This implies
\[ \left( V_n \right)^n \leq \left( M + \frac{1}{n} \cdot V \right)^n \leq \sum_{k=1}^{n} \left( \frac{1}{n} \right)^{k} \cdot M^{k} \cdot \frac{1}{2^{n-k}} \cdot V \]
\[ \leq \sum_{k=1}^{n} \left( \frac{1}{n} \right)^{k} \cdot C \cdot \frac{1}{2^{n-k}} \cdot V \equiv C \cdot \left( 1 - \frac{1}{n} \right)^{n} \cdot V. \]

Since \( \left( 1 - \frac{1}{n} \right)^{n} \) tends to zero if \( n \to \infty \), it follows that \( \left( V_n \right)^n \leq V \) if \( n \) is sufficiently large. By Lemma 1, \( A \) is an \( \mathbb{A} \)-algebra and the theorem is proved.

Remark. The metrizability of \( A \) is only used to prove \( \mathcal{m} \)-convexity. We regard as an example the algebra \( \mathcal{B} (\mathcal{R}) \) of test functions, which is \( \mathcal{m} \)-convex. Then our theorem shows that \( \mathcal{B} (\mathcal{R}) \) is an \( \mathbb{A} \)-algebra.

References


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$1 < p < \infty$. Moreover, their biorthogonal sequences have analogous properties (with $Z$ replaced by $Z'$ defined in (2.47)). This is all we need to complete the proof of Theorem A.

The bases in $B'_{i,j}(Q)$ as defined in Section 10. The Schauder decomposition into dyadic blocks is here the same as in Section 9. However, in the finite-dimensional subspaces corresponding to the dyadic blocks we construct new bases such that Theorem 10.19 holds.

Section 11 indicates some applications of the main results and contains bibliographical comments.

7. Spline functions. We reduce the approximation problems with boundary conditions on $d$-dimensional cubes to approximation of vector-valued functions. Approximating by spline functions we are led naturally to vector-valued splines. The space of values is denoted by $X$ and as in Section 2 it is equal either to $B$ or to $W_p(Q)$, where $Q$ is a parallelepiped in $R^{d-1}$.

The real-valued B-splines supply the basic tool for our construction. We recall some of their definitions, fundamental properties and results rephrasing them in the vector-valued setting. Most of the proofs will be omitted as they follow by repeating step by step the argument applied in the corresponding real-valued case. For detailed discussion of the real-valued $B$-splines we refer to H. B. Curry-J. J. Scheunberg [18] and C. de Boor [6] and [7].

For given positive integer $r$ and partition $\Pi = (t_j)$ such that $t_j \leq t_{j+1}$, $t_j \leq t_{j+r}$ for $j = 0, \pm 1, \ldots, A = \lim\sup_{j \to \pm \infty} t_j = B$, the $B$-spline $N^r_j$ is defined by the formula

$$N^r_j(t) = (t_{j+r} - t_j) \delta_{j+r} \cdots \delta_{j} \delta_{j-r-1},$$

where the square bracket denotes the divided difference of $(t - s)^{r-1}$ taken at $t_0, \ldots, t_{j+r}$. The function $N^r_j$ is of the class $C^{r-1}$ at $t_j$, $j \leq j + r$ with $a = \max\{t_j, t_{j+r}\}$, where $C^1$ denotes now the class of functions with discontinuities of the first kind.

For $I = (a, b) = (A, B)$ the non-trivial restrictions $N^r_j|_I$ are linearly independent over $I$.

Moreover, the $B$-splines have the following properties:

(7.1) $N^r_j \geq 0$,

(7.2) $\text{supp} N^r_j = (t_j, t_{j+r})$,

(7.3) $\int_{t_j}^{t_{j+r}} N^r_j(t) \, dt = \frac{t_{j+r} - t_j}{r}$,

(7.4) $\sum_{j} N^r_j(t) = 1$, $t \in (A, B)$,

(7.5) If $M^r_j = r(t_{j+r} - t_j)^{-1}N^r_j$ and

$$t_j < t_{j+r}, \ t_{j+r} < t_1,$$

then

$$D^r_j = M^r_j - M^{r-1}_j,$$

(7.6) for $1 < p < \infty$ we have in the norm of $J_p(t_j, t_{j+r})$

$$r^{-1/2} ||N^r_j||_p \leq ||N^r_j||_p \leq ||N^r_j||_p.$$
Note that (7.9) and (7.11) imply
\[ 1 \leq \frac{N + r - 1}{r} |P \cap I| \leq \mu I, \]
where \( |P \cap I| = \max \{|i, r - r - 1, r - 1, \ldots, 0| \leq h \leq |I| \} \) and \( 1 \leq p \leq \infty \) one has
\[ q \leq (\lambda |P \cap I|)^{1/p} \|f\|_p(\lambda |I|). \]

Proposition 7.13 (Bernstein type inequality). Let \( P, I = (a, b), r \) and \( r \) satisfy conditions (7.9) and (7.11). Then there is \( C = C(r, I) \) such that for \( f \in S_p^p(I \times X), \ k = 0, \ldots, r - r, \ 0 \leq \lambda h \leq |I| \) and \( 1 \leq p \leq \infty \) one has
\[ \|f\|_p(\lambda |I|) \leq C \cdot h |P \cap I|^{1/p} \|f\|_p(I \times X). \]

Inequality (7.14) can be proved in exactly the same way as Lemma 9.2 in Ciesielski [12]. The intermediate step in the proof is an application of Bernstein's inequality for the \( X \)-valued splines. It can be obtained from Corollary 7.10 and formula (7.5).

Beside the Bernstein type inequality the orders of approximation by \( X \)-valued splines are important in our considerations.

In what follows the space of \( X \)-valued splines of order \( r \) on \( Y \), corresponding to a \( P \) satisfying (7.9) and (7.11), is denoted by \( S_p^p(I, X) \). The best approximation of \( f \in W_p^p(I \times X) \) by \( S_p^p(I, X) \) is
\[ E_{p, r}^p(f; I \times X) = \inf \{ \| f - g \|_p(I \times X) : g \in S_p^p(I \times X) \}. \]

For the definition of the norm we refer to (2.2).

Proposition 7.15. Let \( P \) satisfy (7.9) and (7.11). Then there is a constant \( C = C(I, \gamma) \) such that
\[ E_{p, r}^p(f; I \times X) \leq C \omega_{p, r}(f; I \times X), \]
holds for \( f \in W_p^p(I \times X), 1 \leq p \leq \infty \).

Proof. This result for real-valued functions is well-known and the proof as presented by De Vore [20, Theorem 4.1, p. 136] can be easily adapted to the case of \( X \)-valued splines.

Let us now consider a sequence of partitions \( P_n, I \), \( I = (a, b) \), of the same type as \( P \) satisfying (7.11) and (7.9), and such that \( P_n \subset P_{n+1} \) and \( P_{n+1} \setminus P_n \) is a one-point set.

For simplicity we set
\[ E_n(f) = E_{p, r}^p(f; I \times X), \quad \omega_n(f; I \times X) = \omega_{p, r}(f; I \times X). \]

Proposition 7.16. There is a constant \( C = C(r, I) \) such that for \( 1 \leq k \leq r - r' \)
\[ \omega_k(f; 1, I \times X) \leq C \omega_{p, r}(f; I \times X) + \sum_{j=1}^{n} 2^{k-1} E_j(f) \]
holds for \( f \in L_p(I \times X), 1 \leq p \leq \infty \).

To prove this we apply standard argument and use (7.14) (cf. [10], Theorem 10).

The last problem we want to discuss in this section is the best approximation of \( X \)-valued functions with small support by \( X \)-valued splines. The theorem we are going to prove exhibits the local character of the best spline approximation.

For the proof of the theorem it is convenient to have the following simple abstract

Lemma 7.17. Let \( Y \) be a Banach space and \( B \) its subspace, and let \( T \) be linear operator such that
\[ T = T_0, \quad T(B) \subseteq B, \]
\[ \hat{Y} = \ker T, \quad \hat{B} = B \cap \hat{Y}. \]

For given \( y_0 \in \hat{Y} \) let \( s_0, s_0 \) denote the best approximations to \( y_0 \) in \( E, \hat{B} \), respectively. Then
\[ \| y_0 - e_0 \| \leq \| y_0 - e_0 \| \leq (1 + |T|) \| y_0 - e_0 \|. \]

Proof. Put \( u_0 = s - T \hat{x} \). Clearly, \( u_0 \in \hat{B} \). Thus,
\[ \| y_0 - u_0 \| \leq \| y_0 - s \| + \| s - T \hat{x} \| \leq (1 + |T|) \| y_0 - s \|. \]

In what follows we use \( I \) for \( (0, 1) \) and \( J \) for one of the intervals \((-1, 1) \) or \((-1, 2) \). The uniform mesh corresponding to the step \( 2^{-\mu} \), \( \mu \geq 0 \), with multiplicities \( r, \alpha, \beta, \gamma \) at the points \(-1, 0, 1, 2, 3 \), respectively, is denoted by \( P(\alpha, \beta, \gamma) \). It is assumed that \( r \geq 1 \) is fixed and \( 1 \leq \alpha, \beta, \gamma \leq r \). The corresponding spline space is
\[ S_{r, \alpha, \beta, \gamma}(J) = S_{r, \alpha, \beta, \gamma}(J \times X). \]

Moreover, let
\[ \begin{align*}
\hat{S}_{r, \alpha, \beta, \gamma}(I) &= \{ u : u \in S_{r, \alpha, \beta, \gamma}(J) : \supp u \subseteq I \}, \\
S_{r, \alpha, \beta, \gamma}(J) &= \hat{S}_{r, \alpha, \beta, \gamma}(I), \\
\hat{S}_{r, \alpha, \beta, \gamma}(I) &= \hat{S}_{r, \alpha, \beta, \gamma}(I). 
\end{align*} \]

Theorem 7.18. For given \( r \geq 2 \) there is a constant \( C \) such that for \( 2^{\mu} \geq 2^{(r-1)} \) we have
\[ C \cdot \text{dist}(b, S_{r, \alpha, \beta, \gamma}(J)) \leq \text{dist}(b, \hat{S}_{r, \alpha, \beta, \gamma}(I)) \leq C \cdot \text{dist}(b, S_{r, \alpha, \beta, \gamma}(J)) \]
for \( b \in W_p^p(J \times X), 1 \leq p \leq \infty \), with support \( b \subseteq I \). The dist is taken with respect to the norm in \( W_p^p(J \times X) \).

Proof. The first case: \( J = (0, 1) \). Let \( Y = W_p^p(J, X) \), \( Z = W_p^p(J \setminus I, X) \), and let the operator restricting functions from \( J \) to \( J \setminus I \) be denoted by \( R, R : Y \to Z \). Moreover, let the extension operator \( S : Z \to Y \)
be defined similarly as in (2.7). Then, $T = SE$: $X \to Y$ is bounded and $T^* = T, \quad \bar{Y} = \ker T = \{ f \in Y: \supp f \subseteq I \}$. If now $E = \cal E_n(J)$, then one checks easily that $TE \subseteq E$ and $\bar{E} = \bar{Y} \cap E = \cal E_n(I)$. An application of Lemma 7.17 completes the proof in the first case.

The second case: $J = (-1, 2)$. Let $S_1$ and $S_2$ be extension operators defined similarly as in (2.7) extending from $(-1, 0)$ to $(-1, 1/2)$, and from $(1/2, 2)$ to $(1, 2)$, respectively. Moreover, let

$$
S_n(t) = \begin{cases} S_1(t) & \text{for } -1 \leq t < 1/2, \\ S_2(t) & \text{for } 1/2 < t \leq 2.
\end{cases}
$$

We find as in the first case that $T = SE$ has similar properties, i.e. $T$ is bounded, $T^* = T$, $\ker T = \bar{Y}$. Now $E = \cal E_{n, i,J}(J)$. It then follows that $TE \subseteq E$. Moreover, $\bar{E} = \bar{Y} \cap E = \cal E_{n, i,J}(J)$. An application of Lemma 7.17 gives for $h \in \bar{E}$

$$
dist(h, \bar{E}) \leq C_1, \quad \text{for } -1 \leq t < 1/2, \leq 2
$$

(7.19)

Introducing $F = \cal E_{n, i,J}(J)$ and $\bar{F} = \cal E_{n, i,J}(J)$ we find $\dist(h, F) \leq \dist(h, \bar{F})$, and this is the trivial part of the inequalities in question. The opposite inequality is proved as follows. We note that $\bar{F} = F \cap \bar{E}$ and that for $2^p \geq 2(r-1) = 2^p - \bar{E}$ where the sum is the algebraic one. Now, for every $e \in E$ we have a unique representation in terms of the $2$-splines corresponding to $J_1(a, r, y)$

$$
e = \sum_{\gamma} a_{\gamma} N_{\gamma}^y, \quad a_{\gamma} \in X.
$$

We now define

$$
P_e = \sum_{\gamma} a_{\gamma} N_{\gamma}^y, \quad Q_e = \sum_{\gamma} a_{\gamma} N_{\gamma}^y,
$$

with $\sigma = \{ \gamma; N^y_{\gamma} \in F \}$. Since $2^p \geq 2^p - \bar{E}$, it follows that for $j \notin \sigma, N^y_{\gamma} \in \bar{E}$.

Thus, $P + Q = 1$ and $PE \subseteq F, QE \subseteq \bar{E}$. Clearly $P$ and $Q$ are projections and, by Corollary 7.10, they are bounded in $\bar{Y}$, and the bounds for their norms depend on $r$ only:

The dist(h, F) can now be estimated as follows. Let $f \in F, \dot{f} \in \bar{E}$. Then $f - \dot{f} \in \bar{E}$ and consequently

$$
f - \dot{f} = P(f - \dot{f}) + Q(f - \dot{f}),
$$

whence we infer

$$
f - P(f - \dot{f}) = \dot{f} + Q(f - \dot{f}).
$$

Now, the left-hand side is in $F$, and the right-hand side is in $\bar{E}$. Thus, $f - P(f - \dot{f}) \in F \cap \bar{E} = \bar{F}$ and for $h \in \bar{E}$

$$
dist(h, \bar{E}) \leq \|h - [f - P(f - \dot{f})]\| \leq \|h - \dot{f}\| + \|P\| \|f - \dot{f}\|
\leq (1 + \|P\|) \|h - \dot{f}\| + \|P\| \|h - \dot{f}\|.
$$

Since $f \in F$ and $\dot{f} \in \bar{E}$ are arbitrary, we get

$$
\text{dist}(f, \bar{F}) \leq C \text{dist}(h, \bar{E} \cup \bar{F})
$$

Moreover, $E \subseteq F$ implies $\dist(h, E) \leq \dist(h, F)$. Combining the last two inequalities with (7.19) we get the desired result.

