A construction of convolution operators on free groups

by

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Abstract. For a non-commutative free group $G$ and for any $p$, $1 < p < \infty$, we construct a non-negative function $f$ on $G$ which by convolution defines a bounded operator on $l^p(G)$ but unbounded on $l^q(G)$, $q \neq p$.

Let $G$ be a discrete group. A complex function $f$ on $G$ is called a convolution operator on $l^p(G)$, $1 \leq p \leq \infty$, if $f*g$ belongs to $l^p(G)$ whenever $g \in l^p(G)$. The set $C^*(G)$ of all convolution operators on $l^p(G)$ equipped with the operator norm

$$||f||_p = \sup_{g \in l^p \setminus \{0\}} ||f*g||_p$$

is a Banach algebra. We note that $C^1(G)$ and $C^*(G)$ both coincide with the algebra $l^1(G)$ and that $f \in C^q(G)$ if and only if $f^* \in C^q(G)$, where $1/p + 1/p' = 1$, and $f^*$ is the function on $G$ defined by $f^*(x) = f(x^{-1})$.

The aim of this note is to prove the following

Theorem. Let $G$ be a non-commutative free group and let $1 < p < \infty$. There exists a non-negative function $f$ on $G$ such that $f \in C^r(G)$ but $f \notin C^q(G)$ for $1 < q < \infty$, $q \neq p$.

Remarks. 1. The theorem is motivated by a result of Lohoué [3]. For any $p$, $1 < p < \infty$, and any connected semi-simple Lie group $G$ he has constructed a positive measure $\mu$ on $G$ which by convolution defines a bounded operator on $l^p$ and unbounded on $l^q$ for $q \neq p$.

2. It is known that for an amenable group $G$ one always has $C^q(G) = C^r(G)$ for $1 \leq p \leq q \leq 2$ or $2 \leq q \leq p \leq \infty$ ([12], Theorem C) and that a non-negative function $f$ belongs to $C^q(G)$ if and only if $f \in l^p(G)$ ([1], Theorem 3.22), which is in sharp contrast to the theorem above.

We start the proof of the theorem with the following two lemmas:

Lemma 1. Suppose $f$ and $g$ are two functions on $G$ such that for $x_1$, $x_2$ in the support of $f$ and $y_1$, $y_2$ in the support of $g$ the equality $x_1 y_1 = x_2 y_2$ implies $x_1 = x_2$ and $y_1 = y_2$. Then

$$||f*g||_p = ||f||_p ||g||_p, \quad 1 \leq p \leq \infty.$$
The proof is obtained by a simple verification.

**Lemma 2.** Let $G$ be a non-commutative free group and let $a$ and $b$ denote two of the free generators in $G$. For a pair $(m, n)$ of natural numbers let $f_{m,n}$ denote the characteristic function of the set

$$\{a^i b^{j+1}; \ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n\}.$$ 

Then

$$m^{1/p} n \leq \|f_{m,n}\|_p \leq m^{1/p} n + mn^{1-1/p}, \quad 1 \leq p < \infty.$$ 

Proof. For $r \geq n$ define the function

$$g_r = r^{-1/p} \chi_{A_r}, \quad A_r = \{a^i b^{j+1}; \ i = 1, 2, \ldots, m; \ j = 0, 1, 2, \ldots, r-1\},$$

(we use the notation $\chi_A$ for the characteristic function of a set $A \subset G$ and $\delta_r$ for $\chi_{A_r}$). Then $\|g_r\|_p = 1$ and

$$f_{m,n} * g_r \geq mn^{-1/p} \chi_{A_{r+1}}, \quad r = 1, 2, \ldots, n.$$

Therefore

$$\|f_{m,n} * g_r\|_p \leq m^{1/p} n (1 - (1-1/p)^r).$$

and so

$$\|f_{m,n}\|_p \geq mn^{1/p}.$$ 

To show the second estimation we let $g$ be any function in $L^p(G)$ with $\|g\|_p = 1$. For a given integer $j$ define $g_j(x) = g(b^{-j} x)$ if the first letter of the word $b^{-j} x$ is neither $b$ nor $b^{-1}$ and put $g_j(x) = 0$ elsewhere. The functions $\delta_{j} * g_j$ have pairwise disjoint supports and $g = \sum \delta_{j} * g_j$. Thus

$$\sum_j \|g_j\|_p = \|g\|_p = 1.$$ 

Let also $\phi$ be the characteristic function of the set

$$\{a^i; \ i = 1, 2, \ldots, m\}.$$ 

We have

$$f_{m,n} * \phi = \sum_{i=1}^n \phi * \delta_{j} * g_j = \sum_{j=0}^n \phi * g_j + \sum_{i=1}^n \phi * \delta_{j} * \sum_{i=1}^n g_{j+i},$$

thus

$$\|f_{m,n} * \phi\|_p \leq \|\sum_{i=1}^n \phi * g_i\|_p + \|\sum_{j=0}^n \phi * \delta_{j} * \sum_{i=1}^n g_{j+i}\|_p.$$ 

