Reflexive Banach spaces without equivalent norms which are uniformly convex or uniformly differentiable in every direction

by

D. N. KUTZAROVA and S. L. TROYANSKI (Sofia)

Abstract. An example is given of a reflexive Banach space which fails to have either an equivalent norm that is uniformly convex in every direction or an equivalent norm that is uniformly differentiable in every direction.

1. It is shown in [4] that every reflexive Banach space has an equivalent norm that is locally uniformly convex and Fréchet differentiable. This led (see [2]) to the question whether every reflexive Banach space has an equivalent norm which is uniformly convex in every direction or uniformly differentiable in every direction. We give a negative answer to this question:

THEOREM 1.1. There is a reflexive Banach space $Y$ such that:

(i) $Y$ has no equivalent norm which is uniformly convex in every direction.

(ii) $Y$ has no equivalent norm which is uniformly differentiable in every direction.

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2. Definitions and notations. The norm of a Banach space $X$ is uniformly convex in every direction if $\|x_n - y_n\| \to 0$ whenever $(x_n)$ and $(y_n)$ are sequences in $X$ such that $|x_n| = |y_n| = 1$, $|x_n + y_n| \to 2$, and there is a $s$ and $(\lambda_n)$ with $x_n - y_n = \lambda_n s$.

The norm of a Banach space $X$ is uniformly differentiable in every direction if, for every $y \in X$ with $|y| = 1$,

$$\lim_{r \to 0} \sup \{ \|x + ry\| + \|x - ry\| - 2 : x \in X, |x| = 1 \} = 0.$$ 

Let $\Phi_r$ be a family of finite subsets of $I$ such that $\Phi_r$ contains all one-point subsets of $I$ and all subsets of members of $\Phi_r$. We denote by $\lambda_1(\Phi_r)$ the space of all real-valued functions $x$ on $I$ such that

$$|x| = \sup \left( \sum_{i} \left| \sum_{r \in A_i} x(r) \right| \right)^{|A|} < \infty,$$
where the supremum is taken over all finite systems \( \{A_i\}_{i=1}^n \) with each
\( A_i \in \Phi_r \) and \( A_i \cap A_j = \emptyset \) if \( i \neq j \).

Since each \( A_i \) in (1) can be contained in the support of \( w \), we have
\( \|w+y+z\| > (\|x\| + \|y\| + \|z\|)^{\alpha \beta} \), if \( x \) and \( y \) have disjoint supports. Therefore
\( \lambda_{s_1}(\Phi_r) \) is the completion of the space of all \( f \) with finite support.

Also, if \( e_1, \ldots, e_r \) is the natural unconditional basis for \( \lambda_{s_1}(\Phi_r) \) defined by
\( e_i(\theta) = \delta_{i \theta} \), then this basis is boundedly complete, meaning that the series
\( \sum_{i=1}^{\infty} a_i e_i \) converges unconditionally whenever
\[ \sup \left\{ \left\| \sum_{i=1}^{\infty} a_i e_i \right\| : A = \Gamma, |A| < \infty \right\} < \infty. \]

3. Reflexivity of \( \lambda_{s_1}(\Phi_r) \).

**Lemma 3.1.** Let \( \{e_i\}_{i=1}^r \) be the natural basis in \( \lambda_{s_1}(\Phi_r) \) with biorthogonal functionals \( \{e_i^*\}_{i=1}^r \). If \( A \in \Phi_r \), then
\[ \left\| \sum_{i \in A} e_i \right\| = |A|, \quad \left\| \sum_{i \in A^c} e_i^* \right\| = 1. \]

The proof is straightforward.

**Lemma 3.2.** \( \Phi_r \) have the property that, for every \( A \in \Phi_r \) with \( |A| > 2 \), there is a positive integer \( k(A) \) such that \( |B| < k(A) \) if \( B \in \Phi_r \) and \( B \cap A = \emptyset \).

Then \( \lambda_{s_1}(\Phi_r) \) is reflexive.

