The boundary of Taylor's joint spectrum for two commuting Banach space operators

by

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Abstract. In this note it is shown that the boundary $\delta \sigma$ of Taylor's joint spectrum for a pair of commuting operators on an arbitrary Banach space is contained in the union of the joint approximate point spectrum $\text{AP} \sigma$ and the joint approximate compression spectrum $\text{AC} \sigma$, but neither $\delta \sigma \subseteq \text{AP} \sigma \cap \text{AC} \sigma$ nor $\delta \sigma \subseteq \text{AC} \sigma$ is true in general. This is in strict contrast to the case of a single operator where $\delta \sigma \subseteq \text{AP} \sigma \cap \text{AC} \sigma$.


Based on Vasilescu's characterization C. Munno and M. Takaguchi [2] proved that the boundary of Taylor's joint spectrum for a pair of commuting Hilbert space operators is contained in the union of the joint approximate point spectrum and the joint approximate compression spectrum in the sense of A. T. Dash [1]. Since this union is of course contained in Taylor's joint spectrum, the result of Munno and Takaguchi gives an easy characterization of at least an important part of the spectrum. The method of proof in [2] heavily relies on the Hilbert space setting. It is the purpose of this note to show that the above-mentioned result holds true in the Banach space setting, too. As it seems our proof is completely elementary.

Moreover, we shall show that in general neither $\delta \sigma \subseteq \text{AP} \sigma$ nor $\delta \sigma \subseteq \text{AC} \sigma$, but $\text{AP} \sigma \cap \text{AC} \sigma$ is nonempty for two commuting operators.

Let $X$, $Y$, $Z$ denote complex Banach spaces and let $L(X, Y)$ denote the space of all continuous linear operators from $X$ into $Y$, writing $L(X)$ for $L(X, X)$ and $X'$ for the dual space $L(X, C)$ instead. Given $S \in L(X, Y)$ we let $S' \in L(Y', X')$ denote the dual operator.

Let $T = (T_1, T_2)$, $(T_i \in L(X), \ i = 1, 2)$ denote a pair of commuting operators. Consider the sequence

$$0 \to X \to X \oplus X \to X \to 0 \tag{1.1}$$
where $\delta^0_t(x) := T_t x \oplus T_t x (x \in X)$ and $\delta^1_t(x_1 \oplus x_2) := T_t x_2 - T_t x_1 (x_1, x_2 \in X)$. By definition [3], $T_t$ is said to be nonsingular if the sequence (1.1) is exact. Taylor’s joint spectrum $\sigma(T; X)$ is the complement of the set of all $z = (z_1, z_2) \in C^2$ such that $z \cdot T := (z_1 - T_t, z_2 - T_t)$ is nonsingular. An element $z = (z_1, z_2) \in C^2$ belongs to the joint approximate point spectrum $\mathcal{A} \mathcal{P} \mathcal{R}(T; X)$ resp. joint approximate compression spectrum $\mathcal{A} \mathcal{C} \mathcal{A} \mathcal{R}(T; X)$ (Dash [1]) if there exists a sequence $(x_n) \subset X$ resp. $(x'_n) \subset X$ such that

$$
\|x_n\| = 1 \quad \text{and} \quad \|x_n - T_t x_n\| \to 0 \quad \text{as} \quad n \to \infty \quad (i = 1, 2)
$$

resp.

$$
\|x'_n\| = 1 \quad \text{and} \quad \|x'_n - T_t x'_n\| \to 0 \quad \text{as} \quad n \to \infty \quad (i = 1, 2).
$$

Finally, given $T \in L(X, Y)$ let $\ker(T)$ and $\text{im}(T)$ denote the kernel and the range space of $T$, respectively.

2. Main result. The following is our main result:

2.1. Theorem. Let $T = (T_t, T_s) \in L(X)^2$ denote a pair of commuting operators on a complex Banach space $X$. Then

$$
\sigma(T; X) = \mathcal{A} \mathcal{P} \mathcal{R}(T; X) \cup \mathcal{A} \mathcal{C} \mathcal{A} \mathcal{R}(T; X),
$$

where $\sigma(T; X)$ denotes the boundary of Taylor’s joint spectrum.

We repeat again that in contrast to the case of a single operator the left-hand side is in general neither contained in $\mathcal{A} \mathcal{P} \mathcal{R}(T; X)$ nor in $\mathcal{A} \mathcal{C} \mathcal{A} \mathcal{R}(T; X)$.

