The isomorphic problem of envelopes

by

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Abstract. It is shown that there is a separable Banach space $X$ which has no separable isomorphic envelope, i.e. there is no separable space $Y$ such that whenever $Z$ is separable and finitely representable in $X$, $Z$ embeds isomorphically into $Y$. This strengthens Stern's solution to the (isometric) problem of envelopes posed by Lindenstrauss and Pełczyński.

1. Introduction. The notion of an envelope of a Banach space was introduced by Lindenstrauss and Pełczyński in [6]: A Banach space $Y$ is an envelope of a Banach space $X$ if $Y$ is finitely representable in $X$ and each space $Z$, which is finitely representable in $X$ and whose density character does not exceed that of $Y$, embeds isometrically into $Y$. Let us recall that, given $\lambda \geq 1$, a space $Y$ is said to be finitely $\lambda$-representable in $X$ if for each $\varepsilon > 0$ and each finite-dimensional subspace $F \subset Y$ there exists a subspace $E \subset X$ satisfying $d(E, F) < \lambda + \varepsilon$, where $d$ denotes the Banach–Mazur distance. $Y$ is called to be finitely representable in $X$ if it is finitely $1$-representable. The density character of a space $X$ is the least cardinal $\kappa$ such that $X$ possesses a dense subset of power $\kappa$.

Lindenstrauss and Pełczyński [6] proved that $L_p$ is an envelope of $L_q$, $1 < p < \infty$, and posed the problem whether every separable Banach space has a separable envelope. This problem was solved by Stern [10] who showed that there exists an equivalent norm on $L_q$ arbitrarily close to the original one so that the resulting space $X$ has no separable envelope.

Clearly, the notion of an envelope is an isometric one, but it has a natural isomorphic counterpart: Let us say that a Banach space $Y$ is an isomorphic envelope of a space $X$ if $Y$ is finitely representable in $X$ and each space $Z$ which is finitely representable in $X$ and whose density character does not exceed that of $Y$ embeds isomorphically into $Y$.

Note that formally one could still replace the phrase "finitely representable" by "finitely $\lambda$-representable for some $\lambda"", but this would not lead to essential changes. Indeed, a Banach space $Y$ which is finitely $\lambda$-representable in $X$ is $\lambda$-isomorphic to a space which is finitely $(1)$-representable in $X$. 
Roughly speaking, a separable envelope of $X$ is an isometrically universal member within the class of all separable "local subspaces" of $X$, while a separable isomorphic envelope is an isomorphically universal member of the same class.

With the definition above, the isomorphic problem of envelopes arises: Does every separable Banach space have a separable isomorphic envelope? It follows from the results of [6] that each separable $L_p$-space ($1\leq p < \infty$) has a separable isomorphic envelope. So do all separable subspaces of $L_p$-spaces for $2\leq p < \infty$ since each such space $X$ is either isomorphic to $l_p$ or contains an isomorphic copy of $l_p$ (cf. [4], [6]). In the latter case $l_p$ is finitely representable in $X$ ([5]) and $l_p$ will be an isomorphic envelope of $X$.

The space constructed by Stern [10] has an isomorphic envelope, because it is isomorphic to a Hilbert space. Moreover, his method is based essentially on the metric geometry of $l_p$ and does not carry over to the isomorphic context.

The aim of this paper is to show that there exists a separable Banach space without separable isomorphic envelope. The space we present has a simple representation: It is an $l_p$-sum of $l_p$-spaces. This way we also get an alternative counter-example to the isomorphic problem. Our proof involves ultrapowers of Banach spaces, which enable us to describe the structure of a possible envelope (this was also used in [10]), and the local incomparability of $L_p$-spaces for different $p > 2$, established in [9].