**Corollary 7.20.** Let $r \geq 2$ be given, and let $J$ be defined as in $I_1$ in (2.38). Then there is $C_1$ such that

$$
dist(h, \cal E_{n, i,J}(J)) \leq C_1 \text{dist}(h, X; 1/2, J)
$$

holds for $h \in \cal W_p(J; X), 1 \leq p \leq \infty$, with $\supp h \subseteq I$, and for $2^r \geq \|f\|^{-1}$.

This corollary follows immediately by Proposition 7.15 and Theorem 7.18.

8. Fundamental estimates for the spline systems. We should keep in mind that in the definition of $\cal W_p(J; X)$ the space $X$ in general depends on the exponent $p$. To avoid any confusion we shall use occasionally in what follows the symbol $X_p$ for $X$. In the space $L_p(\cal I_1, \sigma I)$ we have the natural scalar product

$$
(f, g)_{L_p(\cal I_1, \sigma I)} = \int (f(t), g(t)) dt.
$$

In what follows we assume that $J = (a, b)$ and that the partition $I$ satisfies condition (7.9). Moreover, let $E = (1-r, \ldots, N-1)$. To each $e \in E$ there corresponds the $X$-valued spline space

$$
S_{n,p}(I; X; e) = \{ a: a = \sum_{\gamma} a_{\gamma} N_{\gamma}^y, \gamma \in \sigma \},
$$

with $\sigma = \{ \gamma; N^y_{\gamma} \in F \}$. Clearly, $S_{n,p}(I; X; e) \subseteq S_{n,p}(I; X; e) = S_{n,p}(I; X; e)$, and, by Corollary 7.19, for each $e \in E$ if it is a closed subspace in $L_p(I; X), 1 \leq p \leq \infty$. In particular there is a unique orthogonal with respect to (8.1) projection $P_{n, p}(I; X; e)$ of $L_p(I; X; e)$ onto $S_{n,p}(I; X; e)$. As previously, in the case $X = H$ we simply write $S_{n,p}(I; e)$ for $S_{n,p}(I; X; e)$. In the real-valued case there is a unique biorthogonal system $(N^y_{\gamma}, f \in e)$ in $S_{n,p}(I; e), i.e., such that

$$
(N^y_{\gamma}, N^x_{\gamma}) = \delta_{i,j} \quad \text{for } i, j \in e
$$

and

$$
S_{n,p}(I; e) = \text{span}\{N_{\gamma}^y, f \in e\} = \text{span}\{N_{\gamma}^x, f \in e\}.
$$
For the kernel of the orthogonal projection operator \( P^O_n(\cdot ; e) \) from \( L_p(I) \) onto \( S^O_n(I; X) \) we have the formula

\[
K^O_n(\xi) = \sum_{j=0}^{\infty} \chi_j \otimes \chi_j^T = \sum_{j=0}^{\infty} a_{ij}^X \chi_j \otimes \chi_j^T.
\]

The matrix \( A^X = (a_{ij}^X)_{i,j} \) is the inverse to the Gram matrix \( G^X = (\langle \chi_j, \chi_k \rangle_{L_p(I)})_{i,j} \). For \( p = 2 \) we check directly

\[
P^O_n(f; X, e)(t) = \int K^O_n(\xi, t, e)(s) ds,
\]

where \( f \in L_p(I; X) \). Since the kernel \( K^O_n(\xi) \) is bounded, formula (8.3) makes sense for all \( p: 1 \leq p < \infty \). Thus (8.3) is the definition of a projection \( P^O_n(\cdot ; X, e) \) from \( L_p(I, X) \) onto \( S^O_n(I; X, e) \). Actually by interpolation (application of Hölder inequality we get

\[
\|P^O_n(\cdot ; X, e)\|_{L_p(I, X)} \leq \sup_{t \in I} \|K^O_n(\xi, t, e)\|_{L_p(I, X)}
\]

and it should be mentioned that the symmetry of the kernel \( K^O_n(\xi) \) was used here. Now, properties (7.3), (7.4), formula (8.2) and inequality (8.4) give

\[
\sup_{t \in I} \|K^O_n(\xi, t, e)\|_{L_p(I, X)} \leq r^{-1} \sup_{i < j} \|a_{ij}^X\|_{L_p(I, X)}
\]

PROPOSITION 8.6. Let \( r \geq 1 \) and \( I = [a, b] \) be given, and let \( II \) satisfy (7.9). Then there are constants \( C_0 < \infty \) and \( 0 < \gamma < 1 \) such that

\[
|a_{ij}^X| \leq C_0 (s_{ij}^r - t_i^{-2r} (s_{ij}^r - t_i)^{-1} (s_{ij}^r - t_i)^{-1} (s_{ij}^r - t_i)^{-1})^{\gamma}, \quad i, j \in \mathbb{N}.
\]

This result is due to de Boor [7], Corollary 2, p. 17. The version presented here follows from Corollary 10.10 with \( p = 2 \) and from a slightly modified result of Demko (19), Theorem 2.2. The modification means simply that Demko's theorem can be extended by the same proof to the case of matrices with entries indexed by pairs of elements from a countable metric space. In our case the metric space is the set \( e \) with the natural distance induced from the real line.

Particular cases of Proposition 8.6 were known earlier (c.f. [10] and [21]).

The inequalities (8.4), (8.5) and (8.7) give now

PROPOSITION 8.8. Let \( I = [a, b] \), \( r \geq 1 \), and let \( II \) satisfy (7.9) and (7.11). Then there is a constant \( C_0 \) such that

\[
\|P^O_n(\cdot ; X, e)\|_{L_p(I, X)} \leq C_0 < \infty
\]

holds uniformly in \( p, 1 \leq p < \infty \).
We now define for $-r < k < r, n \geq 2 - r + |k|$

$$f_k^{(n)} = \frac{[D_k^{(n)}]}{[H^{-1}p_n]} \quad \text{for} \quad 0 \leq k < r,$$

$$f_k^{(n)} = \frac{[D_k^{(n)}]}{[H^{-1}p_n]} \quad \text{for} \quad -r \leq k < 0.$$ Integration by parts gives immediately for $|k| < r$

$$\int f_k^{(n)}(s) \, ds = \delta_{k,0} \quad i, j \geq 2 - r + |k|.$$ Defining for $|k| < r, n \geq 2 - r + |k|$

$$K_k^{(n)}(s, t) = \sum_{i=0}^{n-1} f_{s-i}^{(r-1)}(s) f_{t-i}^{(r-1)}(t),$$

we check using (8.9) that

$$P_k^{(n)}(f; X)(t) = \int K_k^{(n)}(s, t) f(s) \, ds$$

is a projection in $L_p(I; X)$.

**Proposition 8.10.** Let $r \geq 1$ be given. Then there are constants $C_r < \infty$ and $g_r, 0 < g_r < 1$, such that

$$|K_k^{(n)}(s, t)| \leq C_r g_r^{n-k+1} \quad s, t \in I,$$

holds for $n \geq 2 - r + |k|$ with $|k| < r$.

Moreover, for $t \in I, -r < k < r$ and $n \geq 2 - r + |k|$ we have

$$|P_k^{(n)}(t)| \leq C_r g_r^{n-k},$$

where $C_r = C_r g_r^{n-k+1} \quad s, t \in I,$

The proof of (8.11) is given in [14], and for the proof of (8.12) we refer to [15] and for a much simpler proof to [12].

Case (ii). Now we do not have ready results. The basis in question has to be constructed. We apply the same technique starting with orthogonal projections. Referring to the notation introduced earlier we recall that now $E = \{1, \ldots, n-1\}, Y = \{Y_{\infty-1}, n-1\}, N = n$ and $P = I_n$. In analogy to the case (i) we introduce for $n \geq 1$ the notation

$$Q_n^0(s; X) = F_{\mu_0}(s; X; e), \quad Q_n^0 = Q_n^0(I; \mathcal{H});$$

$$S_n^0(I; X; e) = S_n^0(I; X; e), \quad S_n^0 = S_n^0(I; 1; e).$$

Now $\dim S_n^0 = n$ and, clearly,

$$S_n^0(I; X; e) = S_{n+1}^0(I; X; e)$$

and $\{Q_n^0(s; X), n \geq 1\}$ is the corresponding family of orthogonal projections. We define the new orthonormal system as follows: $g_0^0 = X^0_0, X^0_{\infty-1}(I), g_0^0 = S_n^0(I; e)$ and $g_k^0$ is orthogonal in $L_p(I)$ to

$$S_{n-1}^0(I; e), \quad |g_k^0 ||_p^0 (I) = 1.$$ Again,

$$Q_n^0(f; X)(t) = \int L_k^0(s, t) f(s) \, ds,$$

where

$$L_k^0 = \sum_{i=1}^n g_i^0(t) g_i^0(s).$$

Moreover, we introduce for $n \geq 1$, $-r < k < r$,

$$g_k^0 = \begin{cases} D_k g_k^0 \quad &0 \leq k < r \quad \text{for} \quad \frac{1}{2} - |k| < \frac{1}{2}, \\
H^{-1} p_n \quad &-r \leq k < 0 \quad \text{for} \quad \frac{1}{2} - |k| < \frac{1}{2}.
\end{cases}$$

It then follows that for $|k| < r$

$$g_k^0, g_{r-1}^0 = \delta_{k,0} \quad i, j \geq 1.$$ Thus the operators

$$Q_n^0(f; X)(t) = \int L_k^0(s, t) f(s) \, ds,$$

with

$$L_k^0 = \sum_{i=1}^n g_i^0(s) g_i^0(t),$$

are projections for $|k| < r$.

**Lemma 8.13.** There are constants $C_r < \infty$ and $g_r, 0 < g_r < 1$, such that for $|k| < r$ and $n \geq 1$

$$|P_k^{(n)}(s; t)| \leq C_r g_r^{n-k+1} \quad s, t \in I.$$

In particular the operators $Q_n^0: L_p(I) \rightarrow L_p(I)$ are bounded uniformly in $n$ and $q, 1 < p < \infty$.

**Proof.** It is sufficient to prove (8.14) for $0 \leq k < r$. The proof goes by induction in $k$. For $k = 0$ (8.14) follows by Proposition 8.6. Suppose now that (8.14) holds for some $h, 0 \leq k < r - 1$. Now, for $f \in S_n^0(I; e)$ by the Markov inequality for algebraic polynomials we obtain

$$|P_k^{(n)}(f; X)(t)| \leq C_n \frac{1}{|I_n|} \int_{I_n} f(s) \, ds, \quad s, t \in I_n,$$

where $I_n = (a_{n-i-1}, a_{n-i})$ and $C_n$ is a constant depending on $r$ only. In particular (8.15) implies

$$|D_k(f; e) | \leq C_n \frac{1}{|I_n|} \int_{I_n} f(s) \, ds, \quad s \in I_n.$$
Applying (8.16) to
\[ f(s) = L_a^{k+1}(s, t), \]
and then using (8.14) we obtain
\[ |D_a L_a^{k+1}(s, t)| \leq C_n g^{(k-1)}(s). \]
If now \( t > s \), then (8.17) gives
\[ |L_a^{k+1}(s, t)| = \int_s^t |D_a L_a^{k+1}(s, u)| du \leq C_n g^{(k-1)}(s). \]
However for \( t < s \) we have by (8.17)
\[ |L_a^{k+1}(s, t)| = \int_s^t |D_a L_a^{k+1}(s, u)| du \leq C_n g^{(k-1)}(s) + |D_a L_a^{k+1}(s)|. \]
Since for \( 0 \leq t \leq s \) we have \( g^{(k)} \leq g^{(k-1)} \), it is therefore sufficient to prove
\[ |D_a L_a^{k+1}(s)| \leq C_n g^{(k-1)}. \]
In what follows we denote by \( N_{a,j}^{(k)} \) the \( j \)th B-spline corresponding to the dyadic partition \( D_a \). The biorthogonal functions in \( S_a(I; \varepsilon) \) are denoted as \( N_{a,j}^{(k)}(s, t) = 0, \ldots, -1 \). It is convenient to introduce also the operator
\[ G^k(s) = \int_s^{s+1} f(t) dt. \]
Now,
\[ L_a^{k+1} = D^k \sum_{j=0}^{n-1} (1, H^{(j)}(a), H^{(j)}(b)) \]
\[ = D^k \sum_{j=0}^{n-1} (G^k, N_{a,j}^{(k)}(a), N_{a,j}^{(k)}(b)) \]
Moreover, let
\[ G^k = \sum_{j=0}^{n-1} b_j N_{a,j}^{(k)}. \]
Since \( (G^k)^{(k)}(s) = s^{k-1}N_a^{(k)} \) and \( G^{k+1} \in S_a(I) \), it follows that the \( b_j \)'s are uniquely determined. For later convenience we introduce
\[ g = \sum_{j=0}^{n-1} b_j N_{a,j}^{(k)}. \]
According to (8.20) we have
\[ (G^k)^{(k)}(s) = \sum_{j=0}^{n-1} b_j N_{a,j}^{(k)} = \sum_{j=0}^{n-1} (g, N_{a,j}^{(k)}). \]
and therefore for \( j \in s \)
\[ b_j = (G^k, N_{a,j}^{(k)}) - (g, N_{a,j}^{(k)}). \]
Using (8.22) and (8.19) we get
\[ D^k N_{a,j}^{(k)} = D^k \sum_{j=0}^{n-1} b_j N_{a,j}^{(k)} + \sum_{j=0}^{n-1} (g, N_{a,j}^{(k)}) \]
\[ = D^k \sum_{j=0}^{n-1} (g, N_{a,j}^{(k)}) N_{a,j}^{(k)} - g. \]
Notice that \( s_{n-r+1} < \langle 0, s_{n-r+1} \rangle \).
Moreover, we are going to show that
\[ \|G^k\|_{\infty} = \|G^k\|_{\infty}(I) \leq C_n g^{(k-1)}, \quad k \geq 1. \]
In order to see this we define \( H' = \{ a \} \) for \( j = 0, \pm 1, \pm 2, \ldots \) as follows
\[ a_{r-1} = \left\{ \begin{array}{ll}
0, & \text{for } j = 0, \\
\pm 1, & \text{for } j = r, \ldots, 2r - 2, \\
\text{arbitrary increasing} & \text{for } j \geq 2r - 1.
\end{array} \right. \]
The corresponding B-splines are denoted by \( N_{r-1}' \). Clearly, \( N_{a,j}^{(k)} = N_{r-1}' \) for \( j \leq -1 \), and for \( 0 < s < s_{n-r+1} \) we have
\[ (G^k)^{(k)}(s) = \sum_{j=-1}^{-r} b_j N_{r-1}'(s). \]
An application of Corollary 7.10 gives
\[ n^{-k} \sim \|G^k\|_{\infty}(0, s_{n-r+1}) \sim \max_{1 \leq r < r_1} |b_r|, \quad \max_{1 \leq r < r_1} |b_r| \leq \| G^k \|_{\infty}(0, s_{n-r+1}). \]
However, on \( 0 < s < s_{n-r+1} \) we have, by (8.20),
\[ G^k = \sum_{j=0}^{n-1} b_j N_{a,j}^{(k)} \]
and, moreover,
\[ G^k = \sum_{j=0}^{n-1} b_j N_{a,j}^{(k)} + \sum_{j=0}^{n-1} b_j N_{r-1}'(s). \]
The properties of the B-splines imply now \( b_j = b_j' \) for \( j = 1 - r, \ldots, -1 \), whence we infer (8.24). Using (8.24) we shall estimate the right-hand side of (8.23). It follows by the definition that