The proof requires some preparations which we state in some elementary lemmas.

2.2. Lemma. Let $(S_n) \subset L(X, Y)$ denote a sequence of operators between Banach spaces $X$ and $Y$. Assume that $(S_n)$ tends to a topological monomorphism $S \in L(X, Y)$ in the uniform operator topology. Then there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that

$$
\|S_n x\| \geq c \|x\| \quad (x \in X, n \geq n_0).
$$

This follows easily from the triangle inequality and the fact that $S$ is assumed to be a topological monomorphism.

By duality we obtain

2.3. Corollary. Let $(T_n) \subset L(Y, Z)$ denote a sequence of operators between Banach spaces $Y$ and $Z$. Assume that $(T_n)$ tends to a surjection $T \in L(Y, Z)$ in the uniform operator topology. Then there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that

$$
T_n (K_Y(0, 1)) \subseteq K_Z(0, c) \quad (n \geq n_0),
$$

where $K_Y(0, 1)$ denotes the open 1-ball in $Y$ centered at 0.

Putting together we obtain

2.4. Lemma. Let $(S_n) \subset L(X, Y)$, $(T_n) \subset L(Y, Z)$ denote sequences of operators between Banach spaces $X, Y, Z$ such that $\ker(T_n) \subseteq \text{im}(S_n)$ (n \in \mathbb{N})$. Assume that $(S_n)$ tends to a topological monomorphism $S \in L(X, Y)$ in the uniform operator topology. Then

$$
\ker(T) \subseteq \text{im}(S).
$$

Proof. Let $y \in \ker(T)$. Then we have

$$
\|T_n y - y\| = \|T_n y - T_n T_n^{-1} y\| < \|T_n - T_n^{-1}\| \cdot \|y\| \to 0 \quad \text{as} \quad n \to \infty.
$$

By 2.3 we know that $T_n$ is a surjection for $n$ sufficiently large. Consequently we find $x_n \in \ker(T_n)$ (n \in \mathbb{N}) such that

$$
\|y - x_n\| \to 0 \quad \text{as} \quad n \to \infty.
$$

But as $\ker(T_n) \subseteq \text{im}(S_n)$ by assumption, we find $x_n \in X$ such that $y_n = S_n x_n$ and $\|y_n\| = \|S_n x_n\| \geq c \|x_n\|$ by 2.2 at least for $n$ sufficiently large. As $\{x_n, n \in \mathbb{N}\}$ is bounded, so is $\{x_n, n \in \mathbb{N}\}$. Moreover, $y = \lim S_n x_n$ and

$$
\|S_n x_n - S x_n\| \leq \|S - S_n\| \cdot \|x_n\| \to 0 \quad \text{as} \quad n \to \infty.
$$

Therefore $y \in \text{im}(S)$, because $S$ was assumed to be a topological monomorphism. Hence $\ker(T) \subseteq \text{im}(S)$.

Proof of 2.1. By doing a translation, it is easy to show that $(0, 0) \in \sigma(T; X)$ implies $(0, 0) \in \mathcal{A} \mathcal{P} \mathcal{R}(T; X)$ and $\delta^0_t(0) = \delta^0_t(0)$ for all $t$. Thus take a sequence $z_n = (z_n, z_n) \in (0, 0)$ and let $T_n := z_n - T_n$. Then evidently $\delta^0_t(0) \to \delta^0_t(0)$ as $n \to \infty$ in the uniform operator topology for $i = 0, 1$.

So let us first assume $(0, 0) \in \sigma(T; X) \cap \mathcal{A} \mathcal{P} \mathcal{R}(T; X)$. This in particular implies that $\sigma(T; X) \cap \mathcal{A} \mathcal{P} \mathcal{R}(T; X)$ is a topological monomorphism, for otherwise we would find a sequence $(x_n) \subset X$ such that $\|x_n\| = 1$ and $\|T_n x_n\| \to 0$ as $n \to \infty$ for $i = 1, 2$, contradicting the assumption $(0, 0) \not\in \mathcal{A} \mathcal{P} \mathcal{R}(T; X)$.

