On singular integrals and Orlicz spaces*

by

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Abstract. This paper deals with the boundedness in $L^2(\mathbb{R}^n)$ of singular integral operators with variable kernel when such kernel belongs to an Orlicz space $L_\Phi$. We give necessary and sufficient conditions on $\Phi$ related with the continuity. On the other hand, we show a counterexample concerning with the result given in [2].

1. Introduction. In this paper we deal with singular integral operators of the form

$$
Kf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x, y)f(x-y)\,dy,
$$

(1.1)

where $f$ is a measurable function and $k(x, y)$ is defined for $x, y \in \mathbb{R}^n$, $y \neq 0$ and satisfies the conditions:

(1.2) For $t > 0$, $k(x, ty) = t^{-n}k(x, y)$;

(1.3) $\int_{S_{n-1}} k(x, y')|\Phi(y')|\,d\sigma(y') = 0$ for each $x \in \mathbb{R}^n$.

Calderón and Zygmund studied the continuity in $L^p(\mathbb{R}^n)$ of such operators and obtained that if the kernel $k(x, y)$ belongs to $L^q(S_{n-1})$ in the variable $y$ and it has bounded norm as function of $x$, then $K$ is bounded in $L^p(\mathbb{R}^n)$ if and only if $q > 2(n-1)/n$ (see [1] and [2]).

We present here an improvement to this theorem in the case which the kernel belongs to an Orlicz space $L_\Phi(S_{n-1})$. This will be done developing the kernel in spherical harmonics as in [2] and then using an interpolation theorem given by M. Jodeit and A. Torchinsky [3].

On the other hand, we give a counterexample related with $L^p$-case, that is in [2] it has been shown that if $1 < p < 2$, then $K$ is bounded

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in $L^p$ if and only if $q > (n-1)p/(n-p)$. We show that if $p > 1$ and $q < (n-1)p/(n-p)$, then $f \in L^p$ doesn’t imply $Kf \in L^p$ even locally.

2. Preliminaries and statement of results. In this section we are going to use (see [3] and [4]).

A generalized Young function ($GYF$) $\Phi$ is a function defined in $[0, +\infty)$ such that:

(i) $\Phi(t) \geq 0$, $\Phi(0) = 0$;

(ii) $\Phi$ is left continuous;

(iii) $\Phi(x)/x$ is non-decreasing.

A convex GYF will be a Young function. Given a GYF $\Phi$, its regularization $\Phi_s$ is defined as

$$\Phi_s(x) = \int_0^x \Phi(t) dt,$$

$\Phi_s$ is a Young function and it satisfies

$$\Phi_s(ax) \leq a \Phi_s(x) \leq \Phi_s(2ax).$$

Let $(X, \mu)$ be a measure space and let $\Phi$ be a GYF. The Orlicz space $L_\Phi(X, \mu)$ consists of all the $\mu$-measurable functions $f$ (modulo the equivalence relation a.e.) such that

$$\int_X \Phi(|f(x)|) d\mu(x) < \infty$$

for some $s > 0$ (depending on $f$). The norm

$$\|f\|_\Phi = \inf\{\lambda > 0 : \int_X \Phi(|f(x)|/\lambda) d\mu(x) < 1\}$$

turns $L_\Phi$ into a Banach space.

The Young's complement of a GYF $\Phi$, denoted $\Phi^*$, is given by $\Phi^*(x) = \sup\{\gamma \in [0, +\infty) : \Phi(y) \leq \gamma y\}$. If $\Phi$ is a Young function and holds the inequality:

$$\int_X |f(x)g(x)| d\mu(x) \leq 2\|f\|_\Phi \|g\|_\Phi,$$

the inverse of a GYF $\Phi$ is defined by $\Phi^{-1}(y) = \inf\{a : \Phi(a) > y\}$ (inf $\Phi = +\infty$) and it satisfies

$$\Phi(\Phi^{-1}(a)) \leq a \leq \Phi^{-1}(\Phi(a)), \quad a > 0.$$

Also holds the following inequality (Young function):

$$\Phi(\Phi^{-1}(a)) \leq a \leq \Phi^{-1}(\Phi(a)) \leq 2a.$$

We shall now state our results:

**Theorem 1.** Let $k(x, y)$ be a kernel which satisfies (1.2) and (1.3) and $\Phi$ a Young function such that $\Phi(a)/a$ is non-increasing and $\Phi(2a)$

$$\leq N\Phi(a)$$

for some $N > 1$ and for each $a \geq 0$. Then if

$$\sup_{x, y} \Phi(|k(x, y)|) \leq M,$$

there exists a constant $C$ depending on $n$ and $\Phi$ only such that for each $f \in C_0^\infty(R^n)$ we have

$$\|Kf\|_\Phi \leq C \|f\|_\Phi,$$

provided that

$$\int_X \Phi(|\sum_{j=1}^n (x_j - y_j)^{\alpha_j-1}|) d\mu(x) < \infty.$$

**Theorem 2.** Let $\Phi$ be a Young function such that for some $M > 0$

$$\int_M \Phi(t) t^{\alpha_j-1} dt = +\infty.$$

Then there exist $k(x, y)$ satisfying (1.2) and (1.3), and $f \in C_0^\infty(R^n)$ such that $sup_{x, y} \Phi(k(x, y)) < \infty$ but $Kf \notin L^p(R^n)$.

**Theorem 3.** If $n \geq 2$, $p > 1$ and $1 < q < (n-1)p/(n-p-1)$, there exist $k(x, y)$ satisfying (1.2), (1.3) and $sup_{x, y} \Phi(k(x, y)) < \infty$ and

$f \in L^q(R^n)$, such that $Kf \notin L^p(R^n)$ locally.

3. Proof of Theorem 1. This will be done in three steps.

3.1. For each $m \geq 1$, by $Y_m$, $1 \leq j \leq d_m$, we denote a complete orthonormal set of spherical harmonics of degree $m$ in $L^2(Y_m)$.

Let us suppose that $k(x, y)$ has the form:

$$(3.1.1) \quad k(x, y) = \sum_{j=1}^{d_m} \sum_{m \geq 1} a_m(y) Y_m(y') |y'|^\alpha, \quad y = y/|y|,$$

where the sum is finite. As in [2], it may be seen that the integral in (1.1) exists for $f \in C_0^\infty(R^n)$. We write

$$f(x) = \sum_{y \in \Phi_m} f(y' \Phi_m \Phi(y))$$

and set $a_m = \sum_{y \in \Phi_m} b_m(y') = a_m(x)/a_m(x)$ so that $\sum_{y \in \Phi_m} b_m(x) = 1$. Now, taking a sequence $\gamma_m$ of positive numbers, we have

$$(3.1.2) \quad \|Kf\|_\Phi^2 = \left[ \sum_{m \geq 1} a_m^2 \sum_{j=1}^{d_m} b_m(x) f_m(x) \right]^2$$

$$\leq \left[ \sum_{m \geq 1} a_m^2 \gamma_m \right] \left[ \sum_{j=1}^{d_m} b_m(x) \right].$$
If we can select this sequence such that
\[ \sum_{m} a_m(\gamma) \gamma_m \leq CM', \]
where \( C \) is a constant depending on \( n \) and \( \Phi \), then integrating (3.1.2) we obtain
\[ \|Kf\|_2 \leq CM' \sum_{m} \gamma_m \sum_{j=1}^{a_m} \|f_j\|_2. \]
From Plancherel's theorem and the following facts:
(i) \( \int_{\mathbb{R}^n} f_j(s) = C_m Y_m(s', \gamma) f(s), \) where \( C_m = C_m s_m^{-\gamma/m} \) \( (\gamma \) denotes the Fourier transform),
(ii) \( \sum_{j=1}^{a_m} Y_m(j) = h_m s_m^{-1}, \) where \( h_m \leq C_m s_m^{-1} \) (see [5]),
we get
\[ \|Kf\|_2 \leq CM \left( \sum_{m} \gamma_m s_m^{-1} \right)^{1/2} \|f\|_2. \]
In consequence we are going to show that there exists a sequence \( \gamma_m \) which satisfies (3.1.3) and
\[ \sum_{m} \gamma_m s_m^{-1} < \infty. \]
(3.1.4)

