Translation invariant complemented subspaces of \( L^p \)

by

J. BOURGAIN (Brussels)

Abstract. \( G \) be a compact Abelian group and \( \Gamma \) the dual group of \( G \). Assume \( A \subseteq \Gamma \) and \( L^p_A \) complemented in \( L^p(G) \) for some \( 1 < p < \infty \) \((p \neq 2)\). A necessary and sufficient condition on \( A \) is given in order that \( L^p_A \) should be isomorphic to the space \( L^p[0,1] \).

Introduction. Throughout this note \( G \) will be a compact Abelian group and \( \Gamma \) the dual group of \( G \). For \( A \subseteq \Gamma \) and \( 1 \leq p \leq \infty \), \( L^p_A \) will have the usual meaning. The results presented here are a continuation of [2] (see also [3] for more details) in which more particularly the Cantor group was considered.

Our purpose is to show that if \( 1 < p < \infty \) \((p \neq 2)\) and \( L^p_A \) complemented in \( L^p(G) \), then the analytic condition for \( A \) considered in Th. 1 of [2] is necessary and sufficient in order that \( L^p_A \) should be linearly isomorphic to the Banach space \( L^p \). From the descriptive point of view, our approach is satisfactory if the elements of \( G \) are of bounded order.

Notations and preliminary results. Let \( \gamma_1, \ldots, \gamma_s \) be a finite sequence in \( \Gamma \). Define by \( V_k(\gamma_1, \ldots, \gamma_s) \) for \( k = 1, 2, \ldots \) the set of characters \( \gamma \) which can be written in the form

\[
\gamma = \gamma_1^{a_1} \cdots \gamma_s^{a_s},
\]

where \( a_s \in \{0, 1, \ldots, k\} \) for \( s = 1, \ldots, r \).

Similarly, let \( W(\gamma_1, \gamma_s) \) be the set of characters \( \gamma \) of the form

\[
\gamma = \gamma_1^{a_1} \cdots \gamma_s^{a_s},
\]

where \( a_s \in \{0, 1, \ldots, s\} \) for \( s = 1, \ldots, r \).

For \( k = 1, 2, \ldots \), we agree to say that a subset \( A \) of \( \Gamma \) has property \((k)\) provided there exist a sequence \( (\gamma_s) \) of distinct characters and a sequence of characters \( (\delta_r) \) in \( \Gamma \) such that for each \( r \)

\[
V_k(\gamma_1, \ldots, \gamma_s), \delta_r \subseteq A.
\]
Analogously, we define property (J) for a subset $A$ of $\Gamma$ replacing (1) by
(2)
\[ W(y_1, \ldots, y_r). \beta \in A. \]
Thus property (1) corresponds to the property stated in Th. 1 of [2].
Although the next results are in [2] explicitly stated for the Cantor group, they extend to any compact Abelian group.

**Proposition 1.** If $p > 2$ and $L_p^p$ embeds in $L_{p'}^p$, then $A$ has property (1).

**Proposition 2.** Property (1) is "primary", i.e. if $A$ has (1) and $A = A_1 \cup A_2$, then either $A_1$ or $A_2$ has (1).

In fact, Prop. 2 is purely combinatorial and related to Hindman's theorem (see [1], [7]).

**Translation invariant complemented subspaces.** In this section we state the new results. Proofs will be outlined in the next section.

**Proposition 3.** If $A \subset \Gamma$ has (L), then $L_p^p$ embeds in $L_{p'}^p$ for all $1 < p < \infty$.

**Proposition 4.** Assume $A \subset \Gamma$ has (1) and $L_{p_0}^p$ complemented in $L_k^p(\delta) (p \neq 2)$. Then $A$ has (L).

**Theorem 5.** Assume $A \subset \Gamma$, $1 < p < 2$, and $L_p^p$ complemented in $L_k^p(\delta)$. Then t.f.a.e.

(i) $A$ has (1).

(ii) $A$ has (L).

(iii) $L_p^p$ is isomorphic to $L_k^p$ (assuming $A$ countable).

