related to, but simpler than the problem above, which we feel are interesting even without their relation to Ulam's conjecture.

4.1. Approximate isometries and measure. We move the problem of §3 to a more amenable environment, hoping that a solution to the new problem would help in the old situation.

CONJECTURE. There is a constant \( c > 1 \) such that if \( S \) is a subset of some \( \mathbb{R}^m \), \( g: S \to \mathbb{R}^m \) is an \( \delta \)-isometry, and \( T = \{ x \in \mathbb{R}^m | d(x,g(S)) < c\delta \} \), then \( \lambda_\delta(T) \geq \lambda_\delta(S) \).

4.2. Neighborhoods in bricks. Let \( B \) be a brick (rectangular parallelepiped) in \( \mathbb{R}^n \), \( S \) and \( T \) isometric subsets of \( B \). It would seem that not too much more of \( S \) than of \( T \) can be near the boundary of \( B \). Precisely:

CONJECTURE. There is a \( c_\delta > 0 \) depending only on \( n \) such that for any \( \delta > 0 \)

\[ \lambda_\delta(\{ x \in B | d(x,S) \leq \delta \}) \leq \lambda_\delta(\{ x \in B | d(x,T) \leq c_\delta \delta \}). \]

In fact we guess that the best value for \( c_\delta \) is \( 1 + \sqrt{n} \), corresponding to \( S \) a small corner of the brick \( B \).

To see the connection between this conjecture and the problem described at the beginning of the section, let \( S' \to B \) be a subset of the brick \( B \) and \( g: S' \to \mathbb{R}^m \) an approximate isometry. Let \( f: S' \to \mathbb{R}^m \) be an isometry within, say, \( \delta \) of \( g \). Set \( T = f(S') \cap B \) and \( S = f^{-1}(T) \). Then the above conjecture implies that the \((1+c_\delta)\delta\) neighborhood of \( g(S') \) in \( B \) has measure at least to the measure of \( S' \).

References


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(1455)
is finite. If \( q(x) \) is an increasing function satisfying \( q(0) = 0 \), then the space \( \text{BMO}_q \) consists of all functions \( f \) for which

\[
I(f - I(f)) = 0 = \| q(I) \|.
\]

Spanne [10] has shown that if \( q(x) = x^a \), \( 0 < a < 1 \), then \( \text{BMO}_q \) coincides with the space of functions which are Lipschitz continuous of order \( a \).

Let \( f \) be holomorphic in \( D \). Let \( B(t) \) be Brownian motion in \( D \), and let \( \sigma \) be the first exit time of \( B(t) \) from \( D \). A theorem of Paul Lévy states that \( f(B(t)) \) is Brownian motion with a new time scale \( \sigma = \frac{1}{t} \int_0^t |f'(B(\tau))|^2 d\tau \).

Let \( \tilde{B}(t) \) be the motion obtained from this time change, and let \( \tilde{\sigma} \) be the image of \( \sigma \). That is, \( \tilde{\sigma} = \frac{1}{\sigma} \). By an abuse of notation, \( E^\tilde{\sigma} \) will mean that \( B(0) = x \) (so \( \tilde{B}(0) = f(x) \)).

\( q(x) \) is said to be regular if

\[
\theta \int_0^\infty (q(t)^a)^2 dt < \infty.
\]

For example, \( q(x) = x^a \), \( 0 < a < 1 \) is regular.

The following two theorems are not too difficult, and their proofs will be omitted.

**Theorem 1.** \( f \in \text{BMO} \) if and only if

\[
\sup_{|I|} E^\tilde{\sigma} < \infty
\]

and this supremum is comparable to \( ||f||_2^2 \).

**Theorem 2.** Let \( q(x) \) be regular. \( f \in \text{BMO}_q \) if and only if

\[
E^\tilde{\sigma} = 0 = \| q(1 - |x|)^2 \|.
\]

These theorems were motivated by a theorem of Burkholder [1] stating that \( f \in L^p \) if

\[
E^{\tilde{\sigma}} < \infty.
\]

For \( I \subseteq \partial D \) let \( B(I) \) be a square in polar coordinates with one edge coinciding with \( I \). Thus,

\[
B(I) = \{(r, \theta) : \theta \in I, 1 - |I|/2 \pi < r \leq 1\}.
\]

A measure \( \mu \) on \( D \) is said to be a Carleson measure if

\[
\mu(B(I)) = 0 = \| \mu(I) \|.
\]