We now introduce some notation. Given $p$, $1 \leq p < \infty$, a set $I$ and a family of Banach spaces $\{X_i\}_{i \in I}$, we denote by $\{\sum_{i \in I}^{\ell} X_i\}_{\ell \in I}$ the space of all families $(x_i)_{i \in I}$ with $x_i \in X_i$ and $\|\sum_{i \in I}^{\ell} x_i\| = \left( \sum_{i \in I}^{\ell} \|x_i\|^p \right)^{1/p} < \infty$. If $I = \mathbb{N}$ and $X_i = X$, we use the notation $\ell_p(X)$. Furthermore, given a Banach space $X$ and a $\sigma$-additive measure $\mu$ on a measure space $(\Omega, \Sigma)$, we denote by $L_p(\mu, X)$ the space of $X$-valued $\mu$-measurable $\sigma$-additive functions $f$ for which $\int_{\Omega} \|f\|^p d\mu$ is finite. We write $L_p(\mu, X)$ instead of $L_p(\mu, X)$ if $\mu$ is the Lebesgue measure on $[0, 1]$.

Next recall the definition of an ultrapower, introduced in [2]: Let $U$ be an ultrafilter on a set $I$, and let $\{X_i\}_{i \in I}$ be a family of Banach spaces. Denote by $N_U$ the closed subspace of $\{\sum_{i \in I}^{\ell} X_i\}_{\ell \in I}$ which consists of all families $(x_i)$ satisfying $\lim_{\ell \to \infty} \sup_{i \in I} \|x_i\| = 0$. Then the ultrapower $(X_U)$ is defined to be the quotient space $\{\sum_{i \in I}^{\ell} X_i\}_{\ell \in I}/N_U$, equipped with the usual quotient norm. If $(x_{i\ell})$ denotes the equivalence class determined by $(x_i)$, then the norm can be computed as $\|x_{i\ell}\| = \lim_{\ell \to \infty} \|x_i\|$. If all of the spaces $X_i$ are identical with some $X$, we speak of an ultrapower, $(X_U)$. Given operators $T_i: X_i \to X_{i\ell}$ with $\sup_{i \in I} \|T_i\| < \infty$, we can define their ultraproduct $(T_{i\ell})_{\ell}: (X_{i\ell})_{\ell} \to (Y_{i\ell})_{\ell}$ in a canonical way by setting $(T_{i\ell})_{\ell}(x_i)_{\ell} = (T_i x_i)_{\ell}$. Finally, we say that an ultrafilter $U$ on a set $I$ is countably incomplete if there exists a sequence of sets $D_n \in U$ with $\bigcap_{n=1}^{\infty} D_n = \emptyset$. For the basic facts concerning ultraproducts of Banach spaces we refer to [2], [11] and the survey [3].

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2. The counter-example. We first state the main result, which will be proved through a series of lemmas. $Q$ denotes the set of rational numbers.

**Theorem.** Let $2 < a < b < r < \infty$ and let $X = \bigcup_{\ell \in \mathbb{Q} \cap [0,1]} L_p(\ell_p, X)$. Then $X$ has no separable isomorphic envelope.

Throughout this section $a$, $b$, $r$ and $X$ will be fixed as defined in the Theorem. The first two lemmas concern the local structure of those subspaces of $X$ which correspond to subsets of the interval $[a, b]$. More precisely, we shall investigate the question whether or not $l_p$ is finitely representable in these spaces. In this connection we shall frequently use a result of Krivine [5], stating that if, for some $\lambda$, $l_p$ is finitely $\lambda$-representable in a Banach space $Z$, then $l_p$ is finitely representable in $Z$. The first lemma is just a reduction result which will simplify the proof of Lemma 2.

**Lemma 1.** Let $A = [a, b]$ be a closed set and let $p \in (a, b)$. Assume that $l_p$ is finitely representable in $\{\sum_{\ell \in Q}^{\ell} \ell_p(\ell_p, X)\}$. Then there exists an $\ell \in A$ such that $l_p$ is finitely representable in $\ell_p(l_p)$. Let $\mu$ be a $\sigma$-additive measure on $[a, b]$ and let $X$ be a Banach space.

**Proof.** If $l_p$ is finitely representable in a direct sum of two Banach spaces, then $l_p$ is finitely representable in at least one of the summands. Indeed, it is a standard trick, that if a direct sum of two Banach spaces contains an isomorphic copy of $l_p$, then so does one of the summands. By means of ultrapowers, this fact is easily localized (cf. [3], Ch. 6 for this kind of argument), and an application of Krivine’s result yields the desired statement.

