Multivariate interpolation II of Lagrange and Hermite type

by

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Abstract. We present the investigation of second pointwise nature multivariate interpolation (MI-II) introduced in [3]. The Lagrange case of this interpolation in algebraic form was found independently by the authors of [2].

Introduction. In this paper we give the remainder formula, the Lagrange and Newton forms and a recurrence relation for the interpolant polynomial. Further we bring an example of application: "Star" numerical integration and a formula for the main determinant (Vandermonde) of this interpolation.

For the similar aspects of ("dual") Multivariate Interpolation I (MI-I) of Lagrange and Hermite type we refer to [5]-[8], see also [1] and [3], [4], [10].

Let \( t_0, \ldots, t_r \in \mathbb{R} \) and let \( m(t_n), n = 0, \ldots, r, \) be the multiplicity of \( t_n \), that is \( m(t_n) \) is the cardinality of the set \( \{ m| t_m = t_n, m = 0, \ldots, r \} \). Then the familiar univariate Lagrange-Hermite interpolant to \( f \) at knots \( t_0, \ldots, t_r \) is the unique polynomial \( P_f \) of degree not exceeding \( r \), with

\[
P_f(t) = f^{(m)}(t_n), \quad n = 0, \ldots, r, \quad m = 0, \ldots, m(t_n) - 1.
\]

For distinct knots, i.e., when \( m(t_n) = 1, n = 0, \ldots, r \), this polynomial can be written in the Lagrange form:

\[
P_f(t) = \sum_{n=0}^{r} f(t_n) \prod_{m=0}^{r} \frac{t-t_m}{t_n-t_m}.
\]

In the general case \( P_f \) can be written in the Newton form (which uses divided differences),

\[
P_f(t) = \sum_{n=0}^{r} (t-t_0) \cdots (t-t_{n-1})[f(t_0, \ldots, t_n)].
\]

Hence the remainder of interpolation has the following representation:

\[
f(t) - P_f(t) = (t-t_0) \cdots (t-t_r)[f(t_0, \ldots, t_r)].
\]

We will use this formula for the knots \( x_0, \ldots, x_r \), which lie on some line \( f \) in
Let \( u \in \mathbb{R}^k \) be a unit direction-vector of \( l \). Then formula (3) is modified as follows:

\[
    f(x) - P_f(x) = \rho(x, x_0) \cdots \rho(x, x_r) x + x_0, \ldots, x_0 \right) f,
\]

where \( x \in l, \rho(x, x_r) \) is the signed distance (with respect to \( u \)) of \( x \) and \( x_r \). A convenient way of introducing a divided difference here is its well-known Hermite–Genocchi representation,

\[
    Q^{r+1}(\{v_0, \ldots, v_r\}) = \sum_{\mathcal{O}^{r+1}} \frac{\prod_{i=0}^{r+1} v_i}{m!} \frac{\partial^{r+1}}{\partial v^{r+1}} f(v_0, v_1, \ldots, v_r),
\]

where \( Q^{r+1} = \{v_0, \ldots, v_r\} \) is the set of \( r+1 \) points \( \{v_0, \ldots, v_r\} \) and \( D_v \) denotes the directional derivative.

Finally, let us mention the following recurrence relation for interpolated polynomials,

\[
    P_{f}(t) = \left( t - t_0 \right) P_{f}(t_0) + \left( t - t_r \right) P_{f}(t_r),
\]

where \( t \neq t_r \) and \( P_{f} \) interpolates \( f \) at \( \{t_0, \ldots, t_r\} \), \( n = 0, r \).

\[ \Box \]

**Multivariate interpolation II and the Lagrange form.** We start with the following notation.

\[ \mathcal{P}^m_k := \text{collection of subsets of } \{0, \ldots, n\} \text{ of cardinality } m \] For \( q \in \mathcal{P}^m_k \) and \( i = (l_1, \ldots, l_m) \in \mathcal{P}^m_k \), \( q = (l_1, \ldots, l_m) \in \mathcal{P}^m_k \) provided \( q \neq i \). For \( x, y \in R^k \), \( y = (y_1, \ldots, y_k) \) and the multiindex \( \alpha = (x_1, \ldots, x_k) \) we use the following standard notation:

\[
    \begin{align*}
        \langle x, y \rangle &= \sum_{i=1}^{n} x_i y_i, \\
        |x| &= \sqrt{\langle x, x \rangle}, \\
        x^\alpha &= x_1^{a_1} \cdots x_k^{a_k}. \\
    \end{align*}
\]

We denote by \( \pi_m = \pi_m(R^k) \) the set of \( k \)-variate polynomials of total degree not exceeding \( m \).