Proof. Recall that uniform convexity of \( l_1 \) implies that \( \|x+y+z\| \)
for \( x, y, z \) in the unit ball of \( l_1 \), that \( x, y, z \) are nearly equal. Thus there is a positive number \( \epsilon \) such that, if \( \epsilon \), \( \epsilon \), and \( \epsilon \), \( \epsilon \), \( \epsilon \), \( \epsilon \), and \( \epsilon \) are in the unit ball of \( \lambda_{s_1}(\Phi_r) \) and have disjoint supports in \( \Gamma \), and if \( |B| \) is a finite system with each \( B_j \in \Phi_r \), \( B_1 \cup B_2 = \emptyset \) if \( i \neq j \), and
\[ \left\{ \sum_{i \in A} (x_i + y_i + z_i) |\gamma_i| \right\}^{\alpha \beta} \]
(2)
\[ \left( \sum_{i \in A} (x_i + y_i + z_i) |\gamma_i| \right)^{\alpha \beta} > 5/2 \]
if \( A_i \) is obtained from \( (B_j) \) by deleting each \( B_j \) that either contains no point in the support of \( x_i \) or contains one point in the support of \( x_i \) and no point in the support of \( x_i \).

Suppose \( \lambda_{s_1}(\Phi_r) \) is not reflexive. Then there is (see (3)) a bounded sequence \( \{x_i\} \) and a positive integer \( \epsilon \) such that
\[ \text{dist}(\text{conv} \{x_1, \ldots, x_n\}, \text{conv} \{x_i : i > n\}) > \epsilon > 1. \]

Since each \( x_i \) can have finite support, we can replace \( x_i \) by a subsequence \( \{y_i\} \) for which all followers of \( y_i \) are approximately equal on the union of the supports of \( x_i \) and its predecessors, for each \( x_i \). Since the basis \( e_i \) is boundedly complete, there is no loss of generality if we assume that the sequence \( \{x_i\} \) which satisfies (3) also has the property that all followers

of \( x_i \) are zero on the union of the supports of \( x_i \) and its predecessors, for each \( i \).

Let
\[ \tau(1 - \frac{1}{\epsilon}) = \liminf \left\{ \|w\| : w \in \text{conv} \{x_i : i > n\} \right\}. \]

and choose \( M > \tau(1 - \frac{1}{\epsilon}) \) if \( w \in \text{conv} \{x_i : i > M\} \). Since \( \tau > 0 \), there is a sequence \( \{w_i\} \) that satisfies (3), with \( w_i \in \text{conv} \{x_i : M < i \leq P_i\} \), \( w_i \in \text{conv} \{x_i : P_i < i \leq P_{i+1}\} \), etc., and
\[ \|w_i\| < \frac{1}{\epsilon} \quad \text{for each } i. \]

Let \( S_1 \) and \( S_2 \) be the supports of \( w_i \) and \( x_i \), and let \( K \) be an upper bound for \( \sum |A_i| \) for all finite systems \( \{A_i\} \) with each \( A_i \in \Phi_r \), \( A_i \cap A_j = \emptyset \) if \( i \neq j \), and each \( A_i \) containing at least two points of \( S_1 \cup S_2 \). Choose any \( \epsilon > 0 \) and let
\[ w = (1/2K) \sum_{i=1}^{\infty} w_i. \]

Then \( w(\gamma) < \frac{1}{\epsilon} K \) for each \( \gamma \in \Gamma \). Because of (3), there is a finite system \( \{A_i\} \) with each \( A_i \in \Phi_r \), \( A_i \cap A_j = \emptyset \) if \( i \neq j \), each \( A_i \) containing at least two points of \( S_1 \cup S_2 \), and
\[ \left\{ \sum_{i \in A} (\sum_{j \in A} w^i_j + w^i_j(\gamma)) |\gamma_i| \right\}^{\alpha \beta} > 5/2. \]

This is false since \( \|w_i\| < \frac{1}{\epsilon} \), \( \|w_i\| < \frac{1}{\epsilon} \) and
\[ \sum_{i \in A} |w(\gamma)| < K(\frac{1}{\epsilon} K) = \frac{1}{\epsilon} K \]
imply the left member of (4) is less than \( 5/2 \).

4. Proof of Theorem 1.1. Let \( A = \bigcap_{i=1}^{n} \bigcup_{i=1}^{n} \{A_i \} \). That is,
\( A \) is the family of all sequences \( \gamma = \gamma \in \mathbb{N} \) of positive integers such that,
\( 1 \leq \gamma_i \leq n+1 \).
We denote by \( \mathcal{P} \) the family of all finite subsets of \( A \) which have the property that, if \( A \in \mathcal{P} \), then there is a positive integer \( \epsilon \) such that,
\( \gamma_i = \gamma_i^{(1)} \) for \( 1 \leq i \leq n \), \( \gamma_i = \gamma_i^{(2)} \) for \( 1 \leq i \leq m-1 \). \( A \) family similar to \( \mathcal{P} \) is considered in (II). In the sequel, we shall use the following.