3.2. Let us consider for each \( m \geq 1 \) a normalized spherical harmonic \( X_m. \) If \( g \in L^p(\mathbb{R}^n) \), we put\[ g(x) \sim \sum_{m} a_m X_m(x), \quad \text{with} \quad a_m = \frac{1}{\mathcal{A}_m} \int g(x) X_m(x) \, dx. \]
It is well known that the following inequalities are valid (see [5]):
(i) \( |a_m| \leq C m^{-\frac{n-1}{2}} \|g\|_{L^2(\mathbb{R}^n)}. \)
(ii) \( |a_m| \leq C m^{-\frac{n+1}{2}} \|g\|_{L^2(\mathbb{R}^n)}. \)
We need to interpolate the two last inequalities, for which we use the following result of M. Jodeit and A. Torchinsky (see [3], p. 255):
(3.2.1) \textit{Theorem}. Let \((X, \mu)\) and \((Y, v)\) be two measure spaces. Let \( T \) be a linear operator defined for all \( \mu \)-measurable functions on \( X \), which satisfies
\[ \|Tf\|_{L^p(X, \mu)} \leq M_1 \|f\|_{L^p(X, \mu)}, \quad \text{and} \quad \|Tf\|_{L^p(Y, v)} \leq M_2 \|f\|_{L^p(X, \mu)}, \]
Then if \( \Phi \) is a GYF such that \( \Phi(s)^{1/2} \) is non-increasing, we have \( \|Tf\|_{L^p(Y, v)} \leq C \|f\|_{L^p(X, \mu)}, \) where \( \Phi \) is the GYF defined by
\[ \Phi(s) = \begin{cases} 0 & \text{for } s = 0, \\ \sup_{s > 0} \Phi(s) & \text{for } s \neq 0 \end{cases} \]
and \( C \) is a constant which depends on \( M_1, M_2 \) and \( \Phi \) only.