Using this, one can find a sequence $(A_n)$ of closed subsets of $A$ such that, for all $n$, $A_n \supseteq A_{n+1}$, diam $(A_n) < 1/n$, and $l_p$ is finitely representable in $\{\sum_{\ell \in Q}^{\ell} \ell_p(\ell_p, X)\}$. Define $s \in A$ to be the unique common point of the sets $A_n$, i.e. $\{s\} = \bigcap_{n} A_n$. Fix $m \in N$ and $\varepsilon > 0$. By a result of Pelczynski and Rosenthal [9], there exists a $\delta > 0$ (depending on $m$ and $\varepsilon$ only) such that the following holds: If $|s - t| < \delta$, then every $m$-dimensional subspace of $l_p$ is $(1 + \varepsilon)$-isomorphic to a subspace of $l_p$. Now choose $n$ so that $1/n < \delta$ and let $E$
be an $m$-dimensional subspace of $\left( \sum E_n \right)_{\sigma_{rad}}$, with $d(E_n, E_p) < 1/\epsilon$. There exist $m$-dimensional subspaces $E_i \subset L_p$ such that $E \subset \left( \sum E_i \right)$. Each of the spaces $E_i$ is $(1 + \epsilon)$-isomorphic to some subspace of $L_p$, thus $\left( \sum E_i \right)$, and in particular $E_i$ is $(1 + \epsilon)$-isomorphic to a subspace of $L_p(\mathbb{N})$. Consequently, $L_p(\mathbb{N})$ contains a $(1 + \epsilon)$-isomorph of $C_p$. Since $\epsilon$ and $m$ were arbitrary, this concludes the proof.

**Lemma 2.** Let $A = [a, b]$ be a closed set and let $p \in [a, b] \setminus A$. Then $L_p$ is not finitely representable in $\left( \sum E_i \right)_{\sigma_{rad}}$.

**Proof.** In view of Lemma 1 it suffices to show that for $p, q \in [a, b]$, $p \neq q$, $L_p$ is not finitely representable in $L_q(\mathbb{N})$. For technical reasons, we shall actually prove that $L_p$ is not finitely representable in $L_q(\mathbb{N})$.

We assume the contrary, i.e., for each $n$ there exists a subspace $E_n \subset L_q(\mathbb{N})$ with $\dim E_n = n$ and

$$d(E_n, E_p) < 1 + 1/n.$$  

The elements of $E_n$ are vector-valued functions, so let us consider the set of "norm functions" $S_n : L_p \to \mathbb{R}$:

$$S_n = \{ f \in L_p : \text{There exists an } x \in E_n, x = f(t) \text{ with } \|x(t)\| = f(t) \text{ a.e.} \}.$$  

We shall apply the method of weighted $L_p$-norms, developed by Pekarzyaki and Rosenthal in [9], to these sets $S_n$. Let $\Phi$ be the set of all measurable functions $\phi$ on $[0, 1]$ satisfying $\phi(t) > 0$ for all $t \in [0, 1]$ and $\int_{[0, 1]} \phi = 1$. First we show that, taken on $S_n$, the ratio of the $L_p$-norm to weighted $L_p$-norm tends to infinity with $n \to \infty$, more precisely

$$\liminf_{n \to \infty} \frac{\inf_{\phi \in \Phi} \sup_{x \in E_n} \|f\|_{L_p} \phi^{(n)}}{\sup_{x \in E_n} \|f\|_{L_{\phi^{(n)}}}} = \infty.$$  

Assume that this is not the case. Then there exist a constant $C$ and, for each $n$, a function $\phi_n \in \Phi$ so that for all $f \in E_n$,

$$\|f\|_{L_p} \leq C\|f\|_{L_{\phi^{(n)}}}.$$  

Let $\mu_n$ be the probability measure on $[0, 1]$ defined by $d\mu_n = \phi_n \, dt$. We get for $f \in S_n$,

$$C^{-1} \|f\|_{L_p} \leq \|f\|_{L_{\phi^{(n)}}} \leq \|f\|_{L_{\phi_n}} \leq \|f\|_{L_{\phi^{(n)}}} = \|f\|_{L_p}.$$  

Thus

$$C^{-1} \|f\|_{L_p} \leq \|f\|_{L_{\phi^{(n)}}} \leq \|f\|_{L_p} \quad (f \in S_n).$$

We now define an operator $T_n : E_n \to L_p(\mu_n)$ by setting $(T_n x)(t) = \phi_n^{-1/(m+1)} x(t)$, for $x \in E_n$, $t \in [0, 1]$. Then (3) shows that

$$C^{-1} \|x\| \leq \|T_n x\| \leq C \|x\| \quad (x \in E_n).$$

Since $L_p(\mu_n)$ is isometric to $L_p$, the last inequality together with (1) implies that $L_p$ is finitely $C$-representable in $L_p$. But this is known to be impossible [9]. We have got a contradiction which proves (2).