But

$$\|\sum_{i=1}^n \phi * g_i\|_p \leq \|\phi\|_1 \|\sum_{i=1}^n g_i\|_p \leq m \|\sum_{i=1}^n g_i\|_p$$

and

$$\|\sum_{j=0}^n \phi * \delta_{j} * \sum_{i=1}^n g_{j+i}\|_p \leq mn^{1-1/p} \left( \sum_{i=1}^n \|g_i\|_p \right)^{1/p} \leq mn^{1-1/p}.$$ 

Also the functions $\phi * \delta_{j} * \sum_{i=1}^n g_{j+i}, \ j \neq 0$, have pairwise disjoint supports, thus

$$\|\sum_{j=0}^n \phi * \delta_{j} * \sum_{i=1}^n g_{j+i}\|_p = \sum_{j=0}^n \|\phi * \delta_{j} * \sum_{i=1}^n g_{j+i}\|_p,$$

and since $\phi$ and $\delta_{j} * \sum_{i=1}^n g_{j+i}, \ j \neq 0$, satisfy the assumption of Lemma 1, we have

$$\|\sum_{j=0}^n \phi * \delta_{j} * \sum_{i=1}^n g_{j+i}\|_p \leq m \sum_{j=0}^n \|\delta_{j} * \sum_{i=1}^n g_{j+i}\|_p \leq mn^{1-1/p} \sum_{j=0}^n \|g_{j+i}\|_p \leq mn^p.$$ 

All these together give

$$\|f_{m,n} * \phi\|_p \leq m^{1/p} n + mn^{1-1/p}.$$ 

Therefore

$$\|f_{m,n}\|_p \leq m^{1/p} n + mn^{1-1/p}.$$ 

and the lemma follows.

Proof of the theorem. We start the proof with a simple observation that for non-negative functions $f_1$ and $f_2$ on $G$

$$\|f_1 + f_2\|_p \geq \max \{\|f_1\|_p, \|f_2\|_p\}, \quad 1 \leq p < \infty.$$ 

Thus $f_1 + f_2$ belongs to $L^p(G)$ if and only if $f_1$ and $f_2$ both are in $L^p(G)$.

Fix an exponent $p$ and choose two sequences $m_k, n_k$ of integers and a sequence $\alpha_k$ of positive numbers such that the sequence $\alpha_k m_k^{1/p} n_k$ is unbounded for every $q < p$ but the series

$$\sum_{k=1}^\infty \alpha_k (m_k^{1/p} n_k + n_k^{1-1/p})$$

is convergent (put for example $m_k = 2^k$, $n_k = 2^{kp}$ and $\alpha_k = k^{-2} m_k^{-1/p} n_k^{-1})$.

Also define a function $f_k$ by

$$f_k = \sum_{k=1}^\infty \alpha_k f_{m_k n_k},$$

where $f_{m,n}$ are the functions from Lemma 2. Since the series $\sum_{k=1}^\infty \alpha_k \|f_{m_k n_k}\|_p$ is
convergent, $f_i$ is in $C^q(G)$. On the other hand, since all functions $a_k f_{n_k}$ are non-negative,

$$||f_i||_q \geq \sup_k ||f_{n_k}||_q = \infty$$

for all $q < p$ and so $f_i \notin C^q(G)$.

For the exponent $p'$, where $1/p + 1/p' = 1$, let $f_2$ be a function constructed in the same way as $f_1$ with $p'$ in place of $p$. Then $f_2 \in C^p(G)$ but $f_2 \notin C^q(G)$ for $q > p$. Consequently the function $f = f_1 + f_2$ has the property claimed in the theorem.

**Corollary.** Let $G$ be a non-commutative free group and let $1 < p < \infty$, $p \neq 2$. There exists a non-negative function $f$ on $G$ such that the operator $g \mapsto f * g$ is bounded on $l^p(G)$ but the operator $g \mapsto g * f$ is unbounded.

Proof. The operator $g \mapsto g * f$ is bounded on $l^p(G)$ if and only if $f^* \in C^q(G)$ which is equivalent to $f \in C^p(G)$, where $1/p + 1/p' = 1$.

**References**


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