Fefferman [2] showed that \( f \in \text{BMO} \) if \( d\mu = |f'(x)|^2(1 - |x|) dxdy \) is a Carleson measure. In Section 2, we prove the following

**Theorem 3.** Let \( q(x) \) be regular. Then \( f \in \text{BMO} \) if \( d\mu = |f'(x)|^2(1 - |x|) dxdy \) satisfies

\[
\frac{1}{|I|} \mu(B(I)) = 0 = \| q(I) \|.
\]

This theorem was conjectured by Sarason [9]. Probability often clarifies potential theory. In Section 3, we prove the following potential theoretic criterion for \( \text{BMO} \), due to Hayman and Pommerenke [4].

**Theorem 4.** Let \( G \subseteq C \) have the property that every function \( f \) analytic in \( D \) with values in \( G \) belongs to \( \text{BMO} \) if there exist constants \( B_{\delta} > 0 \) such that for all \( \omega_{\delta} \in G \),

\[
\text{cap}(G \cap \{ |\omega - \omega_{\delta}| < R \}) > \delta.
\]

2. Proof of Theorem 3. Let \( G(x, y) \) be the Green's function for the disc. From potential theory [5], we know that \( G(x, y) dy \) is the expected amount of time that Brownian motion starting at \( x \) spends in \( y \). Since \( B(t) \) is obtained from \( f(B(t)) \) by the time change \( T_1 = \int_0^t |f'(s)|^2 ds \), we can write

\[
E^{\tilde{\sigma}} = \int_D |f'(y)|^2 G(x, y) dy.
\]

Therefore, by Theorem 2, it suffices to show:

**Lemma.** Let \( q \) be regular. Then

\[
\int_D |f'(y)|^2 G(x, y) dy = 0 = \| q(1 - |x|) \|
\]

iff

\[
\frac{1}{|I|} \int_{\partial D} |f'(x)|^2(1 - |x|) dx = 0 = \| q(I) \|.
\]

Proof of (1) \( \Rightarrow \) (2). This part of the lemma is true even without the assumption of regularity. The result is known [Sarason 3], and we will omit the proof.

Proof of (2) \( \Rightarrow \) (1). There will be two steps. In Step 1, we will modify the Green's function \( G(x, y) \). In Step 2, we will approximate the modified Green's function by the function \( (1 - |y|) \) on a union of Carleson rectangles.

Step 1. Note that for \( r \) close to 1,

\[
G(x, re^{\theta}) = \mathcal{P}(x, e^{\theta})(1 - r)
\]

where \( \mathcal{P} \) is the Poisson kernel. Here "\( \mathcal{P} \)" means "comparable to". With each point \( x \in D \setminus \{ 0 \} \) we will associate an interval \( I_x \subset \partial D \). Let \( I_x \) be the interval centered at \( |x| \) with length \( |I_x| = 2\pi(1 - |x|) \). Let \( \delta = (1/2)|I_x| \),

4 — Studia Mathematica 72, 1
and assume without loss that $I_n$ is centered at 1. Let $N$ be the largest integer such that $2^N - \delta < \pi$. Define the intervals $I_n$, $0 \leq n \leq N + 1$ by

$$I_n = I_n, \quad I_n = [-2^n \delta, 2^n \delta], \quad n \leq N, \quad I_N = \partial D.$$

In their proof of duality for $BMO$, Fefferman and Stein approximated $\mathcal{P}(x, e^d)$ by the indicator functions of the intervals $I_n$. We intend to approximate $\partial G(x, y)$ by $(1 - r)^n$ times the indicator functions of $E(I_n)$. This procedure will give the desired result, near the edge of $D$. In this step, we will "hollow out the center" of $G(x, y)$ using the following idea. Note that

$$|f'(x)|^2 = \int_{\partial D} |f'(y)|^2 \mathcal{P}(x, y) dy \leq \int_{\partial D} |f'(y)|^2 \mathcal{P}(x, y) dy.$$

So, in the integral $\int_{\partial D} |f'(y)|^2 G(x, y) dy$ we can "project" $|f'(y)|^2 G(x, y)$ onto a region near the boundary.

