $(p, q)$-convex where $0 < q < p \leq \infty$ and $q < \infty$ if there exists a constant $K < \infty$ so that

$$\left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \leq K \left( \sum_{i=1}^{n} |x_i|^q \right)^{1/q}$$
for every choice of vectors \( \{x_i\}_{i=1}^n \) in \( L \). The smallest possible value of \( K \) is called the \((p, q)\)-convexity constant. As usual, for \( p = \infty \) we suppose

\[
\left( \sum_{i=1}^n |x_i|^p \right)^{1/p} = \sqrt[n]{\sum_{i=1}^n |x_i|^p}.
\]

When \( q = 1 \) with \( 1 \leq p < \infty \) we find the concept of \( p\)-convex Banach lattice (cf. Lindenstrauss–Tzafriri [9]), and if \( p = \infty \) the concept of upper \( q\)-estimate (cf. Lindenstrauss–Tzafriri [8] and Kalton [6]). Observe that for \( p = 1 \) we can define \((1, q)\)-convexity in each quasi-normed space \((X, \|\cdot\|)\) replacing \( \sum_{i=1}^n |x_i| \) by \( \sum_{i=1}^n x_i \). In this case we say that the space \( X \) is \( q\)-convex, and it is clear that \( q\)-convexity is equivalent to \( q\)-normability.

Now, we define the \( q\)-convexification of a quasi-Banach lattice \( L \) as in Lindenstrauss–Tzafriri [9], but for all \( s > 0 \).

We denote, as usual, by \( +, - \) and \( ||\cdot|| \) the algebraic operations and the quasi-norm of \( L \). Let \( s \in (0, +\infty) \) for \( x \) and \( y \) in \( L \) and for any scalar \( a \), we define

\[
x(+)y = (x^{10} + y^{10})^{1/10}, \quad a(-)x = ax - x
\]

where \((x^{10} + y^{10})^{1/10}\) is the element in \( L \) corresponding to the function

\[
f(t_1, t_2) = \|t_1^{10}\|\|t_2^{10}\|^{1/10} \strut\text{sign}(t_1^{10})\text{sign}(t_2^{10})
\]

and \( a^s = |a|^s |a|^{s-1} \) (cf. Popa [12] and Cuartero–Triana [2]).

\((L, \|\cdot\|, \langle\cdot,\cdot\rangle)\) is a vector lattice denoted by \( L_s \), in which we can define a lattice quasi-norm \( \|x\|_s = \|x^s\|_s^{1/10} \) (by Hölder's inequality we obtain \( \|L(\cdot,\cdot)\|_s \leq 2^{1/10} M^{1/10} (\|x\|_s + \|y\|_s) \)),

where \( M \) is the multiplier of the quasi-norm \( \|\cdot\| \). \( L_s, \|\cdot\|_s \) is called the \( s\)-convexification of \( L \).

**Lemma.** Let \((L_s, \|\cdot\|_s)\) be a quasi-Banach lattice. Then for every \( 0 < \theta \leq 1 \) and \( x, y \in L \)

\[
\|x\|_s \|y\|_s^{1-\theta} \leq M \|x\|_s \|y\|_s^{1-\theta}
\]

where \( M \) is the multiplier of \( \|\cdot\|_s \).

The proof is similar to that of Proposition 1.4.2 (i) of Lindenstrauss–Tzafriri [9].

**Proposition.** Let \((L_s, \|\cdot\|_s)\) be a quasi-Banach lattice. Then \((L_s, \|\cdot\|_s)\) is also quasi-Banach for every \( 0 < s \leq \infty \).