\[
\begin{equation}
N_{2,1}^{(j)}(t) = \sum_{k=0}^{n} a_{n-k} N_{k,1}^{(j)}(t).
\end{equation}
\]

Proposition 8.6 implies inequality

\[
|\phi_{n-k}^{(j)}| \leq C_n n^{-2-\eta}, \quad i, j \in e,
\]

which together with (8.24) gives

\[
|N_{2,1}^{(j)}(t)| \leq C_n n^{-2-\eta}, \quad j \in e.
\]

This and the properties of the B-splines imply

\[
|D^{k+1} \sum_{j=0}^{\infty} (g_j N_{k,1}^{(j)}(u) N_{2,1}^{(j)}(u))| \leq C_n n^{-2-\eta}.
\]

Finally, \( g \) by definition is a spline with support in \( \langle 0, s_{n-r-1} \rangle \), whence, by (7.14),

\[
|D^{k+1} g(u)| \leq C_n n^r \quad \text{for} \quad 0 < s < 1,
\]

and this gives

\[
|D^{k+1} g(u)| \leq C_n n^{-2-\eta}, \quad s \in I.
\]

The combination of these inequalities and (8.23) give (8.18), and the proof is complete.

**Lemma 8.27.** Let \( r \geq 2 \) be given and let \( t_n = s_{n-r-1} \) for \( n = 2^r + r \), \( 1 < r < 2^r \). Then there exists \( C_r < \infty \), such that

\[
|g_{r-1}^0(t)| \leq C_n n^{-2+2\eta} g_{n-r-1}^0(t)
\]

holds for \(-r < u < t, t \in I, n > 1\).

**Proof.** The function \( g_{r-1}^0 \) is orthogonal to \( S_{n-1}^0(I; e) \) and therefore

\[
g_{r-1}^0 \left( \sum_{j=0}^{\infty} (g_j N_{r-1}^{(j)}(t) N_{n-r-1}^{(j)})
\right.
\]

\[
= \sum_{j=0}^{\infty} (g_j N_{r-1}^{(j)}(t) N_{n-r-1}^{(j)}(t))
\]

where \( e = \{0, \ldots, n-1\} \). However, (7.3) and (7.6) give

\[
|g_j N_{r-1}^{(j)}(t)| \leq \|N_{r-1}^{(j)}(t)\| \leq C_n n^{-2-\eta}.
\]

Thus,

\[
|g_{r-1}^0(t)| \leq C_n n^{-2+2\eta} \max_{2^{r-1} < s < 2^r} |N_{n-r-1}^{(j)}(t)|.
\]

Now, (8.25) and (8.26) imply

\[
|N_{n-r-1}^{(j)}(t)| \leq C_n n^2 n^{-2-\eta}, \quad j \in e, t \in I,
\]

and this proves

\[
|g_{r-1}^0(t)| \leq C_n n^{2+2\eta} g_{n-r-1}^0(t).
\]

Inequality (8.25) now follows for \( 0 < u < r \) from (8.26) by (8.16).

In the proof of the remaining case of (8.26) we use the representation

\[
(u - t)^{-k-1} = \sum_{j=1-r}^{n-2} b_j(t) N_{n-r-1}^{(j)}(u).
\]

It now follows by the very definition of \( g_{r-1}^0 \) that

\[
\int (u - t)^{-k-1} g_{r-1}^0(u) \, du = \sum_{j=1-r}^{n-2} b_j(t) N_{n-r-1}^{(j)}(u)
\]

whence, by (8.20), we obtain

\[
|\int (u - t)^{-k-1} g_{r-1}^0(u) \, du| \leq C_n n^{-2+2\eta} \sum_{j=1-r}^{n-2} |b_j(t)|.
\]

To estimate the right-hand side we introduce a new partition \( \Pi' = (s_{n-r-1}^0, \bar{t} = 0, 1, \ldots) \) as follows

\[
\phi_{n-r-1} = \begin{cases}
0 & \text{for } \bar{t} < s < r
\end{cases}
\]

arbitrary increasing for \( j \geq r - 2 - 1 \).

Moreover, for \( 0 < u < s_{n-r-1}^0 \) let

\[
(u - t)^{-k-1} = \sum_{j=1-r}^{n-2} b_j(t) N_{n-r-1}^{(j)}(u),
\]

where \( N_{n-r-1}^{(j)} \) are the B-splines of order \( r \) corresponding to \( \Pi' \).

It now follows by Corollary 7.10 that uniformly in \( t \in I \) and \( n > 1 \)

\[
\sum_{j=1-r}^{n-2} \int (u - t)^{-k-1} \, du \sim n^{-1} \sum_{j=1-r}^{n-2} |b_j(t)|.
\]

However, \( N_{n-r-1}^{(j)} \) for \( j = 1 - r, \ldots, 1 \), and comparing for \( 0 < u < s_{n-r-1}^0 \), the two representations for \( (u - t)^{-k-1} \) we find that \( b_j(t) = b_j(t) \) for \( j = 1 - r, \ldots, 1 \). Consequently,

\[
\sum_{j=1-r}^{n-2} |b_j(t)| \leq n^{-1} \sum_{j=1-r}^{n-2} |b_j(t)| \leq C_r (t + n^{-1})^{-k-1},
\]
and

\[(8.31) \quad \left| \int (u-t)^{-k-1} y_n^0(u) \, du \right| \leq C_n u_n^{-13} a_n (t + n^{-1})^{-k-1}. \]

If now \( t > t_n \), then (8.30) implies (8.28) for \( -r \leq k < 0 \). On the other hand, if \( 0 < t < t_n = s_{n+1} \), then by (8.30) and (8.31)

\[
|y_n^{r,k}(t)| = \left| \int (u-t)^{-k-1} y_n^0(u) \, du \right| \\
\leq C_n \left| \int (u-t)^{-k-1} y_n^{r,k}(u) \, du \right| + C_n \left| \int (u-t)^{-k-1} y_n^0(u) \, du \right| \\
\leq C_n [n^{2r} + \alpha_n n^{13} (t + n^{-1})^{-k-1} y_n] \\
\leq C_n [n^{2r} + \alpha_n n^{13}] + 0 < y_n < \tilde{y}_n < 1
\]

and this completes the proof.

**Lemma 8.32 (Jackson type inequality).** Let \(-r < k < r - 1, 1 \leq p \leq \infty\). Then there is a constant \( C \) such that

\[(8.33) \quad \left\| \frac{Q_n^{r,k} f - f}{n} \right\|_p \leq C \left\| D \left( f - Q_n^{r,k} f \right) \right\|_p, \quad n \geq 1,
\]

holds for \( f \in W_1^r(I) \) with \( f(0) = 0 \) if \( k \geq 0 \), and for \( f \in W_1^r(I) \) with \( f(1) = 0 \) if \( k < 0 \).

**Proof.** The proof is based on the idea of G. Freud and V. Popov in [23]. For the proof let

\[ S_n^{r,k} = \text{span} \{ y_n^{r,k}, j = 1, \ldots, n \}. \]

It follows, by Lemma 8.13, that with some \( C \), \( 0 < C < \infty \)

\[
\left\| \frac{Q_n^{r,k} f - f}{n} \right\|_p \leq C \left\| f - h \right\|_p, \quad n \geq 1,
\]

holds for \( f \in L_p(I) \) and \( h \in S_n^{r,k} \).

In order to get (8.33) it is important to construct a suitable \( h \). This is done below. We let

\[
g = D (f - Q_n^{r,k} f), \quad a_j = \int_{s_{j+1}}^{s_{j+1}} g, \quad I_{n,j} = \langle s_{n-j}, s_{n+1} \rangle,
\]

\[
b_j = M_{n,j} a_j, \quad G = \int_{s_j}^{s_{j+1}} g, \quad H = \int_{s_j}^{s_{j+1}} h,
\]

where \( M_{n,j} \) is the corresponding to the \( n \)th partition, B-spline normalized in \( L_1 \)(cf. (7.5)).
Moreover, for $|k| < r$ we have

$$(\lambda_n^{(k)}, \lambda_n^{(k)}) = \delta_{kn}, \quad i, j \geq n.$$  

A direct proof of these formulas is omitted.

**Proposition 8.35.** Let $|k| < r$. Then there are constants $C_k < \infty$, $0 < q < 1$, depending on $r$ only such that

$$|M_n^{(k)}(x, t)| \leq C_k n^{-q(r-k)}, \quad n \geq 1, \quad s, t \in I.$$  

In particular, the projections

$$H_0^1(I) \rightarrow L_p(I), \quad 1 \leq p \leq \infty,$$

are bounded uniformly in $n$ and $p$.

This proposition follows from (8.11) and (8.14).

**Proposition 8.36.** (Bernstein and Jackson type inequalities). Let $-r < k < r - 1$, $1 \leq p \leq \infty$. Then, for $f \in L(I)$ we have

$$(8.37) \quad \|DH_n^{(k)}f\|_p(I) \leq C_k n^{\alpha - k(p - 1)}\|f\|_p(I), \quad n \geq 1.$$  

Moreover, for $f \in H^m(I)$ with $f \in W^m_p(I)$, we get

$$(8.38) \quad \|f - H_n^{(k)}f\|_p(I) \leq C_k n^{m + \alpha} \|f - H_n^{(k)}f\|_p(I), \quad n \geq 1.$$  

In both inequalities $C_k$ depends on $r$ only.

Inequality (8.37) follows from (7.34); (8.38) is a consequence of Lemma 8.33 and of Lemma 4.8 in [17].

**Proposition 8.39.** Let $-r \leq k < r$. There are constants $C_{k,n}, 0 < q < 1$, such that

$$|a_n^{(k)}(t)| \leq C_k n^{-1/2 - q\gamma} \|a_n^{(k)}\|_N^{(k)}, \quad n \geq 1, \quad t \in I.$$  

This proposition follows by (8.28) and (8.12).

Applying similar arguments as in the proof of Theorem 7.1 in [12] and using the last proposition we obtain:

**Proposition 8.40.** Let $r \geq 1$ be given. There is $C_{k,n} < \infty$ such that, if $1 \leq p \leq \infty$, $N = 2^m$, $0 \leq \gamma \leq 1, \ldots, \gamma = n$, then

$$\|\sum_{n=1}^{2^N} |a_n^{(k)}(t)| \|_N(I) \leq C_k N^{\gamma(r - 1/2)} \|a_n^{(k)}\|_N, \quad n \geq 1,$$

holds for $-r \leq k < r$ and

$$N^{\gamma(r - 1/2)} \|a_n^{(k)}\|_N \leq C_k \|\sum_{n=1}^{2^N} |a_n^{(k)}(t)| \|_N(I)$$  

holds for $|k| < r$.

**Lemma 8.41.** Let $|k| < r$ and let $J, m$ be defined as before. Then for some $C_{k,n} < \infty$

$$(8.42) \quad \|f - H_n^{(k)}f\|_p(I) \leq C_{k,n} m \|f\|_{L^2(I)}, \quad n \geq 1,$$

holds for $f \in L^p(I)$, $1 \leq p \leq \infty$, $n \geq 1$.

Proof. It follows by Proposition 8.35 that it is sufficient to prove (8.42) for $n = 2^m, \mu \geq 0$. In the rest of the proof we assume therefore that $n$ is of such form. We now distinguish four cases. At first let $(k_n^{(k)}) = (f_n^{(k)})$ and let $0 \leq k < r$. In this case inequality (8.42) follows by Theorem 4.1 of [12]. In case $-r < k < 0$ we define

$$\hat{B}_{k,n} = \text{span}\{(f_n^{(k)})_{n \geq n_0\ldots n_m)}}$$

where the order of the splines is $r - k$ and the multiplicities at $0, 1$ are $a = r, \beta = r$, respectively. To obtain (8.42) it is now sufficient to apply Corollary 7.50 and Proposition 8.35. The last proposition is used to pass from the best approximation to $(f - H_n^{(k)}f)_{L^2(I)}$.