Let $m$ be a measure with support $Q$ contained in a disc $D(e, r)$ with center $e$ and radius $r$. Let $\mathcal{P}_m(x, y) \in \partial D(e, r)$ be the Poisson kernel with respect to $\partial D(e, r)$. The "projection" of $m$ onto $\partial D(e, r)$ will mean the measure $\mu$ on $\partial D(e, r)$ defined by

$$\mu(dy) = \int_{\partial D} \mathcal{P}_m(x, y) m(dx).$$

"Hollowing out" $Q$ means replacing $m$ by its "projection". Often, we shall speak of "projecting" a function $f(x)$ and "hollowing out" its domain. In these cases, we are referring to the measure $f(x) dx$. Also, the "mass" of a function is its Lebesgue integral.

Let $p(d e, d r)$ be a probability measure with support on the set

$$\{(e, r); \ Q = D(e, r)\}.$$

Let $\mu_{\partial D}$ be the "projection" of $m$ onto $\partial D(e, r)$. The "projection" of $m$ with respect to $p$ will mean the measure $\mu$ satisfying

$$\mu = \int_{\partial D} \mu_{\partial D}(d e, d r).$$

Most of the time, both $m$ and $\mu$ will have continuous densities $f(x)$ and $h(x)$, respectively. Then we shall speak of $h(x)$ as the "projection" of $f(x)$. As in the previous part of the proof, assume that $x$ lies on the positive real axis, and

$$s = x, \quad y = re^0.$$

Let $m_{\partial D}(y)$ be as before. Now

$$\frac{\partial}{\partial r} G(x, y) = \frac{1}{|m_{\partial D}(y)|} \frac{\partial}{\partial r} |m_{\partial D}(y)| = \frac{1 - \delta^2}{(y - s)(1 - y^2)} \biggl(1 - \frac{r^2}{1 + r^2 B^2 - 2rR \cos \theta}\biggr) \frac{1}{1 + r^2 B^2 - 2rR \cos \theta}.$$

When $r = 1$, equality holds in the above formula. Fix $K > 1$. We wish to find a region in $D$ for which

$$\left| \frac{\partial}{\partial r} G(x, re^0) \right| \leq K \left| \frac{\partial}{\partial r} G(x, re^{0}) \right|_{r = 1},$$

That is, the radial derivative of $G$ should be bounded by $K$ times its value on the boundary. We will compare each of the expressions

(a) \quad 1 + r^2 B^2 - 2rR \cos \theta,

(b) \quad r^2 + B^2 - 2R \cos \theta

with its value on the boundary. First, expression (a):

$$1 + r^2 B^2 - 2rR \cos \theta = 1 + B^2 - 2R \cos \theta + R(1 - r^2) \cos \theta - R(1 + r^2) \cos \theta + 2R(1 - r^2).$$

The order of magnitude of the second term is bounded by the order of magnitude of the first term if

$$(1 - r)^2 \leq \max(1 - \cos \theta, 1 - B^2).$$

That is, \quad $$(1 - r)^2 \leq \max(1 - \cos \theta, 1 - B^2).$$

The argument for expression (b) is similar:

$$r^2 + B^2 - 2rR \cos \theta = 1 + B^2 - 2R \cos \theta + (1 - r^2) \cos \theta - (1 + r^2) \cos \theta + 2R(1 - r^2).$$

This again leads to (a). Let $Q$ be the region determined by (*). Next, we will find the regions $Q_1$, $Q_2$, and $Q_3$ of whose diameters has endpoints $P_{e1}, P_{e2}$ on the real axis. We will estimate the location of these points. Let $y$ be real, and

$$1 - y = e, \quad 1 - x = A.$$
Let $S_n$ be the circle with center $1 - 2^n \delta$ and radius $2^{n-1} \delta$. Then, for $n$ large enough, $S_n \subset Q_n$. Note that for $C$ large enough, $\bigcup_{n=1}^\infty S_n$ covers $E = \{re^{i\theta} 1 - r \geq C \max(1 - R, \theta)\}$. Let $M$ be a constant to be specified later. Now, for $y \notin S_M$,

$$G(x,y) < K_M \sum_{n=1}^{\infty} 2^{-n}I(y \in Q_n).$$

Since the $S_n$ cover $Q$, by changing $K_M$, we find, for $y \notin S_M$,

$$G(x,y) < K_M \sum_{n=1}^{\infty} 2^{-n}I(y \in S_n \cap E).$$

We wish to use inequality (A) to "project" $2^{-n}I(y \in S_n)$ onto $D \setminus S_n$. Let $z_n$ be the center of $S_n$. Now, $2^{-n}I(y \in S_n)$ has mass $2^n \delta^3$. We claim that the projection of $2^{-n}I(y \in S_n)$ is the same as if the mass were concentrated at $z_n$.