**Lemma 4.1.** \( \mathcal{P} = \bigcup_{A_i} \{A_i \} \). Then for some \( A_i \),
\[ \text{sup} \{ \text{dist}(A_i : A \in \mathcal{P}) : A \in \mathcal{P} \} = \infty. \]

Proof. For \( \delta = (\delta_i)_{i=1}^{\infty} \), we define \( \pi_n(\delta) = \delta_n P_n(\delta) = (\delta_n^{(1)})_{i=1}^{\infty}. \)
Suppose that for every \( I \)

\[
\omega_I = \sup \{|A| : A \in \mathcal{A}_I, A \in \mathcal{P}_A \} < \infty.
\]

We shall define inductively an increasing sequence of nonnegative integers \( \{n_I\}_{I=1}^{\infty} \) and a sequence of finite systems \( \{\delta^i\}_{i=1}^{\infty} \) with \( \delta^i \in \{1, 2, \ldots, i+1\} \), such that

\[
p^{-1}_i((\delta^i)^{2k+1}_i) \cap \bigcup_{k=1}^i \mathcal{A}_i = \emptyset.
\]

Let \( n_0 = 0 \). Suppose \( \{n_I\}_{I=1}^{n_k}, \{\delta^i\}_{i=1}^{2k+1} \) with the above property have been found. If \( p^{-1}_i((\delta^i)^{2k+1}_i) \cap \bigcup_{k=1}^i \mathcal{A}_i = \emptyset \), we take \( n_{k+1} = n_k \). If \( p^{-1}_i((\delta^i)^{2k+1}_i) \cap \bigcup_{k=1}^i \mathcal{A}_i \neq \emptyset \), we put \( n_{k+1} = \max(n_k + 1, m_{k+1}) \). There is \( \delta_{k+1} \in \mathcal{A}_{k+1} \) such that \( \delta^i = \pi_i(\delta_{k+1}) \) if \( 1 \leq i \leq n_k \). Put \( \delta^i = \pi_i(\delta_{k+1}) \) for \( n_k + 1 \leq i \leq n_{k+1} - 1 \). From (5) it follows that

\[
\left| \pi_{n_{k+1}+1}(\pi_{n_k+1}^{-1}((\delta^i)^{2k+1}_i) \cap \mathcal{A}_{k+1}) \right| \leq m_{k+1} \leq n_{k+1}.
\]

Consequently there is a positive integer \( n_{k+1} \), \( 1 \leq n_{k+1} \leq n_{k+1} - 1 \), such that \( p^{-1}_i((\delta^i)^{2k+1}_i) \cap \mathcal{A}_{k+1} = \emptyset \). From the construction, it is clear that \( n_k \to \infty \). For \( \delta = (\delta^i)_I \), we have \( \delta \in \mathcal{A}, \delta \notin \mathcal{A}_I, I = 1, 2, \ldots \), which contradicts the assumption \( \mathcal{A} = \bigcup_{I=1}^\infty \mathcal{A}_I \).

It follows from Lemma 3.2. that both \( \lambda^*_A(\mathcal{P}_A) \) and \( \lambda^*_A(\mathcal{P}_A) \) are reflexive.

Denote by \( Y \) the cartesian product of \( \lambda^*_A(\mathcal{P}_A) \) and \( \lambda^*_A(\mathcal{P}_A) \). It is simple to show that \( Y \) has the properties of Theorem 3.1, using Lemma 4.1, and the following

Proposition 4.2 \((\mathbb{C})\). Let \( X \) be a Banach space with unconditional basis \( \{v_n\}_{n=1}^\infty \). Then

(i) if the norm in \( X \) is uniformly convex in every direction, then for every \( \varepsilon > 0 \) there is a decomposition \( \Gamma = \bigcup_{I=1}^\infty \mathcal{I}_I^\varepsilon \) such that

\[
\inf \| \sum_{k=1}^\infty ||u_k|| : A \in \mathcal{I}_I^\varepsilon, |A| = 1 \| \geq \varepsilon^{-1}.
\]

(ii) if the norm in \( X \) is uniformly differentiable in every direction,

\[
\varepsilon^{-1}.
\]

(1) In [3] it is shown that the conditions in Proposition 4.2 are not only necessary but sufficient as well.

References


DEPARTMENT OF MATHEMATICS AND MECHANICS
UNIVERSITY OF NOVA

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