Let \( L_0, \ldots, L_r \) be \( (k-1) \)-dimensional hyperplanes in \( R^k \) and let the equation

\[
    \lambda_1 x_1 + \cdots + \lambda_k x_k + \lambda_{k+1} = 0
\]

determine \( L_n, n = 0, \ldots, r \). We briefly write for \( i = (l_1, \ldots, l_m) \in \mathcal{P}^m_k \),

\[
    \{l_k\} := \bigcap_{i=1}^{n} L_i.
\]

Let us call \( L_0, \ldots, L_r \) admissible if \( x_i := \{l_k\} \) \( \forall i \in \mathcal{P}^m_k \) is a point in \( R^k \). The admissibility of \( L_0, \ldots, L_r \) is clearly equivalent to

\[
    d_i(l) := \det \begin{vmatrix} x_1 & x_2 & \cdots & x_r \end{vmatrix} \neq 0 \quad \forall i \in \mathcal{P}^m_k.
\]

In what follows it is assumed that the hyperplanes \( L_0, \ldots, L_r \) are admissible. Let the knot \( x_i, i \in \mathcal{P}^m_k \) belong to \( m(x_i) \) hyperplanes from \( L_0, \ldots, L_r \) i.e., \( m(x_i) \) is the multiplicity of \( x_i \) and \( m(x_i) \geq k \). We say that \( L_0, \ldots, L_r \) are in general position if

\[
    m(x) = k \quad \forall i \in \mathcal{P}^m_k.
\]

Denote also by \( \rho(x, L_n), n = 0, \ldots, r \), the signed distance of \( x \) from \( L_n \),

\[
    \rho(x, L_n) = \lambda_1 x_1 + \cdots + \lambda_k x_k + \lambda_{k+1} \sqrt{\lambda_1^2 + \cdots + \lambda_k^2}.
\]

Now we are in a position to present the basic

**Theorem 1.** Let \( L_0, \ldots, L_r \) be admissible \((k-1)\)-dimensional hyperplanes and \( \{x_i \in I\} \) be the set of all distinct points from \( \{x_i \in I\} \); then we have:

(i) For an arbitrary set of real numbers

\[
    I = \{q \mid q \in J, |x| \leq m(x_i) - k\}
\]

there is a unique polynomial \( P_J \in \pi_{n-k+1} \), such that

\[
    D^\alpha P_J(x_i) = \gamma_q \quad \forall q \in J, |x| \leq m(x_i) - k.
\]

(ii) If \( L_0, \ldots, L_r \) are in general position, then we have the analog of Lagrange form (1) for \( P_J \), namely

\[
    P_J(x) = \sum_{i=r}^{n} p_i(x) \prod_{\beta=0}^{r} \frac{\rho(x, L_\beta)}{\rho(x_i, L_\beta)}.
\]

**Proof.** (ii) can be readily checked from formula (7). To prove (i), we consider the polynomials

\[
    P_{\lambda, \beta} = \lambda x_1^{\beta} \cdots \lambda x_k^{\beta} \prod_{\alpha=0}^{m(x_i) - k} \rho(x, L_\alpha),
\]

where \( \beta \in \mathcal{P}^m_k \). If \( \beta \leq m(x_i) - k \). They have the following properties:

\[ \Box \]
and

$$D^k P_{f,i}(x) = \begin{cases} 1 & \text{if } i = j, x = \beta, \\ 0 & \text{if } i = j, |a| \leq |\beta|, x \neq \beta, \\ 0 & \text{if } i \notin I_{n}, i \neq j, |a| \leq m(x) - k. \end{cases}$$

This clearly gives us a way of construction of $P_f$. On the other hand, dim $\mathcal{N}_{n,i} = \# I$ and that completes the proof.■

This theorem was presented by the author in [5], [7]. Part (ii) was found independently by W. Dahmen and C. A. Micchelli in [2].