Now, we put
\[ h(x, \cdot) \sim \sum_{m} a_m(\gamma) Y_m = \sum_{m} a_m(\gamma) Y_m, \]
with \( Y_m = \sum_{j=1}^{a_m} Y_m(j) \), where \( a_m(\gamma) \) and \( b_m(\gamma) \) are defined as in (3.1).
Let us fix \( x \in \mathbb{R}^n \), we define \( T_{x, \gamma} g = \langle b_m(\gamma) s_m^{-\gamma(n-1)/m+1} \rangle_{m=1} \), where \( g \in L^p(\mathbb{R}^n) \)
and \( b_m(\gamma) \) are the coefficients of \( g \) in the system \( Y_m \) defined before.
Taking on the set \( N \) of natural numbers the measure \( \mu_m = \gamma_m^{-2} \) and observing now that \( T_{x, \gamma} \) satisfies the hypothesis of (3.2.1) and \( T_{x, \gamma} \Phi(s) = (a_m(\gamma) s_m^{-(\gamma(n-1)/m)+1})_{m=1} \)
for each \( s \), we have
\[ (3.2.3) \quad \|a_m(\gamma) s_m^{-\gamma(n-1)/m+1}\|_{L^q(\mathbb{R}^n)} \leq C \|h(x, \cdot)\|_{L^q(\mathbb{R}^n)}, \]
where \( C \) depends on \( n \) and \( \Phi \) only.
Let \( A \) be a GYF. From (3.2) we obtain
\[ (3.2.3) \quad \sum_{m} a_m(\gamma) \gamma_m \leq \sum_{m} a_m(\gamma) s_m^{-\gamma(n-1)/m+1} \gamma_m s_m^{-1} \leq 2 \|a_m(\gamma) s_m^{-\gamma(n-1)/m+1}\|_{L^q(\mathbb{R}^n)} \|h(x, \cdot)\|_{L^q(\mathbb{R}^n)}. \]
Since \( B(x) = 2A(x) \) is a GYF, we have as a consequence from the definition of the norm in \( L^q \) that
\[ (3.2.4) \quad \|a_m(\gamma) s_m^{-\gamma(n-1)/m+1}\|_{L^q(\mathbb{R}^n)} = \|a_m(\gamma) s_m^{-\gamma(n-1)/m+1}\|_{L^q(\mathbb{R}^n)} \|h(x, \cdot)\|_{L^q(\mathbb{R}^n)}. \]
Now we choose \( A(\gamma) = \frac{1}{2} \Phi(\gamma^{1/2}) \) (see lemma below), then using (3.2.3) and (3.2.4) in (3.2.3) we obtain
\[ \sum_{m} a_m(\gamma) \gamma_m \leq C \|h(x, \cdot)\|_{L^q(\mathbb{R}^n)} \|h(x, \cdot)\|_{L^q(\mathbb{R}^n)}. \]
Hence is enough to show that there exists \( \gamma_0 \) such that \( \|h(x, \cdot)\|_{L^q(\mathbb{R}^n)} < \infty \) and (3.1.4) is valid. In order to do this we need two lemmas.
(3.2.5) \textit{Lemma}. Let \( \Phi \) be a Young function such that \( \Phi(s)^{1/2} \) is non-increasing, then
(i) \( A(\gamma) = \frac{1}{2} \Phi(\gamma^{1/2}) \) is a GYF;
(ii) \( \frac{\alpha^{1/2}(1-\alpha)}{\alpha^{1/2}} \leq (1-\alpha)^{1/2} \leq 2(\alpha^{1/2}-1)^{-1}. \)

Proof. (i) \( \Phi \) is a Young function, then \( A(\gamma) = \frac{1}{2} \Phi(\gamma^{1/2}) \) if \( s \neq 0. \)
Hence it is sufficient to see that \( g(s) = \alpha \Phi(\gamma^{1/2}) \) is non-increasing. If \( s > 0, \alpha > 0, \) for each \( y > 0 \) we take \( y' = (s + \alpha s^{1/2}) y, \) then
\[ y' \geq y \quad \text{and} \quad \Phi(y')/y' \leq \Phi(y)/y. \]
Hence \( (s + \alpha)\Phi(y) \geq \alpha\Phi(y') \); then
(x+e)^{1/2}y - (x+e)\Phi(y) \leq \omega^2y - x\Phi(y) \leq g(x).

Now taking sup over y we obtain \(g(x+e) \leq g(x)\).

(ii) Since \(\Phi\) and \(\Phi\) are strictly increasing, \(A^{-1}(x) = (B^{-1}(2x)^{-1})^{-2}\).

Hence the estimates follow using (2.5) with \(A\) and \(\Phi\).

3.2.6 Lemma. Let \(\Phi\) be as in (3.2.5). \(\Phi\) satisfies (2.7) if and only if there exists a sequence \(e_n > 0\) such that

\[
S_1 = \sum_{m=0}^{\infty} \Phi(e_m)m^{n-2} < \infty \quad \text{and} \quad S_2 = \sum_{m=0}^{\infty} \Phi(e_m)^2m^{n-2} < \infty.
\]

Proof. Condition (2.7) is equivalent to

\[
\int_0^{\infty} x^{n-2}\Phi(x^n) dx < \infty.
\]

Hence if (2.7) holds, we put \(e_n = m_{n_2}\).