**Proof.** Let \( \{x_n\} \) be a Cauchy sequence in the positive cone \( L^*_s \). We now distinguish two cases:

(a) \( 0 < s < 1 \). Since \( \|x_n - x_m\|_s \leq \|x_n - x_m\|_s^{1/s} \) \( (m, n \in \mathbb{N}) \) there is \( x \in L^* \) such that the sequence \( \{x_n\} \) converges to \( x \) in \( L \). We shall prove that \( \{x_n\} \) converges to \( x \) in \( L^*_s \); indeed, let \( M \) be the multiplier of the quasi-norm \( \|\cdot\|_s \); for every \( n \in \mathbb{N} \)

\[
\|x_n - x_m\|_s \leq M \|x_n - x_m\|_s^{1/s} + \|x_n - x\|_s^{1/s} + \|x - x_m\|_s^{1/s}.
\]

When \( s \geq 1/2 \), \( \|x_n - x_m\|_s^{1/s} \leq \|x_n - x_m\|_s \), consequently \( \lim x_n = x \) in \( L^*_s \). If \( s \geq 1/2^{s+1} \) we repeat the procedure \( k \) times.

(b) \( 1 < s < \infty \). Now

\[
\|x_n - x_m\|_s \leq M^2 \|x_n^{1/s}\|_s \|x_n - x_m\|_s^{1/s} + \|x_n - x_n^{1/s}\|_s^{1/s}.
\]

When \( s \leq 2 \) then \( \|x_n - x_m\|_s^{1/s} \leq \|x_n - x_m\|_s \) and so \( \{x_n\} \) is a Cauchy sequence in \( L^* \). If \( s \leq 2^{s+1} \) we shall repeat the procedure \( k \) times. Hence, there is \( x \in L^* \) so that \( \lim x_n = x \) in \( L^*_s \) and in \( L^*_s \). In order to complete the proof, we can use Theorem 16.1 of Aliprantis–Burkinshaw [1].

It is easily verified that if \( L \) is \((p, q)-\)convex for \( 0 < q \leq p < \infty \) then \( L^* \) is \((sp, sq)-\)convex for every \( 0 < s < \infty \). In particular, \( L \) is \((p, p)-\)convex if and only if \( L_{10} \) is normable.

The property of being \((p, p)-\)convex for some \( p > 0 \) or, equivalently, of having a Banach \( s\)-convexification for some \( s > 0 \), has been characterized by Kalton [6] by means of \( L\)-convexity: a quasi-Banach lattice has this property if and only if \( x \in L^* \) with \( \|x\| = 1 \) and \( 0 < x_i \leq u (1 \leq i \leq n) \) satisfy

\[
\frac{1}{n} (x_1 + \ldots + x_n) \geq (1 - \varepsilon) u
\]

then

\[
\max_{1 \leq i \leq n} \|x_i\| \geq \varepsilon.
\]

In contrast with the Banach case, there are \((p, q)-\)convex quasi-Banach lattices for some \( q \geq p > 0 \) which are not \((s, s)-\)convex for any \( r > 0 \), i.e., which are not \( L\)-convex. An example of this with \( q = \infty \) are the spaces \( L^p(q) \) where \( q \) is a suitable pathological submeasure (Kalton [6]). A different example has been supplied by us to G. Pisier (1):

(1) During a collaboration supported by the “Programa general de relaciones científicas hispano-francesas”.
Let \((E, \| \cdot \|)\) be a Banach space of Rademacher type \(\geq 1\) and consider it canonically imbedded in the Banach lattice \(\mathcal{F}(K)\) of continuous functions over the unit ball \(K\) of the dual \(E^*\) with its \(w^*\)-topology. Consider the ideal \(L\) generated by \(E\) in \(\mathcal{F}(K)\) endowed with the quasi-norm \(\cdot \| \cdot \|\) defined by

\[
\| \varphi \| = \inf \left\{ \sum_{i=1}^n \| x_i \|^{1/p} : (x_i)_{i=1}^n \in E \text{ and } \sum_{i=1}^n |x_i|^{1/2} \right\}.
\]

It is easily verified that \((E, \| \cdot \|)\) is a \((2, \rho)\)-convex quasi-normed lattice.

With a suitable choice of \(E\), the completion of \(L\) cannot be \((r, s)\)-convex for any \(r > 0\). Suppose, to the contrary, that it is \((r, s)\)-convex for some \(r > 0\). Consequently there exists a constant \(K\) such that

\[
\| \sum_{j=1}^n |y_j|^p \|^{1/p} < K \sum_{j=1}^n |y_j|^{s/p}
\]

for every \((y_j)_{j=1}^n \in E\).