We denote by $P_f$ the above unique polynomial for which

$$D^k P_f(x) = D^k f(x), \quad \forall i \in I, |a| \leq m(x) - k.$$  

This we shall briefly write

$$P_f = f(L_0, \ldots, L_r).$$

Let us call $L_0, \ldots, L_r$ interpolatory hyperplanes.

If $L$ is an $n$-dimensional hyperplane in $\mathbb{R}^k$, then $f|_L$ denotes the restriction of $f$ to $L$ and is considered as an $n$-variate function.

**Remark 1.** Let $i \in I_n$, $n < k$,

$$P_{f,i} = f(L_{m}, m = 0, \ldots, r).$$

Then we have on the $(k-n)$-dimensional hyperplane $\{L_i\}$,

$$P_{i,j} = f(L_{m}, m = 0, \ldots, r).$$

Of course, interpolatory hyperplanes here are $(k-n-1)$-dimensional and are contained in $\{L_i\}$.

3. The Newton form, remainder formula and a recurrence relation. Let us first choose the directional vector of the line $l_i = [L_i]$, $i = (i_1, \ldots, i_{k-1}) \in I_{k-1}$, as follows

$$u_i = \begin{pmatrix} e_{i_1} & \cdots & e_{i_k} \\ \lambda_{i_1} & \cdots & \lambda_{i_k} \\ \vdots & \cdots & \vdots \\ \lambda_{i_1} & \cdots & \lambda_{i_k} \end{pmatrix},$$

where $e_1, \ldots, e_k \in \mathbb{R}^n$, $(\alpha_{ij}) = \delta_{ij}, n, m, i = 1, \ldots, k$. Denote for $i \in I_{k-1}$,

$$c(n, i) = |n| \cdot |\lambda|, \quad \lambda = (\lambda_{i_1}, \ldots, \lambda_{i_k}) \in \mathbb{R}^k.$$  

Now we present the Newton form of $P_f$ (cf. (2)).

---

**Theorem 2.** Let the $(k-1)$-dimensional hyperplanes $L_0, \ldots, L_r$ be in general position and let

$$P_{f,i} = f(L_0, \ldots, L_i).$$

Then

$$P_f(x) = \sum_{s \in I_{k-1}} \sum_{m=0}^{r} \prod_{n=0}^{s-1} c(m, n) \varphi(x, L_m)^{r-m} \nu_i(x, L_m).$$

**Proof.** Let $P_{f,i}$ be the interpolating polynomial satisfying

$$P_{f,i} = f(L_0, \ldots, L_i), \quad n = k-1, \ldots, r, \quad P_{f,k-1} = 0.$$  

We use the Lagrange form, and taking into account the above relation we obtain

$$P_{f,i} - P_{f,i-1} = \sum_{s \in I_{k-1}} \sum_{m=0}^{r} \prod_{n=0}^{s-1} c(m, n) \varphi(x, L_m)^{r-m} \nu_i(x, L_m).$$

**Applying Remark 1 to the line $l_i = [L_i], i \in I_{k-1}$**, we obtain (interpolatory hyperplanes in this case are zero-dimensional, i.e., they are knots)

$$P_{f,i-1} = f(L_{m}, m = 0, \ldots, n-1).$$

Hence according to (4)

$$f(x_{0,n}) - P_{f,i-1}(x_{0,n}) = \sum_{m=0}^{n} \varphi(x_{0,n}, x_{0,m}) \nu_i(x_{0,n}, L_m).$$

Finally we notice that

$$\frac{\varphi(x_{0,n}, x_{0,m})}{\varphi(x_{0,n}, L_m)} = \frac{1}{\cos(\lambda, \lambda^*)} = c(m, i).$$