In the case $(k_n^{(k)}) = (f_n^{(k)})$ and $0 \leq k < r$ the argument is similar and it is omitted. It remains to prove (8.42) in the case of the system $(g_n^{(k)})$ for $-r < k < 0$. To this case there corresponds $J = (0, 2), \gamma = (0, 1)$. For the given $f \in L^2(I)$ we use $f$ to define the Steklov mean; here $f$ is the extension by zero of $f$ to $[0, \infty)$. The definition is as follows: for $h > 0, t \in I$

$$g(t) = \sum_{k=1}^{\infty} (t^{k+1})^p / \int_0^1 \int_0^1 f(t + \rho \gamma(t + \rho + s) \rho \gamma) \rho \gamma = 0$$

It then follows that $g \in W^m_p(I)$, $g \gamma = 0$, and that for the restriction $g_r$ of $g$ to $I$ we have

$$(8.43) \quad \|f - g_r\|_p(I) \leq C_{k,n} m \|f\|_{L^2(I)}.$$  

On the other hand Proposition 8.35 gives

$$(8.43) \quad \|f - g_r\|_p(I) \leq C_{k,n} m \|f - H_n^{(k)}g_r\|_{L^2(I)}$$

and Propositions 8.34 and 8.36 imply

$$(8.43) \quad \|g_r - H_n^{(k)}g_r\|_p(I) \leq C_{k,n} m \|g_r - H_n^{(k)}g_r\|_{L^2(I)}$$

with $g_r := D^m g_r \in m = - k$. However, Proposition 8.35 implies

$$(8.43) \quad \|g_r - H_n^{(k)}g_r\|_p(I) \leq C_{k,n} m \|g_r - H_n^{(k)}g_r\|_{L^2(I)}.$$

Thus, the combination of these inequalities with (8.43), $k = n^{-1}$, gives (8.42), and this completes the proof.

**Remark 8.44.** It should be clear that all the results on the operators $H_n^{(k)}$, $|k| < r$, can be extended directly to the $X_r$-valued case.
The last lemma of this section is needed in the next section in the proof of the unconditionality of the systems \((h_k^{(N)}), |k| < r\). The next lemma is preliminary to the last one. We need some more notation. For a linear operator \(T : L_2 \to L_2\) we denote by \(T^*\) its Hilbert space adjoint. For \(-r \leq k < r\), \(r \in \mathbb{C}\) and \(f \in C(I)\) we define
\[
A_{\mu}^{(r)} f = \sum_{N=1}^{\infty} \left( \int \mu h_{k}^{(N-r-1)}(1 \cdot) - h_{k}^{(N-r-1)}(0_1) \right) h_{k}^{(N)}.
\]
In particular, for \(|k| < r\) we have the relation
\[
A_{\mu}^{(r)} = H^{(r)} - H^{(r)}.
\]

**Lemma 8.45.** Let \(-r < k < r, N = 2^\mu, \mu = 0, 1, \ldots\). Then for some \(C_r < \infty\)
\[
\|A_{\mu}^{(r)}1_k\|_{L_2} \leq C_r N^{-1/2}
\]

**Proof.** Notice that \(-r \leq k < r\) implies \(-r \leq -k - 1 < r\). Thus by Proposition 8.40
\[
\|A_{\mu}^{(r)}1_k\|_{L_2} \leq \sup_{N \in \mathbb{N}} \|A_{\mu}^{(r)}1_k\|_{L_2} \leq C_r N^{-1/2}.
\]

**Lemma 8.46.** Let \(|k| < r\). Then \((A_{\mu}^{(r)})^* = A_{\mu}^{(r)}\) and for some \(C_r < \infty\) we have for \(f \in L_2(I), \mu > 0, k > 0\)
\[
\|A_{\mu}^{(r)}f_{k}\|_{L_2} \leq C_r 2^{r-k-1} f_{k}(I).
\]

**Proof.** The case \(k = 0\) is trivial since \((A_{\mu}^{(0)})^* = (A_{\mu}^{(0)}), \mu = 0, 1, \ldots\) is an orthogonal family of orthogonal projections in \(L_2(I)\). For later use let \(N = 2^\mu, M = 2^\nu\) and \(g = d_{\mu}^{(r)}\). By duality we reduce the proof to the case where \(\mu \leq \nu\) \((M \leq N)\), and this inequality is assumed whenever \(-r+1 < k < r\). Each of the following three cases will be treated separately: \(0 < k < r\), \(-r+1 < k < 0\), and \(k = -r+1\).

**Case:** \(0 < k < r\). Using Bernstein’s inequality (7.14) and Proposition 8.34 we obtain
\[
\|A_{\mu}^{(r)}, A_{\mu}^{(r-k)}g\|_{L_2} = \|A_{\mu}^{(r)}g\|_{L_2}
\]
\[
\leq C_r M \|A_{\mu}^{(r-k)}g\|_{L_2} \leq C_r M \|A_{\mu}^{(r-k)}1_k\|_{L_2}.
\]

Applying now Lemma 8.45 we get
\[
\|A_{\mu}^{(r-k)}g\|_{L_2} = \|A_{\mu}^{(r-k)}1_k\|_{O(M^{-1/2})}
\]

\[
= \|f_{\mu} A_{\mu}^{(r-k)}1_k\|_{O(M^{-1/2})} = \|f_{\mu} 1_k\|_{O(M^{-1/2})}.
\]

Moreover, Proposition 8.35 implies
\[
\|A_{\mu}^{(r-k)}(g - (g, 1))\|_{L_2} \leq C_r \|g - (g, 1)\|_{L_2}.
\]

**Hence,**
\[
(g, 1) - g = \sum_{N=1}^{\infty} (f_{\mu} h_{k}^{(N)}) h_{k}^{(r-k)}
\]

and therefore by Propositions 8.40 and 8.35
\[
\|g - (g, 1)\|_{L_2} \leq C_r \|A_{\mu}^{(r-k)}g\|_{L_2}.
\]

Combination of all these inequalities completes the proof.

**Case:** \(-r+1 < k < 0\). The Bernstein’s inequality implies (8.47)
\[
\|A_{\mu}^{(r-k)}g\|_{L_2} = \|D_{\mu}^{(r-k)}g\|_{L_2} \leq C_r \|D_{\mu}^{(r-k)}g\|_{L_2}.
\]

For the latter function we write the following identity (8.48)
\[
H A_{\mu}^{(r-k)}g = A_{\mu}^{(r-k)}(Hg - (g, 1)) f_{\mu}
\]

\[
= A_{\mu}^{(r-k)}(Hg - \lambda) + (g, 1) f_{\mu} + \lambda A_{\mu}^{(r-k)}1_k
\]

where
\[
F_{\mu} = \sum_{N=1}^{\infty} h_{k}^{(N)}(0) h_{k}^{(r-k)},
\]

\[
\lambda = \sum_{N=1}^{\infty} (f_{\mu} h_{k}^{(N)}) h_{k}^{(r-k)}(1).
\]

**Now,**
\[
\lambda - Hg = \sum_{N=1}^{\infty} (f_{\mu} h_{k}^{(N)}) h_{k}^{(r-k)}
\]

and therefore by Propositions 8.35 and 8.40
\[
\|A_{\mu}^{(r-k)}(Hg - \lambda) f_{\mu}\|_{L_2} \leq C_r \|f_{\mu}\|_{L_2}.
\]

Next, Lemma 8.45 gives
\[
\|g, 1)\|_{L_2} = \|f_{\mu} A_{\mu}^{(r-k)}1_k\|_{L_2} \leq C_r \|f_{\mu}\|_{L_2},
\]

and Proposition 8.40 implies
\[
\|\lambda f_{\mu}\|_{L_2} \leq C_r \|h_{k}^{(N-r)}(0)\|_{L_2} \leq C_r M^{-1/2}.
\]

Thus, we have established the desired inequality for all \(k, r, \mu\).
and
\[ |\lambda| \leq C_N N^{-1/2} \| f \|_2. \]
Moreover, Lemma 8.45 gives
\[ \| A_{n,2}^{s-2} f \|_2 \leq C_M M^{-1/2}. \]
Combining (8.47)–(8.53) we obtain the desired inequality.

Case: \( k = -r-1 \). Let \( m = -k = r = -1 \), \( h = A_{r-1}^{s-2} f \) and let \( k_j \) be its extension by zero from \( J = (0, 1) \) to \( J = (0, 2) \). The functions \( k_j \) are splines of order 1 corresponding to the uniform dyadic partition with step \((2N)^{-1}\). By a duality argument it is sufficient to prove our inequality for \( \mu \geq r \) (\( M \geq N \)). Now, by Lemma 8.41,
\[ \| A_{n,2}^{s-2} A_{r-1}^{s-2} f \|_2 = \| A_{n,2}^{s-2} f \|_2, \]
and the improved version of (7.14) (cf. [22], Lemma 9.3) gives
\[ \| A_{n,2}^{s-2} f \|_2 \leq C_N N^{1/2} M^{-1/2} \| f \|_2, \]
and the inequality (9.3) follows.

This completes the proof in the last case.


For fixed integer \( d \geq 1 \) let \( Q = I^d \), \( I = (0, 1) \), and let \( Z, Z' \) be two complementary boundary sets as defined in Section 2 (cf. (2.37) and (2.47)).

Our aim in this section is to construct Schauder bases in the Sobolev spaces \( W^{p+1}_p(I)^{10} \), \( m \geq 0, 1 \leq p < \infty \). The bases we are constructing are tensor products of one-dimensional bases. Therefore we start with the definition of the one-dimensional bases.

**Definition 9.1.** Let the integer \( m = r - 2, r \geq 2, \) and the set \( Z \subseteq \partial I \) be given. Moreover, let \( n(Z) = \{ 0 \}, \{ 1 \} \) and \( n(Z \setminus \partial I) = 2 - r \) for \( Z = \emptyset, \{ 0, 1 \} \). The basic functions are now defined for \( n \geq n(Z) \) as follows
\[ F_n^{(r-2)}(t) = \begin{cases} f_n(t) & \text{if } Z_n = \emptyset, \\ f_n(2t-1) & \text{if } Z_n = \{ 0, 1 \}, \\ f_n(t) & \text{if } Z_n = \{ 0 \}, \\ f_n(2t-1) & \text{if } Z_n = \{ 1 \}. \end{cases} \]

It should be now clear that for \( n \geq n(Z) \)
\[ F_n^{(r-2)}(t) \in W^{p+1}_p(I)^{10} \quad \text{for } 1 \leq p < \infty \]
and
\[ F_n^{(r-2)}(t) \in W^p_p(I)^{10} \quad \text{for } 1 \leq p < \infty. \]

The very definition of the functions \( F_n^{(r-2)} \) implies for \( i, j \geq n(Z) = n(Z) \),
\[ (f_n^{(r-2)}; Z) h_n^{(l; r)} \| h_n^{(l; r)} \|_{(l; r)} = 0. \]
It is sufficient to check that $T_\epsilon$ is of strong type $(2,2)$ and of weak type $(1,1)$ and then to apply Marcinkiewicz’s interpolation theorem.

The strong type $(2,2)$ of $T_\epsilon$ can be proved as follows. Lemma 8.46 says, if the hypotheses of Collatz’s lemma (cf. [26], pp. 102–103) are satisfied. Thus in the notation of Section 8 we have

$$\|f\|_{L_2}^2 (T) \sim \|\mathcal{F}^\epsilon\|_{L_2}^2 (T) + \sum_{\nu = 0}^\infty \|\mathcal{D}^\epsilon\|_{L_2}^2 (T),$$

whence by Khinchin’s inequalities

$$\|f\|_{L_2}^2 (T) \sim \|\mathcal{F}^\epsilon\|_{L_2}^2 (T) + \sum_{\nu = 0}^\infty \|\mathcal{D}^\epsilon\|_{L_2}^2 (T).$$

Now, Proposition 8.80 gives

$$\|D^\epsilon\|_{L_2}^2 (T) \sim \sum_{k = 1}^\infty \|\mathcal{F}(f_k h^{\nu - k}) h^{\nu} \|_{L_2}^2 (T).$$

Consequently,

$$\|f\|_{L_2}^2 (T) \sim \|\mathcal{F}^\epsilon\|_{L_2}^2 (T).$$

The weak type $(1,1)$ of $T_\epsilon$ can be proved with the help of Proposition 8.39 exactly in the same way as it was proved in (13) that this property holds in case $(h^{\nu}) = (f_k^{\nu})$. This completes the proof of the lemma.

We are now ready to pass to the $d$-dimensional case. Let $Z$ be given as in (2.37). For given multi-indices $k = (k_1, \ldots, k_d)$, $0 \leq k_i < m + 1$, $m = (m_1, \ldots, m_d)$, $n = n(Z; k)$,

$$F_{m_i}^{(k)}(Z) = F_{m_i}^{(k)}(Z_1) \otimes \cdots \otimes F_{m_d}^{(k)}(Z_d),$$

$$F_{m_i}^{(0)} = F_{m_i}^{(0)}(Z).$$

It now follows by (9.5) and (2.47) that

$$F_{m_i}^{(0)}(Z) = F_{m_i}^{(0)}(Z_1) ; F_{m_i}^{(k)}(Z_2) = F_{m_i}^{(k)}(Z_1) \otimes \cdots \otimes F_{m_d}^{(k)}(Z_d),$$

where $l_i \geq n(Z; k_i)$, i.e. $j_i \geq n(Z; k_i)$ for $j = 1, \ldots, d$. It follows from (9.10) that the following operation is a projection: for $f \in L_1 (Q)$

$$U_{m_i}^{(k)}(f; Z) = \sum_{k_1, \ldots, k_d} \{ f_{i_1} E_{i_1}^{(k_1)} ; \cdots ; f_{i_d} E_{i_d}^{(k_d)} \} X_{m_i}^{(k)}(Z).$$

Clearly, for $f = f_1 \otimes \cdots \otimes f_d$

$$U_{m_i}^{(k)}(f; Z) = \{ f_{i_1} E_{i_1}^{(k_1)} ; \cdots ; f_{i_d} E_{i_d}^{(k_d)} \} X_{m_i}^{(k)}(Z).$$

We use later on the identification: $U_{m_i}^{(k)} = U_{n_i}^{(k)}$. It is important that Proposition 9.7 in view of (9.11) and (9.12) extends to

**Proposition 9.13.** If $0 \leq k_i < m + 1$ and $f \in W^1_2 (Q; Z)$, then

$$L^\epsilon U_{m_i}^{(k)}(f; Z) = U_{m_i}^{(k)}(L^\epsilon f; Z).$$

The ordering of multi-indexed sequences of functions which was used in (14) is called rectangular.