Now we proceed with the application of the argument from [9]. The proof of Proposition 3.1 of [9] combined with (2) yields the following:

For each $\delta > 0$ and $m \in N$ we can find an $n_0 \in N$ such that whenever $n > n_0$, there exist functions $f_1, \ldots, f_m \in S_n$ of norm one and measurable sets $A_1, \ldots, A_m \subset [0, 1]$ satisfying

$$\frac{1}{n} \int_{[0, 1]} \left| \int_{A_i} \phi_n^{-1} \right| \leq \frac{1}{n} \int_{A_j} \phi_n^{-1} < \frac{1}{n} \int_{A_j} \phi_n^{-1} < \frac{1}{n} \int_{A_j} \phi_n^{-1} = \frac{1}{n} \int_{[0, 1]} \phi_n^{-1}.$$  

Define $B_i = A_i \setminus \bigcup_{j \neq i} A_j$, choose $a_1, \ldots, a_m \in E_n$ so that $\|a_i\| = \int_{[0, 1]} \phi_n^{-1} (a_i, t)$ and put $y_i = x_{n_0} y_i$. The $y_i$'s are disjointly supported vector-valued functions in $L_p(\mathbb{N})$, consequently, they span a subspace isometric to $C_p$. On the other hand, it follows from (4) that

$$\|y_i - y_j\|_{L_p(\mathbb{N})} = \|f_i - f_j\|_{L_p(\mathbb{N})} < (m+1)^{\delta/\epsilon}.$$  

If now $\delta$ is chosen small enough ($\delta < (4m)^{-1} \epsilon$ will suffice), span $(y_i)_{i \in \mathbb{N}}$ is $(1 + \epsilon/m)$-isomorphic to span $(y_i)_{i \in \mathbb{N}}$, hence to $C_p$. (Recalling (1) again, we derive that $L_p$ is finitely representable in $L_p$, contradicting the fact [8] that $L_p$ and all spaces which are finitely representable in it are of type $\mathcal{P}$ while $L_p$ is not (2 $\leq p < \infty$, by assumption). This accomplishes the proof of Lemma 2.

Let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$. In the next lemma, which is the crucial part of the proof of the Theorem, we shall study isomorphic embeddings of $L_p$ into $[X]^{\mathcal{U}}$. We cannot expect a full description of these embeddings, but we will get some information about the location of $L_p$-isomorphs with respect to some natural decomposition of $[X]^{\mathcal{U}}$.

For this, let us first introduce some more notation. Given a set $A \subset [a, b]$, we denote by $P_A$ the canonical projection of $X$ onto the subspace corresponding to the sum $\bigoplus_{i \in A} E_i$. The ultrapower of $P_A$ will be denoted by $Q_A$, thus $Q_A = (P_A)^{\mathcal{U}}$. Furthermore, if $(A_n)$ is a sequence of subsets of $[a, b]$, we define $Q_{A_n}$ to be the ultraproduct of $P_{A_n}$, the sequence of projections $(P_{A_n})$. It is easy to check that $Q_A$ and $Q_{A_n}$ are projections, acting in $[X]^{\mathcal{U}}$.