**Corollary 6.** Assume $A \subset \Gamma$, $1 < p < 2$, $q < \infty$ and $L_{p_0}^q, L_{q_0}^q$ complemented. If $L_p^p$ is isomorphic to $L_k^p$, then $L_{q}^q$ is isomorphic to $L_k^q$.

**Corollary 7.** Assume the elements of $G$ of bounded order and $A \subset \Gamma$. Then t.f.a.e.

(i) There is a sequence $(y_r)$ of distinct characters in $\Gamma$ such that $A$ contains a translate of the subgroup $\{y_1, \ldots, y_r\}$ for each $r$.

(ii) There is a subset $A'$ of $A$ such that for some $1 < p < \infty$ ($p \neq 2$), $L_{p_0}^p$ is complemented in $L_k^p(\delta)$ and isomorphic to $L_k^p$.

(iii) There is a subset $A'$ of $A$ such that for all $1 < p < \infty$, $L_{p_0}^p$ is complemented in $L_k^p(\delta)$ and isomorphic to $L_k^p$.

Referring to the classes $\mathcal{H}$ introduced in [2].

**Corollary 8.** Assume $A \subset \mathbb{Z}$, $1 < p < \infty$ ($p \neq 2$) and $L_{p_0}^p$ complemented in $L_k^p(\delta)$. If $A$ does not belong to any class of finite index, then $A$ contains arithmetic progressions.

**Additional proofs.** For $k = 1, 2, \ldots$, and $(y_r)$ a sequence in $\Gamma$, we say that $(y_r)$ is $k$-disassociated provided

\[ y_1, \ldots, y_r = 1 \text{ and } [y_s]_{k} = 1 \text{ for } 1 < s < r. \]

We omit the proof of following two simple facts.

**Lemma 1.** If $A$ is an infinite subset of $\Gamma$ and $k$ a positive integer, then there is a sequence $(y_r)$ in $\Gamma$ and $\delta \in \Gamma$ such that $(\delta, y_r)$ is $k$-disassociated.

**Lemma 2.** Let $(y_r)$ be a sequence of distinct elements of $\Gamma$ and $k$ a positive integer. Then there exist finite subsets $S_n$ of $N$ satisfying $S_n < \min S_n + 1$, such that $(\delta, y_r)$ is $k$-disassociated, where $\delta = \bigcap y_r$.

The next lemma is easily obtained by standard approximation arguments.

**Lemma 3.** Given $1 < p < \infty$, $\varepsilon > 0$ and positive integers $k, r$, there exists an integer $K$ such that $(y_{\varepsilon, k})_{k, r}$ in $\Gamma$ is $k$-disassociated and if $F \in L_{p_0}^p (|y| \leq k)$, then

\[ \left( 1 - \varepsilon \right) \|F\|_p \leq \sum F_{y_1} \cdots F_{y_r} \leq \left( 1 + \varepsilon \right) \|F_{y_1} \cdots F_{y_r}\|_p. \]

**Lemma 4.** For $1 < p < \infty (p \neq 2)$, there is a constant $c > 1$ such that the orthogonal projection from $(1, y_1)$ onto $(1, y)$ is $L_p$-norm at least $c_0$, whenever $y \in \Gamma$ is of order at least $3$.

**Proof.** By duality, we can take $2 < p < \infty$. Since further

\[ \|a+b\|_p = \|a+b|\|_p \leq \|a+b|\|_p, \]

we can restrict ourselves to $G = \Pi$ and $y = e_1$. Now, one has for $0 < \varepsilon < 1$

\[ 1 + \varepsilon \|a+b\|_p = \|1+\varepsilon b|\|_p = \|1+\varepsilon b|\|_p = \|1+\varepsilon b|\|_p, \]

where

\[ \sigma = \frac{1}{2} \int_0^\infty \left( 1 - \left| \cos \theta \right| \right) \sin \theta \, d\theta > 0. \]

The result is immediate from Lemma 3 and Lemma 4.