**Lemma 9.14.** Let $Z$ be given as in (2.37) and let $0 \leq k \leq m + 1, \ldots, n + 1$. Then $(F_{m_i}^{(k)}(Z), n \geq n(Z; k))$ in the rectangular ordering, is a Schauder basis in $L_2 (Q)$ if $1 \leq p < \infty$, and in $\sum_{k = 0}^m W^1_2 (Q; Z)$ if $k \leq m + 1, \ldots, n + 1$. Then $(F_{m_i}^{(k)}(Z), n \geq n(Z; k))$ is an unconditional Schauder basis in $L_2 (Q)$ if $1 \leq p < \infty$.

This lemma follows from Lemma 9.8 by an argument similar to the one applied in (14).

**Lemma 9.15.** Let $Z$ be given as in (2.37) and let $0 \leq k \leq m + 1, \ldots, n + 1$. Then $(F_{m_i}^{(k)}(Z), n \geq n(Z; k))$ is an unconditional Schauder basis in $L_2 (Q)$ if $1 \leq p < \infty$.

To obtain this result we use Lemma 9.9 and a result of McCarty [31] on products of uniformly bounded commuting Boolean algebras of projections.

**Theorem 9.16.** Let $m \geq 0$ and let $Z$ be given as in (2.37). Then $(F_{m_i}^{(k)}(Z), n \geq n(Z))$ in its rectangular ordering is for each $k, 0 \leq k \leq m$, a Schauder basis in $W^1_2 (Q; Z)$ with $1 \leq p < \infty$.

The proof of this theorem is a direct consequence of Lemma 9.14 and Proposition 9.13. From Lemma 9.15 and Proposition 9.13 follows

**Theorem 9.17.** Let $m \geq 0$ and let $Z$ be given as in (2.37). Then $(F_{m_i}^{(k)}(Z), n \geq n(Z))$ is for each $k, 0 \leq k \leq m + 1$, an unconditional Schauder basis in $W^1_2 (Q)$ with $1 \leq p < \infty$.

Using the notation of Section 2 we now state the fundamental result on orders of approximation.

**Theorem 9.18.** Let $m \geq 0, \mu \geq 0, N = 2\mu, m = (N, \ldots, N)$, and let $Z$ be given as in (2.37). Then for some constant $C = C(m, d)$ we have for $f \in L_2 (Q)$, $1 \leq p < \infty$,

$$\|f - U_{m_i}^{(n)}(f; Z)\|_{p(Z)} = C_{N, \mu} \{ \sum_{k = 0}^m |f_k| \} \{ \sum_{k = 0}^m |f_k| \} \{ \sum_{k = 0}^m |f_k| \},$$

where $f_k$ is the extension of $f$ by zero to $Q_2$.

**Proof.** For given $i = 1, \ldots, d$ we define the following projection

$$U_{m_i}^{(n)}(f; Z) = \sum_{k = 0}^m \{ f_{i_1} E_{i_1}^{(k)} ; \cdots ; f_{i_d} E_{i_d}^{(k)} \} X_{m_i}^{(k)}(Z).$$

where the $E_i$’s are copies of the identity operator acting in $L_2 (Z)$. Then it follows by a telescoping argument (cf. [17], proof of Lemma 5.9) that

$$\|f - U_{m_i}^{(n)}(f; Z)\|_{p(Z)} \leq \sum_{k = 0}^m \|f_k - U_{m_i}^{(n)}(f_k; Z)\|_{p(Z)}.$$
Theorem 9.20. Let \( m \geq 0, \mu \geq 0, N = 2^n, n = (N, \ldots, N) \), \( 0 \leq k \leq k + 1 \leq m \), and let \( Z \) be given as in (23.07). Moreover, let

\[
U_{p, q, f} = U_{p, q}^n(f; Z) .
\]

Then there is a constant \( C = C(m, d) \) such that for \( 1 \leq p \leq \infty \) and \( f \in W^k_p(Q) \), we have Bernstein's and Jackson's inequalities

\[
\begin{align*}
\|U_{p, q, f}^{p, q, k+1}(Q)\| & \leq C N^{k+1} \|U_{p, q, f}^{p, q, k}(Q)\|, \\
\|f - U_{p, q, f}^{p, q, k+1}(Q)\| & \leq C N^{-k} \|f - U_{p, q, f}^{p, q, k}(Q)\|.
\end{align*}
\]

Proof. This result was established for \( Z_j = 0, j = 1, \ldots, d \) and for \( Z_j = (0, 1), j = 1, \ldots, d, \) in [17], Theorem 5.16. Since we have (9.10), Propositions 8.36, 9.13 and Theorem 9.18, similar arguments can be used to establish the theorem in the remaining cases of \( Z \).

Remark 9.23. It is important to realize that we have the following formula for the adjoint operator (in the sense of the Hilbert space \( L_2(Q) \))

\[
U_{p, q}^n(\cdot; Z)^* = U_{p, q}^n(\cdot; Z^*).
\]

10. Spline bases in Besov spaces on cubes with boundary conditions.

Let us start with some obvious conclusions which can be drawn from the results we have already established.

Proposition 10.1. Let \( Q = I^d \) and \( Z \) be given as in (23.07), let \( m \) be an integer. Moreover, let \( 0 < s < m, 1 \leq p, q \leq \infty \). Then \( \{F_{p, q}^n(\cdot; Z)\}, n \geq n(Z) \), in the rectangular ordering, is a basis in \( B^s_{p, q}(Q) \).

Proof. Proposition 2.50 (cf. also Remark 2.51) characterizes \( B^s_{p, q}(Q) \) as an interpolation space between \( W^{s}_{p}(Q) \) and \( W^{s}_{q}(Q) \). Since, by Theorem 9.16, the sequence \( \{F_{p, q}^n(\cdot; Z)\} \) is a Schauder basis in the latter spaces, the proposition follows from the fundamental properties of interpolation spaces (cf. e.g. [3], Theorem 3.1.3).

Proposition 10.2. Let the hypothesis of Proposition 10.1 be satisfied. Then for \( 1 \leq p \leq \infty \) the system \( \{F_{p, q}^n(\cdot; Z), n \geq n(Z)\} \) is an unconditional basis in \( B^s_{p, q}(Q) \).

Proof. Apply similar argument as in the previous proof using Theorem 9.17 instead of 9.16.

The space \( B^s_{p, q}(Q) \), for arbitrary \( Z \) and for any choice of the parameters \( s > 0, 1 \leq p, q \leq \infty \), has an unconditional basis.

We are going to construct such a basis in two steps. The first step gives an unconditional Schauder decomposition into finite-dimensional subspaces, providing at the same time a new equivalent norm in \( B^s_{p, q}(Q) \). In the second step, a recipe is given for choosing in each of the finite-dimensional subspaces a suitable basis so that they will provide a linear isomorphism between \( B^s_{p, q}(Q) \) and certain sequence spaces.

Definition 10.3. For given \( Z, m \geq 0, N = 2^n, n = (N, \ldots, N) \), let

\[
V_{p, q, f} = U_{p, q, f}^n(f; Z) - U_{p, q, f}^n(f; Z), \quad n = (N, \ldots, N) \]

Then \( V_{p, q, f} = U_{p, q, f}^n(f; Z) \), \( 1 = (1, \ldots, 1) \).

Theorem 10.4. Let \( Z \) be given as in (23.07) and let \( 1 \leq p, q \leq \infty, 0 < s < m \). Then \( B^s_{p, q}(Q) \) has an equivalent norm

\[
\|f\|_{B^s_{p, q}(Q)} = \left( \sum_{j=0}^{\infty} \left( \frac{2^n}{\lambda^n} V_{s, p, q}^{n}(f; Q) \right)^2 \right)^{1/2}.
\]

Proof. Theorem 9.18 implies that for some constant \( C = C(m, d) \) we have

\[
\|V_{s, p, q}^{n}(f; Q)\| \leq C \omega_{n, p, q}(f; N^{-1})_{p, q},
\]

whence we infer

\[
\|f\|_{B^s_{p, q}(Q)} \leq C(s, m, d)\|f\|_{B^s_{p, q}(Q)}.
\]

The opposite inequality can be proved as follows. In Section 2 we have defined

\[
Q = \bigcap_{i=1}^{d} I_{i} \quad \text{with} \quad I_{i} = I_{Z_{i}}.
\]

Let now for fixed \( j, 1 \leq j \leq d \),

\[
Q_j = \Delta_{i=1}^{d} I_{i} \quad \text{with} \quad I_{i} = I_{Z_{i}}.
\]

Our aim is to estimate from above

\[
\omega_{n, p, q}(f; N^{-1})_{p, q} = \omega_{n, p, q}(f; Z, X, N^{-1})_{p, q},
\]

where \( f_{Z, X} \) has the obvious meaning. It follows by Proposition 7.16 that

\[
\omega_{n, p, q}(f; N^{-1})_{p, q} \leq C N^{-s} \left( \|f_{Z, X}\|_{p, q} + \sum_{k=0}^{N-1} \|f_{k+1, X}\|_{p, q} \right),
\]

where

\[
E_{k}(f) = \inf \{ \|f_{Z, X} - g_{p, q}(f; I_{j}, X)\|_{p, q} : g \in S_{k}(I_{j}, X) \}
\]

and \( S_{k}(I_{j}, X) \) is the space of \( X \)-valued splines on \( I_{j} \) corresponding to the \( k \)th dyadic partition such that it contains (see below)

\[
\text{span} \{ \left( \sum_{i=0}^{b} f_{i}^{p, q} \right)(\cdot; Z)_{i} f_{j} \in L_{p, q}(Q) \}.
\]

It then follows that (\( n = (N, \ldots, N) \))

\[
E_{2}(f) \leq C \|f - U_{p, q, f}^n(f; Z)\|_{p, q}(Q).
\]
Applying (9.19) we find that
\[ \sum_{j=1}^{d} \| f \|_{V_{pq}^{(d)}(Q)}^{q} \leq C N^{-m} \sum_{j=1}^{d} 2^{m\sigma} \| f \|_{L_{pq}^{\sigma}}(Q). \]
This implies that
\[ \sum_{j=1}^{d} \| f \|_{V_{pq}^{(d)}(Q)}^{q} \leq C \| f \|_{L_{pq}^{\sigma}}^{q}(Q), \]
whence by Theorem 2.28 the proof is complete.

**Corollary 10.6.** For arbitrary boundary conditions \( Z \) we have the following decomposition (cf. Definition 10.3)
\[ f = \sum_{i=1}^{d} V_{ij} f \]
in \( W_{pq}^{m}(Q) \) and \( B_{pq}^{m}(Q) \) for \( 1 \leq p \leq \infty \), \( 1 \leq q \leq \infty \), \( 0 < s < m \). It is unconditional in the Sobolev space for \( 1 < p < \infty \), and in the Besov space for \( 1 < p, q \leq \infty \), \( 0 < s < m \).

This follows from Theorems 9.1 and 10.4.

We now pass to the construction of suitable bases in the finite dimensional subspaces. Let us start with a decomposition of the index set \( N(Z) \)
\[ N(Z) = N(Z_{1}) \times \cdots \times N(Z_{d}), \]
where \( N(Z_{i}) = N_{i} + 1, \ldots, n_{i}, n(Z) \) is given as in Definition 9.1, and \( N(Z) \) is the set of all non-negative integers.

For given \( \mu \geq 1, \mu \in N(Z), \theta \neq \in \subseteq D = \{1, \ldots, d\} \) define
\[ N_{\mu} = \bigcup_{\theta \neq \in \subseteq D} N_{\mu \theta}, \]
\[ N_{\theta} = \{ n \in N(Z); n_{i} \leq 1 \} \text{ for } i \in D \} \] Clearly, all the sets are disjoint and
\[ (10.7) \quad N(Z) = \bigcup_{\mu=1}^{\infty} N_{\mu} = N_{\theta} \cup \bigcup_{\theta \neq \in \subseteq D} N_{\mu \theta}. \]

To each of the components in (10.7) there corresponds a finite dimensional multivariate spline space:
\[ S_{\mu}^{(d)}(Z) = \text{span} \{ P_{\mu}^{(d)}(Z); n \in N_{\mu} \}, \]
\[ S_{\mu \theta}^{(d)}(Z) = \text{span} \{ P_{\mu \theta}^{(d)}(Z); n \in N_{\mu \theta} \}. \]

In \( S_{\mu}^{(d)}(Z) \) the basis will be left as it is.

In \( S_{\mu \theta}^{(d)}(Z) \) there will be constructed a tensor product basis in such a way that its coefficient space will be, uniformly in \( \mu \) and \( 1 \leq p \leq \infty \), linearly isomorphic to \( L_{p}^{\theta} k = \dim S_{\mu \theta}^{(d)}(Z) \). Moreover, the dual basis will have the same property with respect to \( S_{\mu \theta}^{(d)}(Z) \).