Now it remains to sum up (9) using (10) and (11). ■  

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a nonzero vector in $\mathbb{R}^k$ and $L_0 = L_\lambda$ the $(k-1)$-dimensional hyperplane with normal $\lambda$ and passing through $x \in \mathbb{R}^k$ for $n = r+1, \ldots, r+k$. For the convenient presenting of the remainder formula we denote for $i = (i_1, \ldots, i_k) \in I_n$, $n \leq k-1$,

$$\rho := (r-n+k-1, \ldots, r+1, i_1, \ldots, i_\rho) \in I_{k-1} \setminus I_{k-1}.$$
We mean here that \( \emptyset \subseteq \Gamma_0 \) and \( \emptyset^0 = (r + k + 1, \ldots, r + 1) \) and for \( k = 1, \emptyset^0 = \emptyset \). The following theorem gives the remainder formula (cf. (3)).

**Theorem 3.** Let \( L_0, \ldots, L_r, L_{r+k+1}, \ldots, L_{r+k} \) be in general position. Then

\[
D^k [f(x) - P_f(x)] = \sum_{n=0}^{r+k} \sum_{\Gamma \subseteq \Gamma_0} \prod_{m \in \Gamma} c(m, r) q(x, L_m) [x_{\emptyset, k_0}] q = r + k - n, q \in (0, \ldots, r) \backslash \Gamma f.
\]

**Proof.** Let

\[
\bar{P}_f = f(L_0, \ldots, L_r, L_{r+k+1}, \ldots, L_{r+k}),
\]

where \( x \) has been fixed for a moment. We have \( \bar{P}_f(x) = f(x) \) since \( x \) is the common point of \( L_{r+k+1}, \ldots, L_{r+k} \), i.e.,

\[
x = x_{(r+1, \ldots, r+k)} = \bigcap_{m=r+k+1}^{r+k} L_{m,n}.
\]

Using the Newton forms of \( \bar{P}_f \) and \( P_f \) we readily obtain

\[
\bar{P}_f(y) = P_f(y) + \sum_{n=r+k}^{\infty} \sum_{\Gamma \subseteq \Gamma_0} \prod_{m \in \Gamma} c(m, r) q(y, L_m) [x_{\emptyset, k_0}] q = n, q \in (0, \ldots, r) \backslash \Gamma f.
\]

Now we put \( y = x \) in the above relation. Since \( q(x, L_{r+k}) = 0, m = r+1, \ldots, r+k \), we have

\[
f(x) = \bar{P}_f(x) = P_f(x) + \sum_{n=r+k}^{\infty} \sum_{\Gamma \subseteq \Gamma_0} \prod_{m \in \Gamma} c(m, r) q(x, L_m) [x_{\emptyset, k_0}] q = n, q \in (0, \ldots, r) \backslash \Gamma f.
\]

Let us note that the participation of the hyperplane \( L_{r+k} \) in Theorem 3 is symbolic, in fact it is only used to indicate (13).

**Remark 2.** The above method of deriving the remainder formula from the Newton form works in every Lagrange-Hermite interpolation setting. In particular, it can be used for MI-1.

**Corollary 1.** Theorem 2 and Theorem 3 remain valid if we replace the expression “be in general position” by “be admissible” in their hypotheses.

**Proof.** We denote by \( P_f^* \) the formal Newton form (8) for the admissible hyperplanes. Of course for \( P_f^* \) and admissible hyperplanes the remainder formula holds, that is, \( f - P_f^* \) equals to the right-hand side of (12). This gives

\[
D^k [f(x) - P_f^*(x)] = 0 \quad \forall i \in J, \, |x| < m(x_i) - k
\]

since for \( j \in J', \, n < k - 1, \)

\[
D^k [\prod_{m=0}^{r+k} q(x, L_m)] = 0 \quad \forall i \in J, \, |x| < m(x_i) - k.
\]

Thus

\[
"P_f^* = P_f".
\]

Now we present a useful recurrence relation which is the analogue of (5).