**Lemma 5.** For $1 < p < \infty (p \neq 2)$, there is a constant $c_0 > 1$ such that for given positive integers $k, r$, the $L_p$-norm of the orthogonal projection from $V_{p_0}^k(\gamma_1, \ldots, \gamma_r)$ onto $V_{p_0}^k(\gamma_1, \ldots, \gamma_r)$ is at least $c_0$, provided $(\gamma_{\varepsilon, k})_{k, r}$ is a sufficiently disassociated sequence in $\Gamma$ whose elements are at least of order $k+1$.

**Lemma 6.** Assume $L_{p_0}^p$ complemented in $L_k^p(\delta) (p \neq 2)$ and $A$ with property (h). Then for infinitely many characters $y$, the set $y_j A$ has (1).

**Proof.** Remark first that by Lemma 2, the sequence $(y_r)$ in the definition of property (h) can be chosen arbitrarily disassociated. Since
further γ will be obtained as element of V_k(γ_1, γ_2, ..., γ_k), the existence of infinitely many candidates will be automatic. Denote P the orthogonal projection on L^2 and fix r large enough to ensure c_r^2 > ||P||. Take γ_1, ..., γ_r sufficiently dissociated and A_1 = P with (1) such that
V_k(γ_1, ..., γ_r), δ ∈ A_1.

If some γ ∈ {γ_1, ..., γ_k} is of order at most k, then obviously
\[ \bigcap_{j=0}^{k-1} \mathbb{Z}^j \cdot A = \bigcup_{j=0}^{k-1} \mathbb{Z}^j \cdot A \]
has (1), since latter set contains A_1.

Otherwise, Lemma 5 asserts that the L^2-norm of the orthogonal projection from V_{k+1}(γ_1, ..., γ_k) onto V_k(γ_1, ..., γ_k) is at least c_r^2. Fixing δ ∈ A_1, one has
V_k(γ_1, ..., γ_k), δ ∈ V_{k+1}(γ_1, ..., γ_k) \subseteq V_{k+1}(γ_1, ..., γ_k),
where in particular the second set is of |P| complemented in the third. Thus the first and the second set must be different, implying the existence of some \( \xi \in V_{k+1} \setminus V_k \) such that \( \delta \in A_1 \xi \). Applying now Prop. 2, we can fix \( \xi \in V_{k+1} \setminus V_k \) for which \( A_1 \xi \cap A_1 \) has (1). There is a nonempty subset A of \{1, ..., r\} such that
A = \bigcap_{j=1}^{r} \mathbb{Z}^j \cdot A
where \( \gamma = \bigcap_{j=1}^{r} \mathbb{Z}^j \gamma \) and \( \eta \in V_k(\gamma; \delta \notin A) \).

Since
A_1 \cap A_1 \subseteq \bigcup_{j=0}^{k-1} \mathbb{Z}^j \cdot A \cap \bigcup_{j=0}^{k-1} \mathbb{Z}^j \cdot A \end{equation}
the set \( \bigcap_{j=0}^{k-1} \mathbb{Z}^j \cdot A \) has property (1), which concludes the proof.

Lemma 7. Assume L^2 complemented in L^p(G) (p ≠ 2). If A has (1), then A has also (k) for any positive integer k.

Proof. We proceed by induction on k. So assume that under the above hypothesis the implication (1) → (k) holds. Thus in particular A has (k) and hence, by Lemma 6, there is a character γ such that the set
A_k = \bigcup_{j=0}^{k-1} \mathbb{Z}^j \cdot A = \bigcap_{j=0}^{k-1} \mathbb{Z}^j \cdot A
has (1). Observe that L^2 complemented in L^p(G) since A_k is finite intersection of L^p-complemented sets. So, by induction hypothesis, A_k has (k).

Apply again Lemma 6 to obtain a character γ_k ⊄ γ, such that
A_k = 1 = \bigcup_{j=0}^{k-1} \mathbb{Z}^j \cdot A_k = \bigcap_{j=0}^{k-1} \mathbb{Z}^j \cdot A
has (1).

Translation invariant complemented subspaces of L^p

Iteration of this procedure leads to a sequence (\gamma_r) of distinct characters such that for each r
\[ \bigcap_{j=0}^{r-1} \mathbb{Z}^j \cdot A \]
has (1) and hence is nonempty. Thus one can find a sequence (\delta_r) in P satisfying
V_k(\gamma_1, ..., \gamma_r), \delta_r \subseteq A
for each r.