In what follows we consider a uniform partition \( II(\mu) \) with step \( 2^{-n} \) and with multiplicities one at all knots but 0 and 1 where the multiplicities are assumed to be 3r and 2r, respectively, i.e., \( II(\mu) = (t_{\mu})_{j} \)
\[ t_{\mu j} = \begin{cases} (i+3r-1)2^{-n} & \text{for } i \leq -3r, \\ 0 & \text{for } -3r < i \leq 0, \\ 2^{-n} & \text{for } 0 < i < 2r, \\ 1 & \text{for } 2r \leq i < 2r + 2r, \\ (i-2r+1)2^{-n} & \text{for } i \geq 2r + 2r. \end{cases} \]
The B-spline corresponding to the support \( \langle t_{\mu 0}, t_{\mu j + r} \rangle \) is denoted by \( N_{\mu j}^{(d)}(t_{\mu 0}, \ldots, t_{\mu j + r}) \). The function \( N_{\mu j}^{(d)}(t_{\mu 0}, \ldots, t_{\mu j + r}) \) is well defined for \( r, j \) satisfying the condition \( t_{\mu 0} < t_{\mu j + r}. \) With such understanding of the B-splines we can state and prove easily:

**Proposition 10.8.** Let \( H = j, \mu \geq 1, r \geq 2 \). Then we have
\[ \begin{align*}
&\text{span} \{ f^{j \mu \theta}, j = 2r, \ldots, 2^r \} = \text{span} \{ N_{\mu j}^{(d)}(t_{\mu 0}, \ldots, t_{\mu j + r}) \}, \\
&\text{span} \{ f^{j \mu \theta}, j = 2r, \ldots, 2^r \} = \text{span} \{ N_{\mu j}^{(d)}(t_{\mu 0}, \ldots, t_{\mu j + r}) \}, \\
&\text{span} \{ f^{j \mu \theta}, j = 2r, \ldots, 2^r \} = \text{span} \{ N_{\mu j}^{(d)}(t_{\mu 0}, \ldots, t_{\mu j + r}) \}, \\
&\text{span} \{ f^{j \mu \theta}, j = 2r, \ldots, 2^r \} = \text{span} \{ N_{\mu j}^{(d)}(t_{\mu 0}, \ldots, t_{\mu j + r}) \},
\end{align*} \]
(10.9)
where
\[ N_{\mu j}^{(d)}(t_{\mu 0}, t_{\mu j + r}) = \begin{cases} 2^{r \mu} H^{N_{\mu j}^{(d)}} & \text{for } j = 0, \ldots, r - 1, \\
N_{\mu j + r}^{(d)} & \text{for } j = r, \ldots, 2^r - 1. \end{cases} \]

**Proposition 10.10.** There is a constant \( C_{r} \) such that for \( \mu \geq 1 \) and \( 1 \leq p \leq \infty \)
\[ C_{r}^{-1} 2^{-np} \leq \| N_{\mu j}^{(d)}(I) \|_{L_{p}^{\theta}} \leq C_{r} 2^{-np}, \]
\[ j = 0, \ldots, 2^r - 1. \]

Moreover,
\[ \| a \|_{L_{p}^{\theta}} \| \leq \sum_{j=0}^{2^r-1} \| a_s N_{\mu j}^{(d)}(I) \| \leq C_{r} \| a \|_{L_{p}^{\theta}}, \]
where \( \| a \|_{L_{p}^{\theta}} \) is the norm in \( U_{p}^{\theta} \) and
\[ \| N_{\mu j}^{(d)}(I) \|_{L_{p}^{\theta}} = \| N_{\mu j}^{(d)}(I) \|_{L_{p}^{\theta}}. \]

**Proof.** The first two inequalities for \( r \leq j \leq 2^r \) follow by (7.3) and (7.6). Now, \( \text{supp} H^{N_{\mu j}^{(d)}} \leq \text{sup} \| N_{\mu j}^{(d)} \|_{L_{p}^{\theta}} \leq (0, (5r-1)2^{-n}) \) for \( j = 0, \ldots, r - 1, \) and
\[ \| H^{N_{\mu j}^{(d)}} \|_{L_{p}^{\theta}} \leq C_{r} 2^{-np}. \]
Thus,
\[ \|S_{j}^{(n)}\|_{p}(I) \leq C_{r}2^{-\nu p}, \quad j = 0, \ldots, r-1. \]

The opposite inequality follows from Bernstein’s inequality
\[ 2^{m}|\partial^{m}S_{j}^{(n)}|_{p}(I) \geq \|\partial^{m}S_{j}^{(n)}\|_{p}(I) = |\partial^{m}S_{j}^{(n)}|_{p}(I) \geq C_{r}2^{-\nu p}. \]

The right-hand side of (10.11) can be proved in a similar way as in the case of the B-splines (as in Corollary 7.10). The left-hand side can be proved as follows. We notice that there are coefficients \( b_{j-1}, \ldots, b_{-1} \) such that on \( X = \langle 0, 1 \rangle \)
\[ -\sum_{j=-1}^{r} b_{j}N_{j,\nu}^{(n)} = \sum_{j=0}^{r-1} a_{j}S_{j}^{(n)} \]

where \( N_{j,\nu}^{(n)} \) is defined as in Theorem 7.8. Thus by the theorem just quoted
\[ \left\| \sum_{j=0}^{r-1} a_{j}S_{j}^{(n)} \right\|_{p}(I) = \left\| \sum_{j=0}^{r-1} b_{j}N_{j,\nu}^{(n)} + \sum_{j=0}^{r-1} a_{j}N_{j,\nu}^{(n)} \right\|_{p}(I) \geq C_{r} \left( \sum_{j=0}^{r-1} |b_{j}|^{p} + \sum_{j=0}^{r-1} |a_{j}|^{p} \right)^{1/2} \]
\[ \geq C_{r} \left( \sum_{j=0}^{r-1} |b_{j}|^{p} \right)^{1/2} + \sum_{j=0}^{r-1} |a_{j}|^{p} \].

However, by Bernstein’s inequality we obtain
\[ \left\| \sum_{j=0}^{r-1} b_{j}N_{j,\nu}^{(n)} \right\|_{p}(I) = \left\| \sum_{j=0}^{r-1} a_{j}S_{j}^{(n)} \right\|_{p}(I) \]
\[ = 2^{n\nu} \sum_{j=0}^{r-1} \left( |b_{j}|^{p} \right) |N_{j,\nu}^{(n)}(I)| \geq C_{r} \sum_{j=0}^{r-1} |a_{j}|^{p}, \]
and this completes the proof of (10.11).

**Lemma 10.12.** Let \( B_{k} = (b_{ij}^{(k)}) \), \( k = 0, 1 \), be the Gram matrices with
\[ b_{ij}^{(0)} = (N_{j,\nu}^{(n)}, N_{i,\nu}^{(n)}), \quad i, j = 1 - r, \ldots, 2^{n} - 1, \]
\[ b_{ij}^{(1)} = (S_{j}^{(n)}, S_{i}^{(n)}), \quad i, j = 0, \ldots, 2^{n} - 1. \]

Let \( A_{k} = B_{k}^{-1} \), \( k = 0, 1 \). Then there are constants \( C_{2} \) and \( g_{r} \), \( 0 < g_{r} < 1 \), such that
\[ |a_{ij}^{(k)}| \leq C_{2}g_{r}^{-\nu} \quad \text{for} \quad i, j = (1 - k)(1 - r), \ldots, 2^{n} - 1, \]
where \( k = 0, 1. \)

**Proof.** We are going to prove the case \( k = 1 \) only. The other case is quite similar and the proof is omitted. Since (cf. Section 8) for \( n = 2^{n} \)
\[ L_{2}^{0}(t, i) = \sum_{j=1}^{2^{n}-1} a_{ij}^{(1)}S_{j}^{(n)}(t, \lambda), \quad 0 < g_{r} < 1, \]
it follows that
\[ a_{ij}^{(1)} = \int_{I} \left( \int_{I} L_{2}^{0}(t, i)S_{j}^{(n)}(t, \lambda) \right) ds \, dt. \]

Here, \( (S_{j}^{(n)}) \) and \( (N_{j}^{(n)}) \) are the dual bases to \( (N_{j}^{(n)}) \) and \( (N_{j}^{(n)}) \), respectively. Using Corollary 7.10, (10.11) and Proposition 8.6 (suitably adapted in case of \( N_{j}^{(n)} \)) we obtain
\[ |S_{j}^{(n)}(t)| \leq C_{r}g_{r}^{-\nu}, \quad 0 < g_{r} < 1, \]
\[ |N_{j}^{(n)}(t)| \leq C_{r}g_{r}^{-\nu}. \]
These inequalities, Lemma 8.13 and (10.13) give the desired result.

**Definition 10.14.**
\[ (N_{j}^{(n)}) \quad \text{and} \quad (S_{j}^{(n)}) \]
\[ j = 0, \ldots, 2^{n} - 1. \]

**Lemma 10.15.** The following relations hold
\[ (N_{j}^{(n)}) \quad \text{and} \quad (S_{j}^{(n)}) \]
\[ j = 0, \ldots, 2^{n} - 1. \]

Moreover, if \( (f_{i}) \) is one of the systems \( (N_{j}^{(n)}) \), \( i = 0, \ldots, 2^{n} - 1 \)
\[ (N_{i}^{(n)}), \quad i = 1 - r, \ldots, 2^{n} - 1 \]
\[ (N_{j}^{(n)}), \quad j = 0, \ldots, 2^{n} - 1, \] or \( (S_{j}^{(n)}) \), \( i = 0, \ldots, 2^{n} - 1 \)
then for some constant \( C_{3} \) we have
\[ \|\sigma_{i}\|_{H^{-k,1}_{r}} \leq C_{3}\|\sigma_{i}\|_{L^{p}_{r}} \]
where \( l = \dim\text{span}(f_{i}). \) In addition to this we have
\[ (N_{i}^{(n)})_{L^{p}_{r}} \sim (N_{j}^{(n)})_{L^{p}_{r}} \sim 2^{nk}, \quad 1/p - 1/\alpha = 1. \]

**Proof.** The duality relations follow by Definition 10.14, Lemma 10.12 and Definition 10.14 imply for \( k = 0, 1, n = 2^{n} \)
\[ |S_{j}^{(n)}(t)| \leq C_{r}g_{r}^{-\nu}, \quad 0 < g_{r} < 1. \]
This, the duality relations and the properties of the B-splines imply the remaining statements of the lemma.

**Definition 10.16.** Let
\[ f_{n,j}^{(2r)} = N_{n-j}^{(2r)} \text{ for } j = 2, 3, \ldots, 2^r; \]
\[ f_{n,j}^{(2r)} = N_{n-j}^{(2r)} \text{ for } j = 2, 3, \ldots, 2^r; \]
\[ g_{n,j}^{(2r)} = N_{n-j}^{(2r)} \text{ for } j = 1, 2, \ldots; \]
\[ g_{n,j}^{(2r)} = N_{n-j}^{(2r)} \text{ for } j = 1, 2, \ldots; \]

**Definition 10.17.** The function \( F_{n,0}^{(n)} \) for \( n \geq 1 \) is defined by the formula as given in Definition 9.1 if we replace formally \( f_{n,j} \) by \( F_{n,0}^{(n)} \) and \( g_{n,j} \) by \( F_{n,0}^{(n)} \) respectively. Moreover, let \( F_{n,0}^{(n)} = F_{n,0}^{(n)} \) for \( n \geq 1 \) and \( n \geq 1 \).

We are now ready to define our new basis.

**Definition 10.18.** The new system for given \( Z \) is defined according to the decomposition (10.7) as follows
\[ G_{n,m}^{(n)}(Z) = F_{n,m}^{(n)}(Z) \text{ for } n \in N_n, \]
\[ G_{n,m}^{(n)}(Z) = \bigoplus_{m \neq n} F_{n,m}^{(n)}(Z) \bigoplus \bigoplus_{m \neq n} N_{n,m}^{(n)}(Z) \]
for \( n \in N_{n,m}, \) \( \lambda \geq 1, \) and \( n \neq 0 \in D. \)

**Theorem 10.19.** For given \( m \geq 0 \) and \( Z \) we have
\[ G_{n,m}^{(n)}(Z) = \bigoplus_{m \neq n} N_{n,m}^{(n)}(Z), \]
where \( n \in N(\lambda) \). Moreover, for \( m \geq 0, \)
\[ \text{span}[G_{n,m}^{(n)}(Z); n \in N(\lambda)] = \text{span}[F_{n,m}^{(n)}(Z); n \in N(\lambda)] \]
and for \( m \geq 1, f \in L_2(\mathbb{Q}) \)
\[ V_f = \sum_{m \in \mathbb{N}_0} \left\{ f G_{n,m}^{(n)}(Z) \right\} G_{n,m}^{(n)}(Z), \]
where \( V_f \) is given as in Definition 10.3.

Finally, for some constant \( C = C(m, d) \)
\[ \left\| a \right\|_{C^{-1}} \leq 2^{n(d+1)/p} \left\| \sum_{m \in \mathbb{N}_0} a G_{n,m}^{(n)}(Z) \right\|_{L_2(\mathbb{Q})} \leq C \left\| a \right\|_{C^{-1}}, \]
where \( \lambda = \text{cardinality of } N_n. \)

**Proof.** Property (10.20) follows immediately from Definitions 10.18, 10.17, 10.16 and Lemma 10.15. Let us now introduce
\[ S_n(Z) = \text{span}[F_{n,m}^{(n)}(Z); n \geq n \leq 2^n], \]
\[ A_n(Z) = \text{span}[F_{n,m}^{(n)}(Z); 2^n-1 < n \leq 2^n]. \]

It then follows (see the definition of Proposition (10.17)) that
\[ S_{n,R}(Z) = \bigoplus_{n \geq n} A_n(Z) \bigoplus S_{n,R}(Z). \]

Proposition 8.40 gives with \( \lambda = 2^{n-1} \)
\[ \left\| \sum_{n \in \mathbb{N}_0} a F_{n,m}^{(n)}(Z) \right\|_{L_2(\mathbb{Q})} \leq \left( \sum_{n = 2^{-1}}^{2^n-1} \left\| a \right\|_{C^{-1}} \right)^{1/2}. \]

and Lemma 10.15 implies with \( \lambda = 2^{d-1} - n(Z) \)
\[ \left\| \sum_{n \in \mathbb{N}_0} a F_{n,m}^{(n)}(Z) \right\|_{L_2(\mathbb{Q})} \leq \left( \sum_{n = 2^{-1}}^{2^n-1} \left\| a \right\|_{C^{-1}} \right)^{1/2}. \]

Applying a standard lemma on products of operators in \( L_2 \) spaces we find
\[ \left\| \sum_{n \in \mathbb{N}_0} a G_{n,m}^{(n)}(Z) \right\|_{L_2(\mathbb{Q})} \leq \left( \sum_{n = 2^{-1}}^{2^n-1} \left\| a \right\|_{C^{-1}} \right)^{1/2}. \]

where \( \lambda = \text{dim } S_{n,R}(Z). \) Since
\[ N_n = \bigcup_{\lambda \in \mathbb{N}_0} N_{n,\lambda}, \]
it follows that (10.23) is satisfied. Introducing
\[ S_{n,R}(Z) = \bigoplus_{\lambda \in \mathbb{N}_0} S_{n,R}(Z), \]
we find that
\[ S_{n,R}(Z) = S_{n,R}(Z) \bigoplus S_{n,R}(Z) \]
whence (10.21) follows, and this completes the proof.