If $\delta^0_t(0) \not\in X$, we find $x_n \in X'$ such that $\|x_n\| = 1 (n \in \mathbb{N})$ and $\delta^1_t(0) \
ot\in X$ as $n \to \infty$. But this in particular implies that $x_n \in X'$ as $n \to \infty$ for $i = 1, 2$, and hence $(0, 0) \not\in \mathcal{A} \mathcal{C} \mathcal{A} \mathcal{R}(T; X)$, which is used. Therefore we conclude that $\text{im}(\delta^0_t) = X$. But then the sequences $(\delta^0_t)_{n=0}^{n=1}$ fulfill the assumptions of 2.4. Consequently, $\ker(\delta^1) \subseteq \text{im}(\delta^0_t)$, and hence we have the exactness of (1.1) because $\text{im}(\delta^0_t) = \ker(\delta^1)$ is always true. This contradicts our assumptions. Therefore $(0, 0) \not\in \mathcal{A} \mathcal{C} \mathcal{A} \mathcal{R}(T; X)$.

On the other hand, let $(0, 0) \not\in \sigma(T; X) \cap \mathcal{A} \mathcal{P} \mathcal{R}(T; X)$. Thus $\delta^0_t(0) = X$. But by the argument of the first paragraph of the proof, $\delta^0_t$ cannot be a topological monomorphism, for this together with $\text{im}(\delta^1) = X$ implies the exactness of
denote the extension of $T_i$ ($i = 1, 2$) upon $(X_h)$ and $T_i = (T_{1i}, T_{2i})$. Then
\[ \sigma(T_i; (X_h)) = \sigma(T; X). \]

**Proof.** Let $(0, 0) \notin \sigma(T; X)$. Since $\delta_T$ is a topological monomorphism, so is $\delta_{T_1}$ and $\text{im}(\delta_{T_2}) = \text{im}(\delta_T)_h$. On the other hand, $\delta_{T_1}$ is a surjection, because $\delta_{T_1}$ is. Finally
\[ \ker(\delta_{T_1}) = (\ker(\delta_T))_h \]
\[ = \text{im}(\delta_{T_2})_h \]
by the exactness of (1.1)
\[ = \text{im}(\delta_T)_h \]
and thus $(0, 0) \notin \sigma(T_1; (X_h)).$

Conversely, let $(0, 0) \notin \sigma(T_1; (X_h))$. Then $\delta_T$ is a topological monomorphism as a restriction of the topological monomorphism $\delta_{T_1}$. If $\delta_{T_1}(x \oplus y) = 0$, then $\delta_{T_1}(x \oplus y) = 0$, where $\oplus$ denotes the canonical embedding of $X$ into $(X_h)$. Consequently we find $(x_h)_h \in (X_h)$ such that $x \oplus y = \delta_{T_1}(x_h)_h$. But this in particular implies that $x \oplus y$ is an accumulation point of im$(\delta_{T_1})_h$ hence $\ker(\delta_{T_1}) = \text{im}(\delta_{T_1})_h$ as $\delta_{T_1}$ is a topological monomorphism.

As a final step we have to show $X = \text{im}(\delta_{T_1})$. By assumption we have
\[ \delta_{T_1}(K_{X_h}) \ni (0, 1) = K_{X_h}(0, c) \]
for some $c > 0$: For every $y \in K_{X_h}(0, c)$ there exists $(x_h)_h \in K_{X_h}$ such that $\delta_{T_1}(x_h)_h = y$. Thus $\delta_{T_1}$ is nearly open, and hence open by the open mapping theorem. Thus $(0, 0) \notin \sigma(T; X).$}

27. **Corollary.** Let $T = (T_1, T_2) \in L(X)^2$ be as in 2.1. Then for every $z_1 \in \partial \sigma(T_1; X)$ there exist $z_2, z \in \sigma(T_2; X)$ such that
\[ z = (z_1, z_2) \in \text{APr}(T; X) \cap \partial \sigma(T; X), \]
\[ w = (z_1, z_3) \in \text{APr}(T; X) \cap \partial \sigma(T; X). \]

**Proof.** Let $z_1 \in \partial \sigma(T_1; X)$. Then $z_1$ is an approximate eigenvalue and by passing to the ultraproduct we may assume that $z_1$ is an ordinary eigenvalue. Let $X_h = \ker(z_1 - T_1)$. Then $T_1, T_2$ leave $X_h$ invariant, since $T_1$ and $T_2$ commute. Take $z_2 \in \partial \sigma(T_2; X)_h = \sigma(T_2; X)$. Then $(z_1, z_2) \in \text{APr}(T; X) \cap \partial \sigma(T; X)$. By duality we find $w \in \text{APr}(T; X) \cap \partial \sigma(T; X).$}