By symmetry, both $2^{-n}I(y \in S_n)$ and the mass concentrated at $z_n$ have the same projection onto $D \setminus S_n$. The projection onto a larger circle is the same if we first project onto $D \setminus S_n$. This fact holds because a projection is the hitting density of Brownian motion. If $t_1$ is the first hitting time of $D \setminus S_n$ and $t_2$ is the first hitting time of the larger circle,

$$P(B(t_1) \in d\omega) = \int_{S_n} P(B(t_1) \in d\omega)P^n(B(t_2) \in d\omega).$$

Let $\epsilon > 0$ be a small constant which we will specify later. For $0 < u < \epsilon$, let $D_u = \{|x| = 1 - u\}$, and "project" $(1/\epsilon)^2\mu du$ of the mass onto the circles $m_n^u(I_B)$; $t \in (u, u + du)$. Since $m_n^u$ is a conformal transformation, the density of mass will be, for $u < \epsilon$, comparable to $(u/|u|)^{n}m_n^u(y)^{3\delta^3}$. Now, in the region

$$Q_n = \{re^{i\theta} 1 - r < C \max(1 - |s_n|, \theta)\}$$

we have seen that $\|m_n^u(re^{i\theta})\|$ and $\|m_n^u(e^{i\theta})\|$ are comparable. Since the latter equals $\mathcal{P}(x, e^{i\theta})$, the density of mass is comparable to $(1 - r)^{-n}x \times \mathcal{P}(x, e^{i\theta}) 2\delta^3 R_n$ for $re^{i\theta} \in Q_n$.

Note that "hollowing out" $S_n$ puts some mass onto $S_{n+1}$, $m > 1$. We will show that this fact is of no consequence.

Let $y = re^{i\theta}, 1 - r = \delta$. We know that $1 - s_n = 2^n \delta$. Now

$$|m_n^u(y)| \leq \frac{1 - R_n^2}{(1 - rR_n^2)^{3\delta^3} \cos \theta} 2\delta^3 (1 - 3\delta^3/4) \leq 2^n \delta^3 |1 + o(2^n \delta^3/A)|.$$
Also
\[ \mathcal{P}(\delta_0, e^{\delta_0}) \preceq C_2 \delta / 2 e^{\delta_0} \preceq 2^{\sqrt{2} \delta_0}. \]

So,
\[ \mathcal{P}(\delta_0, e^{\delta_0}) \preceq C_3 \mathcal{P}(\delta_0, e^{\delta_0}) \sum_{x=1}^{2^{\sqrt{2} \delta_0}} \mathcal{P}(2^{\sqrt{2} \delta_0}, 2^{\sqrt{2} \delta_0} \delta_0) \preceq C_3 \mathcal{P}(\delta_0, e^{\delta_0}). \]

Thus, we have shown that
\[ \int \int f'(y) G(x, y) dy \leq C \int \frac{1}{|x|} \int |f'(\tau)|^{1/(1-|x|)} d\tau. \]

This proves the lemma, and thus establishes the theorem.

3. Proof of Theorem 4. Given a kernel \( K(x, y) \), the equilibrium measure of a region \( Q \) is the measure \( m(dx) \), supported on \( Q \), which has the greatest mass subject to the condition
\[ \int \int K(x, y) m(dy) = 1, \quad \text{for all } x \in Q. \]

Let \( D(0, R) \) be the disc of radius \( R \) with center \( 0 \), and let \( E = \mathbb{R}^2 \setminus \cup D(0, R) \). Let \( m(dx) \) be the equilibrium measure of \( E \) with respect to the logarithmic kernel \( \log |1/(x-y)| \), and let \( \hat{m}(dx) \) be the equilibrium measure of \( E \) with respect to
\[ G(x, y) = (1/2\pi) \log |(2R - 3y)/(y - x)|. \]

\( G(x, y) \) is the Green’s function of the region \( D(0, R) \). Note that for \( x, y \in D(0, R) \),
\[ \log |1/(x-y)| \leq (1/2\pi) \log |1/(x-y)| \leq (1/2\pi) \log 2R. \]

Recall that
\[ \hat{m}(dx) = \hat{m}(dx)/m(E). \]

Let \( \mu(dx) = m(dx)/m(E) \), \( \hat{\mu}(dx) = \hat{m}(dx)/\hat{m}(E) \).