**Corollary 10.24.** The system \( \{ G_{n,m}^{(n)}(Z); n \geq n(Z) \} \) is an unconditional Schauder basis in \( B_{p}^{2}(\mathbb{Q}), 0 < s < m, 1 < p, q \leq \infty \). Moreover, if
\[ f = \sum_{n \in \mathbb{N}_0} a G_{n,m}^{(n)}(Z), \]
then letting \( s = \frac{d}{d+1} - 1/p \)
\[ \left\| F_{n,m}^{(n)}(Z) \right\|_{B_{p}^{2}(\mathbb{Q})} \leq \frac{1}{2} \left( \sum_{n = 2^{-1}}^{2^n-1} \left\| a \right\|_{C^{-1}} \right)^{1/2}. \]

Remark. The system \( \{ G_{n,m}^{(n)}(Z); n \geq n(Z) \} \) in suitable ordering is a Schauder basis in \( W_{p}^{2}(\mathbb{Q}) \) for \( 1 < p, q \leq \infty. \)

**11. Final comments.** Let us complete the proof of Theorems A and B formulated in the introduction to Part I. Recall that \( M \) is a compact \( d \)-dimensional \( C^\infty \) manifold and \( n \geq 1 \) is a fixed integer. The sequence \( (f_{n})_{n=1}^{\infty} \in C^\infty(M) \) referred to in Theorem A can be obtained as follows.
Let $Q_1, \ldots, Q_N$ be the decomposition of $M$ into non-overlapping $d$-cubes constructed in Section 3. Let $T$ be the isomorphism constructed in Theorem 4.9, i.e.

\[ T : \mathcal{F}(M) \rightarrow \bigoplus_{i \leq N} \mathcal{F}(Q_i)_{x_i} \]

for $\mathcal{F} = W^p_0, 0 \leq k \leq m, 1 \leq p \leq \infty$.

Recall that, if we fix isomorphisms $\phi_i : I^d = Q_0$ for $i = 1, \ldots, N$, then each summand $\mathcal{F}(Q_i)_{x_i}$ is in a natural way identified with the space $\mathcal{F}(I^d)_{\phi_i^{-1}(x_i)}$. Consider, for $i = 1, \ldots, N$, the system

\[ \mathcal{B}_i \subseteq C^m(Q_i)_{x_i} \subseteq T(C^m(M)) \]

which corresponds in this identification to the system

\[ \{ p^{m}(\cdot), \phi_i^{-1}(Z_i) : n \geq n(\phi_i^{-1}(Z_i)) \} \subseteq C^m(I^d)_{\phi_i^{-1}(x_i)} \]

constructed in Section 9 (cf. Theorems 9.16 and 9.17).

The union $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_N$ can be ordered as $(h_{x_i})_{x_i} = \{ \xi \}$, so that $(h_{x_i})_{x_i}$ is the enumeration of $\mathcal{B}_i$ which corresponds to the so-called rectangular ordering of $\mathcal{B}_i$ (as defined in [14]). Put $T_i = T(h_{x_i}) \in C^m(M)$ for $i = 1, 2, \ldots$ Then $(A1)$ follows from Theorems 9.16 and 4.9 and $(A2)$ follows from Theorems 9.17 and 4.9.

Recall that we have fixed a smooth measure $\mu$ on $M$ (cf. Section 4). In particular, on each $Q_i$ one has $dx = \eta_i dx$ for some positive function $\eta_i \in C^m(Q_i)$, where $dx$ is the measure transported from the Lebesgue measure on $I^d$ by means of $\phi_i$. Since $(f_{x_i})_{x_i}$ is, in particular, a Schauder basis in $L_{\mu}(M, d\mu)$, there is a unique $\mu$-biorthogonal sequence $(\eta_{x_i})_{x_i}$ which consists of elements of $L_{\mu}(M, d\mu)$. A different description of this sequence will show that $(A3)$, $(A1')$ and $(A2')$ are satisfied.

By Theorem 4.9 the operator $T_i$ (which is the inverse to $T^*$) induces linear isomorphisms

\[ T_i : \mathcal{F}(M) \rightarrow \bigoplus_{i \leq N} \mathcal{F}(Q_i)_{x_i} \]

for $\mathcal{F} = W^p_0, 0 \leq k \leq m, 1 \leq p \leq \infty$.

For $i = 1, \ldots, N$ the system

\[ \mathcal{B}_i \subseteq C^m(Q_i)_{x_i} \subseteq T_i(C^m(M)) \]

which corresponds to the system

\[ \{ p^{m}(\cdot), \phi_i^{-1}(Z_i) : n \geq n(\phi_i^{-1}(Z_i)) \} \subseteq C^m(I^d)_{\phi_i^{-1}(x_i)} \]

is, by (9.10), $dx$-biorthogonal to $\mathcal{B}_i$.

By Theorem 9.16, the system $\mathcal{B}_i$ forms (in the rectangular ordering) a basis in $\mathcal{F}(Q_i)_{x_i}$ which is unconditional if $1 < p < \infty$ by Theorem 9.17.

It follows that the system

\[ g_i \mathcal{B}_i = \{ g_i \eta_i f_i : f_i \in \mathcal{B}_i \} \]

is $\mu$-biorthogonal to $\mathcal{B}_i$, since the multiplication by $g_i$ defines an automorphism of each space $\mathcal{F}(Q_i)_{x_i}$. This system remains a Schauder basis in $W^p_0(Q_i)_{x_i}$. Finally, since $V_n^{-1} = T^*$, the set

\[ V_n^{-1}(\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_N) \subseteq T(C^m(M)) \]

is $\mu$-biorthogonal to $(f_{x_i})_{x_i}$ and hence, suitably ordered, it coincides with $(g_{x_i})_{x_i}$. Therefore, conditions (A3), (A1'), (A2') are satisfied. Thanks to Theorems 4.9, 9.16 and 9.17. This completes the proof of Theorem A.

As we already mentioned in the introduction the sequence $(f_{x_i})_{x_i}$ is actually a basis in $W^p_0(M)$ for $-m \leq k \leq m, 1 \leq p \leq \infty$ and is an unconditional basis if $1 < p < \infty$. Analogously $(g_{x_i})_{x_i}$ is a basis in $W^p_0(M)$ for $-m \leq k \leq m, 1 < p < \infty$ and is an unconditional basis if $1 < p < \infty$.

The construction of the basis from Theorem B is similar. Instead of the $\mathcal{B}_i$'s one uses the systems $\mathcal{B}_i, i = 1, \ldots, N$ corresponding to

\[ \{ p^{m}(\cdot), \phi_i^{-1}(Z_i) : n \geq n(\phi_i^{-1}(Z_i)) \} \subseteq C^m(I^d)_{\phi_i^{-1}(x_i)} \]

Also Theorems 9.16 and 9.17 are replaced by Corollary 10.24 and the subsequent remark.

One checks first that (B1) and (B2) are satisfied for $0 < s < m$. This is not difficult but rather tedious. It may be more convenient to order the union of the $\mathcal{B}_i$'s a bit differently. Recall that each $\mathcal{B}_i$ is split into blocks $X_{i1}, X_{i2}, \ldots$ with Card$(X_i)$ of the order $2^{d}$. Now we require that the $j$th block of $\mathcal{B}_i$ precede the $(j+1)$st block of $\mathcal{B}_{i+k}$, for $i = 1, \ldots, N - 1$ and the $j$th block of $\mathcal{B}_N$ precede the $(j+1)$st block of $\mathcal{B}_1$. This new ordering will not spoil the properties of the sequence as a basis in $W^p_0(M)$.

Using Corollary 10.24 and Corollary 4.31 we obtain, for $0 < s < m$, formulas similar to (B1) and (B2) with $W^s_0(M)$ replaced by another, explicitly defined, sequence space. It is not difficult to check that the latter space is equal as a set to $W^s_0(M)$, the respective norms being equivalent. We omit this verification.

To complete the proof of Theorem B we need some facts concerning duality and interpolation between the Besov spaces on $M$ (cf. [17]).

Realizing (B2) for $0 < s < m$, we obtain that (B1) holds also for $0 > s > -m$. Moreover, since

\[ B^p_{s+1}(M) = \{ B^{s+1}_{s+1}(M), B^{s+1}_{s+1}(M) \}_{1/2} \]

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and (cf. [3], Theorems 5.0.1 and 6.4.2)

\[ b_{k+1}^{s-1} = b_{k+1}^{s-1} + b_{k+1}^{s-1} \]

we obtain that (B1) is true for \( s = 0 \) as well.

The proof of (B2) for \(-m < s < m\) is analogous.

It should be rather clear that the constant \( C \) in (B1) and (B2) can be taken uniformly bounded for \(|s| \leq m - \varepsilon\), if \( \varepsilon > 0 \) is fixed.

The bases constructed in Theorems A and B satisfy inequalities of Bernstein and Jackson type, i.e. one has

**Corollary 1.12.** There exists \( C < \infty \) such that if \( S_p, N = 2^m \), denotes the \( N^{-1} \) projection operator with respect to \( \{f_{n}^{\infty}_{m}\} \) (or to \( \{g_{n}^{\infty}_{m}\} \) corresponding to \( U_N \) in Theorem 9.20, then

\[ \|S_{p}f\|_{L^{p}(\Omega)} \leq C N^{1}, \quad \text{whenever} \quad -m < k < k + 1 \leq m. \]

Proof. The special case where \( l = 0 \) follows from Theorem A (resp. Theorem B).

Since \( T \) in (11.1) is an isomorphism for \( S = W_{p}^{s}(M), -m < k \leq m, 1 \leq p \leq \infty \), and the basis in \( S(M) \) is composed from bases in the summands in the way described above, it will suffice to verify the analogous statement for the bases in \( S(\Omega) \), for \( i = 1, \ldots, N \).

This, however, has been done in Theorem 9.20 at least in the case \( 0 \leq k \leq k + 1 \leq m \). The verification in the case \(-m \leq k < k + 1 \leq m\) is reduced (by a duality argument) to the case \( 0 \leq k < k + 1 \leq m \) (with \( k' = -k' \leq k' + 1 \leq m \)), since the \( S_{k'} \)'s are isomorphic.

\[ \|S_{k}f\|_{L^{p}(\Omega)} \leq C \|S_{k'}f\|_{L^{p}(\Omega)} \]  

and a similar one for \( I - S_{k} \) (cf. [17]). This completes the proof.

Theorem B simplifies considerably the study of the embedding maps between Besov spaces on \( \mathcal{M} \) by reducing them to diagonal maps between the sequence spaces \( b_{k}^{s} \). In the sequel we always put

\[ \varepsilon = r - 1/p + 1/2, \quad \sigma = r - 1/q + 1/2. \]

Observe first that the inclusion

(11.5)

\[ b_{k}^{s} \subseteq b_{k}^{s} \]

is equivalent to \( r > s, \varepsilon > \sigma \) if \( u > v \) and to \( r > s, \varepsilon > \sigma \) if \( u \geq v \). This follows easily from Theorem B. Recall also that

(11.6)

\[ b_{k}^{s} \subseteq b_{k}^{s} \subseteq b_{k}^{s} \]

(cf. e.g. [17]). (If \( k \) is not an integer, for \( b_{k}^{s} \) one can take the complex interpolation space.)

Let \( \mathcal{M} \) be a quasi-normed operator ideal in the sense of [47]. One can ask whether the embedding map (11.5) belongs to \( \mathcal{M} \). By Theorem B this is equivalent to \( j \in \mathcal{M} \), where \( j \) is the embedding \( b_{k}^{s} \rightarrow b_{k}^{s} \). Clearly, the latter is equivalent to the condition

\[ (D_{1} : b_{k}^{s} \rightarrow b_{k}^{s} \in \mathcal{M}), \]

where \( \lambda = \sigma - \sigma \) and \( D_{1} \) is the diagonal operator (\( D_{1}(a_{m}) = (\lambda a_{m}) \)). It is easy to see that there is a \( \lambda_{*} \in \mathcal{R} \) such that (11.7) holds if \( \lambda > \lambda_{*} \) and (11.7) does not hold if \( \lambda < \lambda_{*} \). The number \( \lambda_{*} \) depends on \( \mathcal{M}, p, q \) but does not depend on either \( u \) or \( v \). This follows from the inclusions

\[ b_{k}^{s} \subseteq b_{k}^{s} \subseteq b_{k}^{s} \subseteq b_{k}^{s}, \]

for \( s > 0, 1 \leq \lambda \leq \infty \). Taking \( u = p, v = q \), we have \( b_{k}^{s} \subseteq b_{k}^{s} \subseteq b_{k}^{s} \subseteq b_{k}^{s} \), hence \( \lambda_{*} \) coincides with the number \( \lambda_{*}(\mathcal{M}, p, q) \) which is the limit order of the ideal \( \mathcal{M} \) (cf. [47]). This generalizes the result of H. König [44] (formulated for Sobolev spaces and \( r \gg s > 0 \)). (This follows from (11.6).)