Let \(e_n\) satisfy (3.2.7), then

\[
g_m = \Phi(e_m)m_{n_2}^n\cdot
\]

Then \(g_m \leq 1\) if \(e_m < m_{n_2}\) and \(g_m \leq m_{n_2}m_{n_2}^{-2}\) if \(e_m \geq m_{n_2}\); in consequence

\[
\sum_{m=0}^{\infty} \sum_{m=0}^{m_{n_2}m_{n_2}^{-2}} \frac{m_{n_2}^n g_m}{m_{n_2}^n} \leq S_1^2 S_2^2 + S_1.
\]

Hence (2.8) holds.

Now we define \(\gamma_m = (\tilde{A}^{-1}[\Phi(e_m)]^{-1})\) with \(e_m\) as in (3.2.7). From (2.4) we get

\[
\sum_{m=0}^{\infty} \tilde{A}(\gamma_m) m^{-2} \leq \sum_{m=0}^{\infty} \Phi(e_m)^2m^{n-2} = S_2 < \infty
\]

and then \(\gamma_m \in L^2(\mathbb{R}^n, \omega(x)) < \infty\). And also from (3.2.5) we have

\[
\sum_{m=0}^{\infty} \gamma_m m^{-2} \leq 4 \sum_{m=0}^{\infty} \Phi(e_m)^2m^{n-2} = S_2 < \infty.
\]

This completes the proof of the theorem when \(k(x, y)\) has the form (3.1.1).

3.3. We now prove the theorem in the general case. Let us first suppose that \(k(x, y)\) is bounded in \(x\) and \(y\). If we take

\[
k_N(x, y) = \sum_{\omega_{y2}^2} \omega_{y2} Y_{y2}(y'),
\]

where \(a_w(\omega)\) are the coefficients of \(k(x, y)\) in \((Y_{y2})\), we have

\[
||k_N(x, \cdot) - k(x, \cdot)||_{L^2(\mathbb{R}^n)} \rightarrow 0, \quad N \rightarrow \infty, \quad x \in \mathbb{R}^n.
\]

Since \(\Phi(\cdot)\Phi(\cdot)\) is non-increasing, there exist \(C_1\) and \(C_2\) such that

\[
||f||_{L^2(\mathbb{R}^n)} \leq C_1 ||f||_{L^2(\mathbb{R}^n)} \quad \text{and} \quad ||f||_{L^2(\mathbb{R}^n)} \leq C_2 ||f||_{L^2(\mathbb{R}^n)},
\]

in consequence

\[
||k_N(x, \cdot) - k(x, \cdot)||_{L^2(\mathbb{R}^n)} \rightarrow 0 \quad \text{if} \quad N \rightarrow \infty.
\]

Let \(s\) be a positive integer, \(s > 0\); by Egdrof's theorem there exists a measurable set \(E_s = \{|x| \leq s\} \) such that \(|\{|x| = s\} - E_s| < \varepsilon\) and (3.3.2) holds uniformly on \(E_s^c\). Hence if \(f_s\) is the characteristic function of \(E_s\), we have

\[
||\phi_s^* k_N(x, \cdot) ||_{L^2(\mathbb{R}^n)} \rightarrow ||\phi_s^* k(x, \cdot) ||_{L^2(\mathbb{R}^n)}
\]

uniformly, \(N \rightarrow \infty\). We put

\[
K_N f(x) = \mathbb{E} f \phi_s^* k_N(x, y) f(x-y) dy, \quad f \in C_0^\infty(\mathbb{R}^n).
\]

Suppose that \(f(y) = \gamma\) if \(|y| > \varepsilon\); then

\[
|K_N f(x) - Kf(x)| \leq C \operatorname{sup} |(\gamma f(y))|(|y| + \varepsilon) ||h_N(x, \cdot) - k(x, \cdot)||_{L^2(\mathbb{R}^n)}.
\]

Now combining this with (3.3.3), we obtain \(K_N f \rightarrow Kf\) p.p.