Let \((y_{j_k})_{k=1}^\infty\) be Gaussian random variables with zero mean and variance 1. Then there is a constant \(C\) such that

\[
\| \sum_{j} y_{j_k} x_{j_k} \|_{L^2(E)} = C \left( \sum_{j} |y_{j_k}|^{s/p} \right)^{1/2}
\]


Moreover, by Khintchine's inequality, there are constants \(C_1\) and \(C_2\) such that

\[
\left( \sum_{j} |y_{j_k}|^{s/p} \right)^{1/2} \leq C_1 \left( \sum_{j=1}^n |y_{j_k}|^{s/p} \right)^{1/2} \leq C_2 \left( \sum_{j=1}^n |y_{j_k}|^{s/p} \right)^{1/2}
\]

where \((e_j)_{j=1}^\infty\) denote the Rademacher functions. Hence,

\[
\| \sum_{j} y_{j_k} x_{j_k} \|_{L^2(E)} \leq C \cdot C_1 \left( \sum_{j=1}^n |y_{j_k}|^{s/p} \right)^{1/2}
\]

and this is false, for example, when \(E = C_p\) with \(p \neq 2\), where \(C_p\) are the Schatten classes of operators. Indeed, if

\[
G_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{j=1}^n y_{i_j} e_i \otimes e_j,
\]

we know that there exists a constant \(\delta > 0\) such that

\[
\| \sum_{i,j} y_{i,j} e_i \otimes e_j \|_{L^2(E)} \geq n^{1/2} \|G_n\|_p \geq \delta n^{1/2} \|H\|_p = \delta n^{1/2} n^{1/p}
\]

which is impossible by (\(\ast\)).

Then it is natural to ask for what values of \(r, s\), \((r, s)\)-convexity implies \((r, s)\)-convexity.

1.3. Proposition. If a quasi-Banach lattice \(L\) is \((p, q)\)-convex, then for every \(r \leq p\), \(L\) is also \((r, s)\)-convex, where

\[
\frac{1}{r} - \frac{1}{s} = \frac{1}{q} - \frac{1}{p} \quad \text{if } p \neq \infty.
\]

In particular,

(a) \(L\) is \((r, r)\)-convex for every \(r \leq p\) if it is \((p, p)\)-convex.

(b) If \(L\) is \(q\)-normable, \(0 < q < 1\), then it is \((p, pq/(p+q-pq))\)-convex for every \(p \in (0, 1]\).

Proof. Let \(r < p\), \(\|x_i\|_{L^p(E)} \leq 1\). By Hölder's inequality, we have for every \(0 < \alpha < 1\), if \(p \leq \infty\)

\[
\left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq K \left( \sum_{i=1}^n |x_i|^{q-p(1-\alpha)} |x_i|^{-q(1-\alpha)} \right)^{1/p} \leq K \left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q}
\]

and if \(p = \infty\)

\[
\left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq K \left( \sum_{i=1}^n \|x_i\|^{q-p(1-\alpha)} \right)^{1/q} \leq K \left( \sum_{i=1}^n \|x_i\|^{q-p(1-\alpha)} \right)^{1/q}
\]

where \(K\) is the \((p, q)\)-convexity constant.

Taking \(\alpha = \frac{r}{q} - \frac{r}{p} = \frac{r}{q} - \frac{r}{q} = 0\), we are done. \(\blacksquare\)

This proposition gives the best possible result, as Example 2.4 of Kalton [6] shows (this follows from the fact that we can identify the \(s\)-convexification of \(L_p(\phi)\) with \(L_s(\phi)\) in the obvious manner).