**Theorem 4.** Let \( L_0, \ldots, L_r \) be admissible and

\[
P_f = f(L_0, \ldots, L_r).
\]

Let also \( L_{a_1}, \ldots, L_{a_k}, i = (i_0, \ldots, i_k) \subseteq \Gamma_{r+k+1} \), be in general position. Then

\[
P_f(x) = \sum_{n=0}^{r+k} q(x_{i_0}) q(x_{i_1}) q(x_{i_2}) \ldots q(x_{i_k}) P^* f(x),
\]

where

\[
P^*_f = f(L_m, m \in (0, \ldots, r) \backslash \Gamma_i).
\]

and of course

\[
x_{i_{0} \ldots k} = x_{i_{0}, \ldots, i_{k-1}, i_{k+1}, \ldots, i_{k}}.
\]

**Proof.** Applying a continuity argument (with the help of Corollary 1) we need to prove (14) for \( L_0, \ldots, L_r \) being in general position. In this case it is not hard to obtain it from the relation

\[
\sum_{n=0}^{r+k} q(x, L_m) = 1.
\]

**Remark 3.** A similar recurrence relation seems not accessible for MI-4.

4. **Application:** “Star” numerical integration. In this section we give an interesting application of MI-1 to the numerical integration on the disk

\[
D = \{ (t_1, t_2) | t_1^2 + t_2^2 \leq 1 \}
\]

in the plane. Let the points \( x_0, \ldots, x_{2q} \) be equidistantly spaced on the circumference

\[
S = \{ (t_1, t_2) | t_1^2 + t_2^2 = 1 \}.
\]

For convenience we put \( x_{2q+1} := x_0, n = 0, \ldots, q - 1 \). Let \( t_1, n = 0, \ldots, 2q \), be the line passing through \( x_{n} \) and \( x_{n+q} \) (with the directional vector \( x_{n+q} - x_{n} \)). These lines form a \( q \)-star (see Fig. 1 for \( q = 3 \)).
Let \( r_1, \ldots, r_q \) be the radii of the concentric circumferences \( S_1, \ldots, S_q \). Then we easily obtain
\[
    r_{i+1} = \sin \left[ \pi/(4q+2) \right] / \sin \left[ (2i+1) \pi/(4q+2) \right], \quad i = 1, \ldots, q-1,
\]
\[
    r_1 = 1.
\]
If we put in (15) the polynomial
\[
    f(x) = f(t_1, t_2) = \prod_{i=1}^{q} (t_1^2 + t_2^2 - r_i^2),
\]
of total degree \((2q-2)\), the following interesting expression for \( c_s \) is obtained:
\[
    c_s = \frac{2\pi}{2q+1} \prod_{i=1}^{q} \left( \frac{r_i^2 - r_f^2}{r_i^2} \right) dr.
\]
For more detailed consideration and a generalization of this numerical integration see [9].

5. A formula for the main determinant (Vandermonde's) of MI-II. First we shall present a quick proof of the following lemma which is interesting in itself (for origins cf. [11], [12]).

**Lemma 1.** Let \( L \) be a \((k-1)\)-dimensional hyperplane, \( P \in \pi_n(R^k) \), and
\[
    (D_{j})^m P(x) = 0 \quad \forall x \in L, \quad m = 0, \ldots, s-1,
\]
where \( \lambda \) has the normal direction of \( L \). Then
\[
    P(x) = P(x, L)^{p_s}(x),
\]
with
\[
    P_s(x) \in \pi_{n-s}(R^k).
\]

**Proof.** Since (17) is independent of the coordinate system, we assume without loss of generality that \( L \) is the hyperplane \( x_1 = 0 \). Next, we can represent \( P(x_1, \ldots, x_k) \) in the form
\[
    P(x_1, \ldots, x_k) = \sum_{m=0}^{s-1} x_1^m P_m(x_2, \ldots, x_k) + x_1^s P_s(x_1, \ldots, x_k),
\]
where
\[
    P_m \in \pi_{n-m}(R^k), \quad m = 0, \ldots, s-1.
\]

Now (16) implies
\[
    P_m(x_2, \ldots, x_k) = 0, \quad m = 0, \ldots, s-1.
\]
To introduce the analogue of Vandermonde determinant of MI-II we first order the sets $I_k$ and $M = \{x = (\alpha_1, \ldots, \alpha_k) | \alpha_i \leq r-k+1\}$, i.e., we assume that
\[ l: \left\{1, \ldots, \binom{r+1}{k}\right\} \rightarrow I_k \]
and
\[ \alpha: \left\{1, \ldots, \binom{r+1}{k}\right\} \rightarrow M \]
are one-to-one.