Since the theorem holds for the kernel \(\phi_s^* k_N(x, y)\), then if \(N \rightarrow \infty\), by Fatou's lemma and (3.3.3) we have

\[
||\phi_s^* Kf||_2 \leq C \operatorname{sup} ||h(x, \cdot)||_{L^2(\mathbb{R}^n)} ||f||_2.
\]

Hence if \(x \rightarrow 0\) and \(x \rightarrow \infty\), we obtain the desired result.

If \(k(x, y)\) is not bounded, we put \(k^{(N)}(x, y) = k(x, y)\) if \(|k(x, y)|\) \(\leq \mathbb{N}\) and \(k^{(N)}(x, y) = 0\) otherwise and define

\[
k_N(x, y) = k^{(N)}(x, y) \mathbb{1}_{|y| < \varepsilon} - |y|^{-1} \sum_{m=0}^{\infty} k^{(N)}(x, y) \mathbb{1}_{|y| < \varepsilon} \int k^{(N)}(x, y) dy.
\]

Since \(|k^{(N)}(x, y)| \leq ||k(x, y)||\), from Lebesgue's theorem we have

\[
k_N(x, y) \rightarrow k(x, y) \quad \text{p.p.}
\]

Furthermore, \(k_N(x, y) \leq k(x, y) + C \omega_{y2}^2 \omega_{y2}^2 = g(x, y)\).

We will prove that \(k_N(x, y) \rightarrow k(x, y)\) in \(L^2(\mathbb{R}^n)\). In fact, using now that there exists \(C > 1\) such that \(\Phi(2x) \leq C \Phi(x)\) if \(0 < \varepsilon < 2\), we
can take \( r \geq 1 \) integer such that \( 2^r < 2^{r'} \) and we obtain
\[
\int_{x_{n-1}} \Phi \left( \frac{|h(x, y') - h(x, y)|}{|x - y'|^{(n-1)} \Gamma(n-1)} \right) \, d\sigma(y') \leq C \int_{x_{n-1}} \Phi \left( \frac{|y - y'|}{|x_{n-1} - y'|} \right) \, d\sigma(y') \leq C \int_{x_{n-1}} \Phi \left( \frac{|y - y'|}{|x_{n-1} - y'|} \right) \, d\sigma(y') \leq C
\]
Then from (3.3.4) we have
\[
\|k(x, \cdot) - k(x, \cdot)\|_{L^q(\mathbb{R}^n)} < \epsilon \|k(x, \cdot)\|_{L^q(\mathbb{R}^n)}
\]
for large \( N \). Now if we argue as in the case where \( k \) is bounded, we obtain the desired result.

4. Proof of Theorem 2. Let \( f \in C^m_\infty(\mathbb{R}^n) \) such that \( f(x) = 1 \) if \( |x| < 1 \), \( f(x) = 0 \) if \( |x| > 2 \) and \( 0 < f(x) < 1 \). We define
\[
k(x, y') = \begin{cases}
\Phi^{-1}_m(|y'|^{-1}) & \text{if } |y'| < |x|^{-1}, \\
-\Phi^{-1}_m(|y'|^{-1}) & \text{if } |y'| > |x|^{-1}
\end{cases}
\]
for \( |x| \) sufficiently large and \( k(x, y') = 0 \) otherwise.