We have also

1.4. Proposition. Let a quasi-Banach lattice \(L\) be \((p_0, q_0)\)-convex and \((p_1, q_1)\)-convex \((p_0 < p_1)\). If

\[
\frac{1}{r} = \frac{1}{p_0} + \frac{1}{p_1} - \frac{1}{q_0} \quad (\theta \in (0, 1))
\]

then \(L\) is \((p_1, q_1)\)-convex.
then \( L \) is \((r, s)\)-convex for every \( s \) with
\[
1 > \frac{\theta}{s} + 1 - \frac{\theta}{q_0}, \quad q_0.
\]

In particular, if \( L \) is \( L\)-convex and \( q\)-normable \((0 < q \leq 1)\), then \( L \) is \((s, s)\)-convex for every \( s \) in \((0, q)\).

**Proof.** Let \( \{x_i\}_{i=1}^n \subset L \) be such that \( \|x_i\| \leq 1 \). By Hölder’s inequality
\[
\left\| \sum_{i=1}^n |x_i|^p \right\|^{1/p} \leq \left\| \sum_{i=1}^n |x_i|^{p_0} \right\|^{1/p_0} \left( \sum_{i=1}^n |x_i|^{q} \right)^{1/q},
\]

where
\[
M = \left( \sum_{i=1}^n |x_i|^{p_0} \right)^{1/p_0} \left( \sum_{i=1}^n |x_i|^{q} \right)^{1/q}.
\]

Then it follows from Proposition 2.2 of Kalton [3] that \( L_{1/p} \) is \((1, s)/r\)-convex whenever
\[
1 > \frac{\theta}{s} + 1 - \frac{\theta}{q_0}, \quad q_0
\]

and so \( L \) is \((r, s)\)-convex.

We can obtain better values for \( s \) than those in Proposition 1.3 if we suppose that \( L \) is \((q, p)\)-convex for some \( q \in (0, +\infty) \).

**1.5. Definition.** Let \( p, q \in (0, +\infty) \). We say that \( L \) is \((p, q)\)-convex if there is a constant \( K \) such that
\[
\left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq K \left( \sum_{i=1}^n |x_i|^q \right)^{1/q}
\]

for every choice of vectors \( \{x_i\}_{i=1}^n \) in \( L \).

**1.6. Proposition.** If a quasi-Banach lattice \( L \) is \((u, r)\)-convex \((u < \infty)\) and \((q, q)\)-convex for some \( q < \infty \), then for every \( s > r \), \( L \) is \((s, r)\)-convex.

In particular, if \( L \) is \(r\)-normable and \((q, q)\)-convex for some \( q < \infty \), then \( L \) is \((s, r)\)-convex for every \( s > r \).

**Proof.** We may assume that \( u = 1 \) (otherwise we can use the fact that \( L_{1/p} \) is \((1, r)/u\)-convex). It is known that a quasi-Banach space \( E \) of Rademacher type \( p \) is \( p\)-convex (cf. Theorem 4.2 of Kalton [4]). Let \( p < \min \{1/r, 2\} \); we shall prove that \( L_p \) is of Rademacher type \( rp \).

Since \( L \) is \((q, q)\)-convex (we may assume without loss of generality that \( q > 1/p \)) there exists a constant \( C \) such that if \( \{x_i\}_{i=1}^n \subset L \)
\[
\left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq C \left( \sum_{i=1}^n |x_i|^q \right)^{1/q}
\]

by Khintchine’s inequality and since \( L \) is \((1, r)\)-convex, there exist two constants \( A, B \) so that this expression is upper bounded by
\[
A \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq A \left( \sum_{i=1}^n |x_i|^q \right)^{1/q} \leq B \left( \sum_{i=1}^n |x_i|^q \right)^{1/q} = B \left( \sum_{i=1}^n |x_i|^q \right)^{1/q}.
\]

Hence \( L_p \) is of Rademacher type \( rp \), and thus \( r\)-convex.

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for every permutation $\sigma$ of the indices $1, 2, \ldots, n$ (see Note in the introduction).