Consequently A has property (k+1).

Proof of Proposition 3. We use the same procedure as in Lemma 7. If \( \gamma_1, ..., \gamma_r \) are already obtained and
A_r = \bigcap_{j=0}^{r-1} \mathbb{Z}^j \cdot A
has (1), Lemma 7 asserts that A_r also has (r+1). In particular, there exists \( \gamma_{r+1} \in \gamma \) such that
A_{r+1} = \bigcap_{j=0}^{r-1} \mathbb{Z}^j \cdot A_r
still has (1).

Proof of Proposition 3. We will construct a system of functions on \( \mathcal{G} \) with spectrum in A which is equivalent to the usual Haar system in L^p(Q, G).

Take first sequences \( \{p_n\}, \{q_n\} \) of trigonometric polynomials with positive spectrum on \( \mathcal{H} \) such that
(i) \( |p_n| + |q_n| \leq 1 \)
(ii) \( \int |p_n|^2 \leq \int |q_n|^2 \), “rapidly enough”
Let further \( p_n, q_n \) be of degree at most m.
Let \( (\gamma_n) \) be a sequence of distinct characters and \( (\delta_n) \) a sequence of characters s.t.
W(\gamma_n, ..., \gamma_r), \delta_n \subseteq A
We will fix our attention to the case where the \( \gamma_n \) are of bounded order. We can then replace \( (\gamma_n) \) by sequences \( (\gamma_n), (\delta_n) \) satisfying
(iii) \( \gamma_1, ..., \gamma_m = 1 \) and \( |\gamma_n| \leq d_n = d_n, ..., \gamma_m = 0 \)
(iv) \( |\gamma_n| \leq d_n \delta_n \delta_n \in A \) provided \( d_n \in \{0, 1, ..., d_n\} (1 \leq s \leq m) \)

Define \( \phi_{m,n} = p_n \circ \gamma_n \) and \( \phi_{m,n} = q_n \circ \gamma_n \). Let further for \( n = \sum_{s=0}^{m-1} 2^s \)
\[ f_{m,n} = \prod_{s=0}^{m-1} \phi_{m,s} \delta_m \]
where the $\alpha_{m,n}$ in $\Gamma$ are chosen such that $Spec_{m,n} \subseteq A$ and the system $(f_{m,n})$ is a martingale difference sequence for the lexicographical order in $L^p(\Omega)$. By (i)

$$1 \geq |f_{m,n}| \geq |f_{m+1,2n} - |f_{m+2,2n}|$$

and by (iii), (ii)

$$\int |g_{m,n}|^2 = \int |g_{m}|^2, \quad \int |g_{m,n}|^2 = \int |g_{m}|^2$$

and

$$\int |f_{m,n}|^2 = \prod_{k=1}^{m-1} \int |g_{k,n}|^2 \sim 2^{m/2}.$$ 

By standard techniques, one can then show that $(f_{m,n})$ contains a sub-system equivalent to the Haar system (cf. [8]).

Proof of Theorem 5. (1) $\Rightarrow$ (2) follows by Prop. 4.

(2) $\Rightarrow$ (3): By Prop. 3, $L^p$ embeds in $\ell^p_2$ and hence also as complemented subspace (see [8]). The isomorphism follows from Pelczynski's decomposition method (see [9]).

(3) $\Rightarrow$ (1) follows by Prop. 1 and duality in case $1 < p < 2$.

Corollary 6 is now straightforward.

Proof of Corollary 7. (iii) $\Rightarrow$ (ii) is obvious and (ii) $\Rightarrow$ (i) is a consequence of Th. 5.

Notice that the orthogonal projection on $g^N[\gamma_1, \ldots, \gamma_p]$ is given by a conditional expectation. The implication (i) $\Rightarrow$ (iii) follows from standard Burkholder–Gundy square-function techniques for martingale difference sequences and Stein's inequality (cf. [6] and [10]).

References