Then for large \( |x| \) and \( \epsilon < 1 \) we have
\[
Kf(x) = \int_{|x'| > 1} k(x, y') f(x - y') \, dy' = \int_{|x'| < 1} k(x, y') f(x - y') \, dy'
\]
\[
\geq C \int_{|x'| < 1} \Phi^{-1}_m(|y'|^{-1}) \, dy' = C \int_{|x'| < 1} \Phi^{-1}_m(|y'|^{-1}) \, dy' = C |x|^m
\]
Furthermore from (3.4) we have
\[
\int_{x_{n-1}} \Phi \left( \frac{|k(x, y')|}{|x_{n-1} - y'|} \right) \, d\sigma(y') \leq C \text{ for each } x \in \mathbb{R}^n.
\]
Hence for \( M_1 > 0 \) sufficiently large we obtain
\[
\int_{|x| > M_1} |Kf(x)| \, d\sigma \geq C \int_{|x| > M_1} \Phi^{-1}_m(|x^{-1}) \, dx' = I.
\]
Now if we set \( \Phi(u) = \Phi(u)^{1/(m-1)} \), from (2.2) and (2.4), we get
\[
I \geq C \int_{M_1} u \Phi(u) \Phi(u)^{-(m-1)} \, du \geq C \int_{M_1} \Phi(u)^{1/(m-1)} \, du = -\infty.
\]
5. Proof of Theorem 3. Let \( \varphi(t) = t^a \) with \( a > 0 \), \( t > 0 \). We define \( f(x) = \varphi(|x|) \) if \( |x| < 1 \) and \( f = 0 \) otherwise, and
\[
k(x, y') = \begin{cases}
|y'|^{-a} & \text{if } |x' - y'| \leq |x|, \ |x| < 1, \\
-|y'|^{-a} & \text{if } |x' + y'| \leq |x|, \ |x| < 1, \\
0 & \text{if } |x| \geq 1,
\end{cases}
\]
where \( \gamma > \beta > 0 \) such that \( \gamma + \beta = (n-1)/q \). Since
\[
\int_{|x'| < 1} |y'|^{-a} \, d\sigma(y') = C |x|^{-\gamma - a} \leq C |x|^{-\beta - a} \leq C,
\]
we have
\[
\int_{x_{n-1}} |k(x, y')|^q \, d\sigma(y') = 2 |x|^{-\gamma - a} \int_{|x'| < 1} |y'|^{-a} \, d\sigma(y') \leq C.
\]
If we set \( \Sigma = \{y' \in \Sigma_{n-1}: k(x, y') \geq 0\} \), then
\[
Kf(x) = \int_{|x'| > 1} k(x, y') f(x - y') \, dy' = \int_{|x'| > 1} f(x - y') \, t \, dt.
\]
Now denoting by \( \langle x', y' \rangle \) the angle between \( x' \) and \( y' \) and putting \( d = |x| \sin \langle x', y' \rangle \), \( \mu = \cos \langle x', y' \rangle \), we have for \( |x| < 1 \) that \( f(x - t) = \varphi_d(t) = \varphi(t + d^2)^{1/2} \) and \( \varphi_d \) denote the characteristic function of the interval \( I_d = [-1/2, 1/2] \). If we denote by \( H \), the Hilbert transform and the truncated Hilbert transform, respectively, we can write
\[
Kf(x) = \pi \int_{\Sigma} \Phi(x, y') H(x, \varphi_d) (\mu) \, d\sigma(y')
\]
and we have \( H(x, \varphi_d) (x) = H(x, \varphi_d) (x) \); \( x > 0 \), if \( x \to 0 \), and then we obtain
\[
Kf(x) = \pi \int_{\Sigma} \Phi(x, y') H(x, \varphi_d) (\mu) \, d\sigma(y')
\]
If \( y' \in \Sigma_{n-1}, |x| < 1 \), we have \( \cos \langle x', y' \rangle > 0 \) and \( \mu \in I_d \), then as \( \varphi_d \) is even, it follows
\[
H(x, \varphi_d) (\mu) \geq H(x, \varphi_d) (\mu).
\]
Furthermore, \( H(x, \varphi_d) \) is the Hilbert transform; then from (5.2) and (5.3) we get
\[
Kf(x) \geq \pi \int_{\Sigma} \Phi(x, y') \, d\sigma(y')
\]
Let us suppose for a moment that there exists \( C_d > 0 \) such that
\[
H(x, \varphi_d) \geq C_d \frac{1}{x} \text{ if } x \to +\infty.
\]
then from (5.4) we obtain for small $|\varepsilon|$

$$K_\varepsilon(\varepsilon) \geq C_\varepsilon \int B(x,y) d^m \mu^{-1} d\sigma(y')$$

$$\geq C_\varepsilon |\varepsilon|^{-\alpha-n} \int |y'-y|^{-\beta-n} d\sigma(y').$$