Thus, the mapping

$$h: w = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes L \rightarrow h(w) = h_{x_1, \ldots, x_n}(y_1, \ldots, y_n) \in L$$

is well defined and satisfies for all $w, w' \in E \otimes L$ and $t \in \mathbb{R}$

(i) $h(w) \geq 0$.
(ii) $h(w) = 0$ if $w = 0$.
(iii) $h(w + w') \leq M[h(w) + h(w')]$ where $M$ is the multiplier of $q$.
(iv) $h(tw) = |t|h(w)$.

(To see that $h$ is well defined, perhaps the simplest way is to consider a Hamel basis $B$ of the space $L$ and recall that every $w$ in $E \otimes L$ can be uniquely written as $w = \sum_{i=1}^{n} x_i \otimes b_i$ with $(x_i)_{B} \in E^{(B)}$. Then if

$$w = \sum_{i=1}^{n} x_i \otimes y_i = \sum_{j=1}^{m} x_j \otimes y'_j,$$

$$y_i = \sum_{k=1}^{m} t_i y_k \quad (1 \leq i \leq n), \quad y'_j = \sum_{k=1}^{m} s_j y_k \quad (1 \leq j \leq m),$$

we have

$$h_{x_1, \ldots, x_n}(y_1, \ldots, y_n) = h_{x_1, \ldots, x_n, y_1, \ldots, y_m}(t_1, b_1, \ldots, t_n, b_n, s_1, \ldots, s_m)$$

and so $0 = h(w) \geq C(|y_1| \vee \ldots \vee |y_n|) \geq 0$. Then $|y_i| = 0$ and $y_i = 0$ ($1 \leq i \leq n$) so $w = 0$.)

Moreover, if $E$ is $p$-convex with constant of $p$-convexity $C$, for every finite collection $(w_1, \ldots, w_n) \subset E \otimes L$,

$$h(\sum_{i=1}^{n} w_i) \leq C(\sum_{i=1}^{n} h^p(w_i))^{1/p}.$$
A direct proof of van der Vaart's theorem

by

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Abstract. The aim of this paper is to give a direct and simple proof of van der Vaart's theorem [3] determining the absolutely continuous component of a signed measure on \( \mathbb{R}^d \) from its characteristic functional.

1. Introduction and results. Let

\[
d\lambda(t) = d\lambda(t_1, t_2, \ldots, t_d) = (2\pi)^{-d/2} dt_1 dt_2 \ldots dt_d
\]

be the modified Lebesgue measure on \( \mathbb{R}^d \), for a \( \lambda \)-integrable function \( f \) on \( \mathbb{R}^d \) define the Fourier transform by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{i\langle \xi, \alpha \rangle} f(\alpha) d\lambda(\alpha), \quad \alpha \in \mathbb{R}^d
\]

where \( \langle \alpha, \xi \rangle \) is the inner product of \( \mathbb{R}^d \), let \( \mathcal{X} \) be the collection of all \( \lambda \)-integrable functions \( \xi \) which satisfy the following conditions:

1. \( \int \xi(t) d\lambda(t) = 1 \).
2. There exists \( a > 1 \) such that

\[
Q(\xi) = \sup_{t \in \mathbb{R}^d} (1 + ||\xi||^a) ||\xi(t)|| < +\infty,
\]

where \( ||\xi|| \) is the Euclidean norm on \( \mathbb{R}^d \), and define

\[
\mathcal{F} = \{ \xi \in \mathcal{X} ; \mathcal{F} \in L^1(\lambda) \}.
\]

Furthermore, for every \( \xi \in \mathcal{F} \) and \( T > 0 \) define \( \xi_T(t) = T^d \xi(Tt) \). Then evidently we have for every \( T > 0 \),

\[
\int \xi_T(t) d\lambda(t) = 1 \quad \text{and} \quad \mathcal{F}_T(\alpha) = \mathcal{F}(\alpha/T).
\]

Let \( \mu \) be a signed measure on \( \mathbb{R}^d \). Then we have the Lebesgue decomposition

\[
d\mu(t) = \frac{d\mu}{d\lambda}(t) d\lambda(t) + d\mu_s(t),
\]

where \( \mu_s \) is the singular component of \( \mu \).

In this paper we shall prove the following theorems.