Finally we state the following

28. **Theorem.** Let $T = (T_1, T_2) \in L(X)^2$ be as in 2.1. Then
\[ \text{APr}(T; X) \cap \text{APr}(T; X) \cap \partial \sigma(T; X) \neq \emptyset. \]
Proof. First we state that there exists a pair \((a, b) \in \text{ext conv } \sigma(T; X)\) such that \(a \in \text{conv } \sigma(T_1; X)\). Here \(K\) denotes the convex hull of a set \(K\) and \(L\) denotes the extreme points of a set \(L\). Suppose that for all pairs \((a, b)\) such that \(a \in \text{ext conv } \sigma(T_1; X)\) and \(b \in \sigma(T_1; X)\) we had \((a, b) \in \text{ext conv } \sigma(T; X)\). Then by Carathéodory's lemma \((a, b)\) is the convex combination of at most \(5\) extreme points \((a_1^i, a_2^i) \in \text{ext conv } \sigma(T; X)\) \((1 \leq i \leq 5)\). But as \(a\) is an extreme point of \(\sigma(T_1; X)\) by assumption, and as \(\sigma(T; X)\) has the projection property, we have \(a = a_1^i\) \((1 \leq i \leq 5)\). Consequently each \((a, a_2^i)\) is an extreme point of the desired form. Now fix an \(a\) with this property.

We next show that
\[
|a| \times C \cap \text{int conv } \sigma(T; X) = \emptyset,
\]
where \(K\) denotes the interior points of a set \(K\). For if there were an \((x, \mu) \in \text{int conv } \sigma(T; X)\), then we would find an open neighborhood of \(a\) in \(\text{conv } \sigma(T_1; X)\). A contradiction, since \(a\) was assumed to be an extreme point.

By Corollary 2.7 we find \(\beta\) such that \((a, \beta) \in \text{APo}(T; X) \cap \partial \sigma(T; X)\). Let \(C(a, \beta)\) denote the connected component of \((a, \beta)\) in \(\sigma(T; X)\). We distinguish two cases:

1° If \(\sigma = C(a, \beta)\) is open relative to \(\sigma(T; X)\), then by a result of Taylor [4], 4.9 we find a spectral projection \(\pi\) with the following properties:

(i) \(\pi \in \{T_1, T_2\}'\) (the bicommutant algebra of \(T_1, T_2\) in \(L(X)\)).
(ii) \(X\) has a direct decomposition \(X = X_1 \oplus X_2\) with
\[
X_1 = \pi X, 
X_2 = (I - \pi) X.
\]
(iii) \(\sigma(T; X_1) = \sigma, \sigma(T; X_2) = \sigma(T; X) \setminus \sigma\).

So without loss of generality we may reduce our considerations to the case where \(\sigma = \sigma(T; X)\) is connected. By 2.7 we find another \((x, \mu) \in \partial \sigma(T; X) \cap ACe(T; X)\). But since \(|a| \times C\) lies in a supporting hyperplane of \(\sigma(T; X)\) by construction, \((a, \beta)\) and \((x, \mu)\) lie in the same component of \(\partial \sigma(T; X)\). But as \(\partial \sigma(T; X) = \text{APo}(T; X) \cup ACe(T; X)\) by 2.1, we get the desired result, because \(\text{APo}(T; X)\) and \(ACe(T; X)\) are compact sets.

2° Next assume that \(C(a, \beta)\) is not open relative to \(\sigma(T; X)\). Then
\[
C(a, \beta) = \bigcap \{K : (a, \beta) \in K, K \text{ closed and open in } \sigma(T; X)\}.
\]

With respect to each such \(K\) we again have a spectral decomposition as in 1°. By 2.7 we find for each such \(K\) an \((x, \mu) \in K \cap ACe(T; X) \cap \partial \sigma(T; X)\).

By the compactness of \(\sigma(T; X)\) we find a cluster point \((a, \mu)\) in \(C(a, \beta) \cap ACe(T; X) \cap \partial \sigma(T; X)\). By the same argument as above, \((a, \beta)\) and \((a, \mu)\) lie in the same component of \(\partial \sigma(T; X)\). Hence we are done.