If $\alpha + \gamma < n$, from (5.1) the last term of the inequality above exceeds $C_\varepsilon |\varepsilon|^{-(\alpha+n+n)} = C_\varepsilon |\varepsilon|^{-\alpha-\beta-n}$. Now, as a consequence of the hypothesis made on $q$ the interval $I = [n/p+n-(n-1)/q, 2n/p]$ is non-empty, hence taking $\alpha > 0$ such that $2\alpha \in I$, we obtain $|K_\varepsilon(\varepsilon)| \geq C_\varepsilon |\varepsilon|^{-\alpha}$ for small $|\varepsilon|$.

Finally, we will show (5.5). As $\varphi_1$ is continuous, bounded and $\varphi_1 \in L^p(E)$ for some $p > 1$ we have

$$I_\varepsilon(x) = \int_t^{+\infty} \varphi_1(t)Q(e,-t)dt - \mathcal{H}p_1(\varepsilon) \quad \text{for each } x \in E,$$

where

$$Q(e,t) = \frac{1}{\pi} \frac{t}{e^t + \varepsilon^t}.$$

Changing variables we obtain

$$I_\varepsilon(x) = \int_t^{+\infty} Q(e,t) \int_t^{+\infty} -\varphi_1(s)X_{[0, t]}(s) ds dt$$

$$= \int_t^{+\infty} Q(e,t) \int_t^{+\infty} -\varphi_1(s) [X_{[0, t]} - X_{[0, t-\varepsilon]}] ds dt,$$

where $X_{[0, t]}$ denote the characteristic function of the interval $[0, t]$. If in the integral above we integrate on $0 < t < \varepsilon$ and $t > \varepsilon$, separately, then we can change the order of integration and get

$$I_\varepsilon(x) = \int_t^{+\infty} -\varphi_1(s) \int_t^{+\infty} Q(e,t)X_{[0, t]} ds ds +$$

$$+ \int_s^{+\infty} -\varphi_1(s) \int_s^{+\infty} Q(e,t)X_{[0, t]} ds ds$$

$$= \int_t^{+\infty} -\varphi_1(s) \int_s^{+\infty} Q(e,t) ds ds + \int_s^{+\infty} -\varphi_1(s) \int_s^{+\infty} Q(e,t) ds ds$$

$$+ \int_s^{+\infty} -\varphi_1(s) \int_s^{+\infty} Q(e,t) ds ds + \int_s^{+\infty} -\varphi_1(s) \int_s^{+\infty} Q(e,t) ds ds$$

$$= \frac{1}{\pi} \int_{+\infty}^{+\infty} -\varphi_1(s) \log \left( \frac{x^2 + (x+s)^2}{x^2 + (x-s)^2} \right) ds.$$

Now if $\varepsilon \to 0$, using Fatou's lemma we obtain

$$\mathcal{H}p_1(\varepsilon) \geq \frac{1}{\pi} \int -\varphi_1(s) \log \left( \frac{x+s}{x-s} \right) ds$$

but if $0 < s < \varepsilon$, we have

$$\log \left( \frac{x+s}{x-s} \right) \geq \frac{2s}{x-s};$$

then

$$\mathcal{H}p_1(\varepsilon) \geq \frac{1}{\pi} \int_{-\varepsilon}^{+\varepsilon} -\varphi_1(s) ds \geq C_\varepsilon \frac{1}{\varepsilon} \quad \text{if } \varepsilon \to +\infty.$$

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References


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