**Lemma 3.** Let $p \in (a, b)$, let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$, and let $Z$ be a subspace of $[X]^{\mathcal{U}}$ isomorphic to $L_p$. Then there exists an element
Then the restriction of \( Q_{[a,b] \setminus I} \) to the closure of \( \text{span}(z_m; n > m) \) is an isomorphism, which means that \( I_\rho \) embeds isomorphically into \( \text{Im} Q_{[a,b] \setminus I} \). It is readily checked that

\[
\text{Im} Q_{[a,b] \setminus I} = \left( \text{Im} P_{[a,b] \setminus I} \right)^\perp.
\]

Therefore \( I_\rho \) is finitely \( \lambda \)-representable for some \( \lambda \) (and hence finitely representable) in

\[
\text{Im} P_{[a,b] \setminus I} = \left( \sum_{\sigma \in [a,b]} P_{(a,b) \setminus [\sigma]} \right)^\perp.
\]

Contradicting Lemma 2. This shows that (8) cannot hold and we can find \( u_{k+1} \), as required. This completes the induction.

Next we define \( s_k = Q_{[a,b] \setminus I} u_k \). According to (6) and (7), we have

\[
|u_k - s_k| \leq \varepsilon^{2k-1}.
\]

As a block-basis of \( (u_k) \), the sequence \( (u_k) \) is equivalent to the unit vector basis of \( I_\rho \). Moreover, its basis constant does not exceed \( \|T\| \|T^{-1}\|^{-1} \). By (9), the choice of \( s_k \) and a well-known perturbation result [(7)], 1.a.9), \( (s_k) \) is equivalent to \( (u_k) \), consequently to the unit vector basis of \( I_\rho \).

On the other hand, the projections \( Q_{x_j} \) form an \( \text{tr} \)-decomposition of \( (X)_X \). Precisely, if \( A_1, \ldots, A_m \) are mutually disjoint subsets of \( [a,b] \) and \( s_k = (s_k(1), \ldots, s_k(m)) \) are elements of \( (X)_X \), then

\[
\sum_{i=1}^{m} Q_{A_i} s_k = \lim_{\delta \to 0} \sum_{i=1}^{m} P_{A_i} s_k = \left( \sum_{i=1}^{m} \|Q_{A_i} s_k\|_2 \right)^\perp = \left( \sum_{i=1}^{m} \|P_{A_i} s_k\|_2 \right)^\perp.
\]

A look at the definition of \( (s_k) \) shows now that it must be equivalent to the unit vector basis of \( I_\rho \), which is a contradiction. This completes the proof of (9) and thus of the lemma.

Proof of the Theorem: Assume that \( Y \) is a separable isomorphic envelope of \( X \). Let \( U \) be a non-trivial ultrafilter on \( N \). By definition, \( Y \) is finitely representable in \( X \), therefore it is isometric to a subspace of \( (X)_X \) (cf. [3], Th. 6.3). In the sequel we thus assume \( Y \subset (X)_X \). It is easily seen that for each \( \sigma \in (a,b) \), \( I_\rho \) is finitely representable in \( X \). Hence \( Y \) contains a subspace isomorphic to \( I_\rho \). An application of Lemma 3 shows that there exists an element \( s_k \in Y \) and a sequence of open intervals \( (I_{k+1}) \) with \( p \in I_{k+1} \), \( I_{k+1} = [a,b] \), \( \lambda_{I_{k+1}}(s_k) \leq \varepsilon^{2k-1} \). Choose \( k \) so that \( u_1, \ldots, u_k \in \text{span}(z_m; 1 \leq m \leq n) \). Suppose that we could not find an \( u_{k+1} \), or equivalently, that for all \( u \in \text{span}(z_m; n > m) \)

\[
|Q_{[a,b] \setminus I} u| \geq 2^{-(k+1)}|u|.
\]
Isomorphic problem of envelopes

(1) The Continuum Hypothesis.

(2) Each Banach space of density character \( \leq \omega_1 \) has an isomorphic envelope of density character \( \leq \omega_1 \).

(3) Each Banach space of density character \( \leq \omega_1 \) has an (isometric) envelope of density character \( \leq \omega_1 \).

Stern proved (1) \( \Rightarrow \) (3). The implication (1) \( \Rightarrow \) (3) was derived from a model-theoretic result of Keisler (cf. [10], Th. 5 and [1], Ch. 6). In exactly the same manner it can be deduced from [1] (combine Prop. 5.1.6 (vi) and Th. 5.1.16) that if we assume the Generalized Continuum Hypothesis, then each non-separable Banach space \( X \) has an envelope \( Y \) of the same density character as \( X \).

References


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