In what follows we assume that the $(k-1)$-dimensional hyperplanes $L_1, \ldots, L_r$ are in general position and that they are given by the following equations
\[ \lambda_1 x_1 + \ldots + \lambda_k x_k = 1, \quad n = 0, \ldots, r, \]
respectively. Let also
\[ d_{k, l} [i] := \begin{vmatrix} 1 & \ldots & 1 \\ \lambda_1^0 & \ldots & \lambda_k^0 \\ \vdots & \ldots & \vdots \\ \lambda_1^0 & \ldots & \lambda_k^0 \end{vmatrix} \]
for $i = (i_0, \ldots, i_k) \in I_{k+1}$, and $d_k [i]$, for $i \in I_k$, be given as in (6). Then we define
\[ V(L_0, \ldots, L_r) := \det \left[ \phi_{\alpha}(x)(\binom{r+1}{k})_{\alpha=0}^{k=r} \right], \]
where
\[ \phi_{\alpha}(x) := x^\alpha. \]

**Theorem 5.** We have
\[ V(L_0, \ldots, L_r) = c \frac{\prod_{i=1}^{r-k} d_k [i]^{r-k+1}}{\prod_{i=1}^{r-k+1} d_{k+1} [i]^{r-k+1}}, \]
where $c$ is independent of $L_0, \ldots, L_r$.

**Proof.** Using Cramer's rule for determining $x_i = L_1 \cap \ldots \cap L_{k+1}$, $i = (i_1, \ldots, i_k) \in I_k$, as the unique solution of the linear system of equations of $L_1, \ldots, L_k$, it is not hard to show that
\[ P_T := V(L_0, \ldots, L_r) \prod_{i=1}^{r-k+1} d_k [i]^{r-k+1} \]
is a polynomial of $x^\alpha = (x_1^\alpha, \ldots, x_k^\alpha)$ for each $n = 0, \ldots, r$. Computing the total degree of $P_T$, then considering it as a polynomial of $\lambda_n^m$, $n = 0, \ldots, r$, $m = 1, \ldots, k$ we obtain the sum
\[ \sum_{n=0}^{r-k+1} \binom{r-k+1}{k} \binom{n+r-k}{k} = \binom{r+1}{k}. \]

Now if for $i = (i_0, \ldots, i_k) \in I_k$ and $n \in \{0, \ldots, r\}$, we have
\[ x_\alpha \in L_i := \bigcup_{n=0}^{r-k+1} \lambda_n^m \sum_{n=0}^{r-k+1} \lambda_n^m = 1, \]
then $x_i \in L_k$. Therefore
\[ x_{i_1, \ldots, i_{r-k+1}} = x_i, \quad m = 1, \ldots, k. \]
It means that in this case $V(L_0, \ldots, L_r)$ will have $(k+1)$ columns equal. Hence
\[ (D_{i_1})^{m} P_T(x) = 0, \quad \forall x \in L, \quad m = 0, \ldots, k-1. \]
Since $L$ is a $(k-1)$-dimensional hyperplane, and
\[ g(\lambda^\alpha_L, L) = c_0 d_{k, l} [i]^{(n, i)}, \]
repeated application of Lemma 1 gives
\[ V(L_0, \ldots, L_r) = c \prod_{i=1}^{r-k+1} d_k [i]^{r-k+1}, \]
where $c$ is a polynomial in $\lambda_n^m$, $n = 0, \ldots, r$, $m = 1, \ldots, k$.

The total degree of the product on the right-hand side of (19), considered as a polynomial of $\lambda_n^m$, $n = 0, \ldots, r$, $m = 1, \ldots, k$, obviously equals $k^2 \binom{r+1}{k}$, i.e., it is the same as for $P_T$. Hence $c$ is a constant. Of course, Theorem 1 (ii) implies $c \neq 0$. \]

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