Limit orders for many important ideals are computed in [47]. We shall discuss briefly the case where \( \mathcal{M} = \mathcal{B}_{0} \) is the ideal of \( t \)-absolutely summable operators (see [47]).

A sufficient condition for (11.5) to belong to \( \mathcal{B}_{0} \) is that \( D_{1} \) in (11.7) admits a factorization of the type

\[ b_{k}^{s} \rightarrow b_{k}^{s} \rightarrow b_{k}^{s} \]

For this it suffices that \( a + b > \lambda, \alpha > 1, \beta > 1, \gamma > 0 \). Numbers \( \alpha, \beta, \gamma \) with these properties exist iff \( \lambda > 1/\gamma \) and \( t > 1/\lambda \). Hence we have obtained

**Corollary 1.11.** If \( r - \sigma > \delta, \gamma \), then the embedding (11.5) is \( t \)-absolutely summable for \( t > 1/\lambda \), where

\[ \lambda = \sigma - \gamma \]

This corollary, the inclusions (11.6) and the well-known relationship between \( t \)-absolutely summable and \( t \)-Radonifying maps (cf. [47]) imply, e.g., the following result concerning random fields on the manifold \( \mathcal{M} \) (abstract Wiener measures).

**Corollary 1.12.** Let \( \eta \) be a cylindrical Gaussian probability on the space \( \mathcal{B}_{k}^{s}(M) \) (or \( W_{p}^{s}(M) \)). If \( u < r - \delta, \gamma \), then \( \eta \) can be uniquely extended to a Radon probability measure on \( \mathcal{B}_{k}^{s}(M) \).

We pass to applications concerning the distribution of the eigenvalues of operators acting in Besov spaces. In the results we shall formulate \( \lambda_{*}(\mathcal{S}) \) will denote the sequence of non-zero eigenvalues of the (bounded linear) operator \( S_{i} : X \rightarrow X \). (They are counted according to their algebraic multiplicities and are ordered so that \( \lambda_{*}(\mathcal{S}) \) decreases as \( n \) tends to \( \infty \).)
First we shall extend a result of B. Carl [37] who considered the case 
\(d = 1, \ p \leq q\).

**Corollary 11.10.** Let \(1 \leq p, q \leq \infty, 0 < r \leq \frac{1}{2} \) and 
\((r-s) > \frac{1}{p} - \frac{1}{p-1} - \frac{1}{q-1}\). Let \(M\) be a compact \(d\)-dimensional \(C^\infty\) manifold and let \(S : B_{p,s}^r(M) \to B_{p,r}^s(M)\) be a bounded linear operator such that \(S(B_{p,s}^r(M)) \subseteq B_{p,r}^s(M)\). Then 
\[\|S\| = d_p(S) = O(n^{-\alpha - \beta})\].

**Proof.** Clearly, without loss of generality we may assume that \(p = q\).

Carl's proof for this case can be carried with some obvious changes. Indeed, now we know by Theorem B that the entropy numbers of the embedding 
\(B_{p,s}^r(M) \to B_{p,r}^s(M)\) decay at the same rate as those of the embedding 
\(B_{p,s}^r(B_{p,r}^s(I)) \to B_{p,r}^s(B_{p,r}^s(I))\).

A stronger result was obtained earlier by A. Pietsch [48] in case \(S\) is a sort of "integral operator with kernel \(K\)". Also he had to consider only the case \(d = 1\) because Theorem B was not available for \(d > 1\) at that time. The following theorem, conjectured at the end of [48] can be obtained using Pietsch's method and Theorem B.

**Corollary 11.11.** Let \(M\) be a compact \(d\)-dimensional \(C^\infty\) manifold. Suppose that \(1 \leq p, q, u, v \leq \infty, r > s, r-s > 1/(p-1/q)\). If \(S : B_{p,s}^r(M) \to B_{p,r}^s(M)\) is the integral operator defined by a kernel \(K\) of class \(\mathcal{K}_p\) then \(d_p(S)\) belongs to the Lorentz space \(l_m\) where
\[\|S\| = d_p(S) = O(n^{-\alpha - \beta})\].

In particular, if \(u = \infty\), then \(d_p(S) = O(n^{-\alpha - \beta})\).

Theorem B enables us also to simplify the estimates of the \(s\)-numbers of the embedding operators by reducing to finite dimensional problems. For the definition of the general notion of the \(s\)-number function we again refer to [47]. Here we shall only describe a typical case in which almost all is known (cf. [43]), namely the Kolmogorov diameters \(d_p(T)\).

Recall that for the operators \(T : X \to Y\) and \(n = 0, 1, 2, \ldots\) one puts
\[d_p(T) = \inf \left\{\|Q_pT\| : Q_p : X \to Y, \dim X = n\right\},\]
where \(Q_p : X \to Y/\mathbb{E}\) is the quotient map. Clearly, \(d_p(T) = 0\) iff \(T\) is a compact operator.

If \(X \subseteq Y\) and \(T\) is the embedding operator, one writes \(d_p(X, Y)\) instead of \(d_p(T)\). The embeddings (11.16) show that
\[d_p(B_{p,s}^r(M), B_{p,w}^s(M)) \leq d_p(W_p^r(M), W_p^s(M)) \leq C d_p(B_{p,s}^r(M), B_{p,w}^s(M)),\]
where \(C\) does not depend on \(n\). In order to estimate \(d_p(B_{p,s}^r(M), B_{p,w}^s(M))\) from above and \(d_p(B_{p,s}^r(M), B_{p,w}^s(M))\) from below one can use Theorem B which reduces the problem to the sequence spaces or, alternatively, to the case where \(d = 1\). In most cases settled so far the best upper and lower estimates coincide up to a constant factor.

### Spline bases in classical function spaces

The diameters \(d_p(B_{p,s}^r(M), B_{p,w}^s(M))\) are now almost completely determined. Let
\[a = \min \{a, 1\} + \alpha(p, q, (\infty - a)\max(1, q/2)\} > 0,\]
where
\[a(p, q) = \begin{cases} 0 & \text{if } 1 \leq p < q \leq 2, \\ 1/2 - 1/q & \text{if } 1 \leq p < 2, \\ 1/2 - 1/q & \text{otherwise.} \]

Then (assuming \(a \geq 0\) if \(a = 0\)) one has for \(n \geq 2\)
\[(1.12) \ Cn^{-a} \leq d_p(B_{p,w}^s(M), B_{p,w}^s(M)) \leq Cn^{-a}(\log n)^p\]
for some \(\beta \leq 3\) and \(0 < C_1 \leq C_2 < \infty\) depending only on \(a - \beta, \sigma, p, q, u, v, \sigma\) and \(\beta\). Moreover, (11.12) obtains with \(\beta = 0\) in each of the cases \(q < p, p < q\).

Let us indicate how these results are deduced from the corresponding finite dimensional facts.

Observe that if \(m, z > 0\), then
\[(11.13) \ d_p(B_{p,w}^s(M), B_{p,w}^s(M)) \geq \min_{\ell} d_p(B_{p,w}^s(M), B_{p,w}^s(M)) (\ell)\]
(recall that \(d_p(B_{p,w}^s(M), B_{p,w}^s(M)) = 0\) if \(h = 0\)). On the other hand, if \(h > 0\) are integers and \(\sum_{n=0}^{m} \hat{h}_n = h, \) then
\[(11.14) \ d_p(B_{p,w}^s(M), B_{p,w}^s(M)) \leq \sum_{n=0}^{m} \hat{h}_n d_p(B_{p,w}^s(M), B_{p,w}^s(M)).\]

This is a version of the method of V. I. Malkin [45].

The lower estimates in (11.12) follow from (11.13) and
\[(11.15) \ d_p(B_{p,w}^s(M), B_{p,w}^s(M)) \geq Cn^{-\alpha(p, q)}\]
if \(N \geq 2n\),
\[(11.16) \ d_p(B_{p,w}^s(M), B_{p,w}^s(M)) \geq Cn^{-\alpha(p, q)}\]
if \(1 < p < q < 2\) and \(N \geq Cn^{-\alpha(p, q)}\).

Inequality (11.15) can be found in [43] or [41], for a slightly weaker form of (11.16) we refer to E. D. Gluskin [41].

Upper estimates in (11.12) follow from (11.14) after a suitable choice of the \(h\)-sequence. One may assume that \(a = 0\).

The case where \(q = p = 2\) is easy, because then \(d_p(B_{p,w}^s(M), B_{p,w}^s(M)) = N^{-\alpha(p, q)}\). (Just put \(h_n = 2^n\) if \(2^{n+1} \leq n\), \(h_n = 0\) otherwise.)

Hence we may assume that \(1 < p < q < 2\). Let \(\epsilon > 0\) and \(\alpha > 0\), \(\alpha \geq a(p, q)\). Given \(r > 0\), let \(b\) satisfy \(b(\epsilon^{r+1} + \ldots + \epsilon^{r}) = n^{-b}\). Put \(h_n = 2^n - 1\), if \(m < n \leq n + 1\). Using these values in (11.14) and B. S. Kashin's estimate (valid for \(2 < p < q < \infty\))
\[(11.17) \ d_p(B_{p,w}^s(M), B_{p,w}^s(M)) \leq C n^{-\alpha(p, q)} (\log n)^{a(p, q)}\].
where $\gamma = (1/p - 1/q)/(1/2 - 1/q)$ (cf. [43] and [41]), we obtain (11.12) if $r = 1$. In some cases, e.g. if $q - r > \gamma/q$, a better choice of $r$ yields (11.12) with $\beta = 0$.

It is not known whether the logarithmic term in (11.17) can be replaced by a constant depending on $(\log N)/\log q$. If this can be done for some $2 < q < \infty$, then one obtains easily that (11.12) holds for this $q$ with $\beta = 0$ except for the case $s(p, q) = (q/2 - 1)(q - \sigma)$ (cf. Added in proof).

In some cases, however, the factor $(\log s)^\beta$ may be necessary. Namely, if $q > 2$ and $w > v \geq 1$, then

$$d_n(b_v^w, b_v^w) \geq C^{-12}(\log n)^{12u}.$$  

This can be deduced using (11.16).

We close the paper with bibliographical comments. The first spline basis in $C(I)$ and $L^2(I)$ was treated by Haar [42], Faber [40], Schauder [51] and Franklin [39]. Extensions of those results to spaces of smooth functions over cubes are given in the papers of Ciesielski [11], Schonsfeld [52], Ciesielski and Domsta [14] and Ryll [50].

The first result on unconditionality of the spline bases concerns the Haar system and it is due to Marcinkiewicz [46]. In Bochkarev [36] we find the proof that the Franklin system is an unconditionally basis in $L^2(I)$, $1 < p < \infty$. For extension of those results to higher order splines we refer to Ciesielski [13].

Special cases of Theorem B were proved in Ciesielski [38], [10] and [12] and by Ropela in [49].

Theorems A and B in a less general form are presented in Ciesielski and Figiel [16].

Added in proof. The correct order of the diameters $d_n(b_v^w, b_v^w)$ in all the remaining cases has been recently found by E. D. Gluskin [33]. In particular, he proved that for $2 < p < q < \infty$ estimate (11.17) holds with $C = c(p, q)$ and $2/3$ replaced by $\beta$. Using this in the argument sketched above (now with $r > 1$), we obtain that the logarithmic factor in (11.12) may be necessary only if $q > \max(2, p)$ and $q(p - \sigma) = \min(1, \gamma)$.

References to Part II

[27] E. C. Ernst, On the convergence of sums of eigenvalues for the logarithmic factor in (11.12) may be necessary only if $q > \max(2, p)$ and $q(p - \sigma) = \min(1, \gamma)$.
О строении безусловных базисов некоторых пространств Кёте

В. П. КОНАДАКОВ (Ростов-на-Дону)

Резюме. В пространствах Кёте числовой последовательности

\[ l_p(a_0(n)) = \left\{ f = (a_n) : \sum \left| a_n(x_0) \right|^p \right\}^{1/p} = \| f \|_p < +\infty, \quad p = 1, 2, \ldots \],

1 < p < \infty, изучено, что для любой последовательности элементов \( (a_n) \), имеющей размерность непрерывную биорготонову систему функционалов, имеющей условие: существует такое семейство в кольце \( \lambda : N \to \mathbb{R}^+, \quad \gamma : N \to \mathbb{N}, \quad \sigma : N \to \mathbb{N}, \quad \phi : N \to \mathbb{R}^+, \quad \psi : N \to \mathbb{N} \),

что

\[ \begin{align*}
\lambda_{m,n} (\psi(m)) & < a_n (\psi(m)) \quad \phi_{m,n} (\psi(m)) (\sigma(m)), \quad n, m \in N.
\end{align*} \]

Это даёт возможность получить простое дополнение квазиуравнениности безусловных базисов в пространствах Кёте \( l_p(a_0(n)) \), \( p = 1, 2, \infty \), имеющих границный безусловный базис, а также доказать квазиуравненности базисов и гипотезу Кёте в некоторых других классах пространств.

Введение. Нашей целью является изучение в пространствах Кёте свойств базисов, основанных на которых считаем квазиуравненностью безусловных базисов (КББ) и характеризацию безусловных базисов дополненных подпространств (проблема Кёте).

Настоящая работа продолжает многочисленные исследования в этой области (см., напр., [1], [2], [3], [4], [5], [9], [10], [17], [19], [20], [28]). Подробнее о вопросах и библиографию можно узнать в обзорах [21], [22].

Важную роль в работе играет теорема 1, которая утверждает, что из матрицы преднорма стандартного базиса ортого всего можно получить матрицу, эквивалентную матрице преднорм любой последовательности элементов, имеющей размерность непрерывную биорготоновую систему функционалов, путём повторения сдвигов столбцов и удаления двух. Это обобщает известную теорему Драгомана-Бесенга [21], [25] для базисов дополненных подпространств линеарных результат В. П. Захарова и автора для \( p \)-абсолютных базисов дополненных подпространств пространств Кёте.

Рассуждение теоремы 1 в комбинации с совершенствованием приёмов работе [17], [20], [7] даёт значительно упрощённое по сравнению