It is a standard fact that \( \mu \) and \( \hat{\mu} \) are the measures for which \( \sup_{x,y} K(x, y) \mu(dy) \) is smallest when \( K(x, y) \) is the appropriate kernel.

Using this fact and inequality (4),
\[ -\log \delta > 1/m(E) = \sup_{x,y} \int \log |1/(x-y)| \mu(dy) \]
\[ \geq \sup_{x,y} \int 2\pi G(x, y) - \log 2R \mu(dy) \]
\[ \geq \sup_{x,y} \int 2\pi G(x, y) \hat{\mu}(dy) - \log 2R = (2\pi \hat{m}(E)) - \log 2R. \]

So,
\[ \hat{m}(E) \geq 2^{\pi \log(2R/\delta)}. \]

From probabilistic potential theory (see Hunt [5]), we know that
\[ P^\infty(0 \text{ hits } E \text{ before hitting } \partial D(0, 2R)) = \int \hat{G}(x, y) \hat{m}(dy). \]

Therefore, it follows that
\[ P^\infty(0 \text{ hits } E \text{ before hitting } \partial D(0, 2R)) \geq \int \hat{G}(x, y) \hat{m}(dy) \]
\[ \geq (1/2\pi) \log 2 \hat{m}(E) \geq \log(2R/\delta). \]

Let \( C_4 \) be the minimum of the last quantity and 1. The expected exit time from \( D(0, R) \) for \( B(0) = z \), is \( C_4 R^2 \) for some constant \( C_4 \), since \( B(0) \) and \( y \neq z \) have the same distribution.

We wish to find \( z \in \partial D(0, 2R) \) such that \( \delta_0, R_0 \) sufficiently small,
\[ P^\infty(0 \text{ hits } E \text{ before hitting } \partial D(0, 2R)) \leq C \log(3R_0/2\delta_0). \]

Let \( G(x, y) \) be the Green’s function for \( D(0, 2R_0) \) and \( m_0 \) and \( \hat{m}_0 \) be as before. Also as before, we deduce from (4) that
\[ \hat{m}_0(E) \leq 2^{\pi \log(3R_0/2\delta_0)}. \]

Let \( n \) be uniform on \( \partial D(0, (3/2)R_0) \). Then
\[ P^\infty(0 \text{ hits } E \text{ before hitting } \partial D(0, 2R_0)) \leq (1/2\pi) \log |1/4 - (1/2) - (1/4)| \hat{m}_0(E) \]
\[ \leq C \log(3R_0/2\delta_0) = \varepsilon, \quad \text{say.} \]
Thus, if $r$ is the first hitting time of $D(o_n, 2R_n)$ and if
\[
G = \{ E \text{ hits } E \text{ before } \delta D(o_n, 2R_n) \},
\]
\[
H = \{ E \text{ hits } E \text{ or } \delta D(o_n, 2R_n) \text{ before } r \},
\]
then
\[
s \geq E^E [ F^{R_n}(G) | 1(H) ] = E^E [ F^{R_n}(G) ] = F^E(G) - \varepsilon.
\]
Thus, inequality (A) is established and the required point $a_n$ exists. Since
\[
D(o_n, R_n/2) \subseteq D(o_n, R_n),
\]
\[
F^E_{\varepsilon}(\text{hit } G \text{ before } \delta D(o_n, R_n/2)) < C/\log(3R_n/2\delta_n).
\]
Let $\tau$ be the first exit time of Brownian motion from $D(o_n, R_n/2)$ starting
from $a_n$. Suppose that $n$ is so large that
\[
C/\log(3R_n/2\delta_n) < 1/2.
\]
Let $F(x)$ be the cumulative distribution function for $\tau$. Let
\[
k(n) = \int_{(F(x) < n)} (1 - F(x)) dx.
\]
By the scaling properties of Brownian motion, $k(n) = \sigma^2 R_n, \sigma > 0$.
Now $k(n)$ is the expectation of the smallest values of $\tau$, so
\[
F^E_{\varepsilon} \geq E^E_{\varepsilon} [ \text{hit } \delta D(o_n, R_n/2) \text{ before } G ] \geq k(R_n) = \sigma^2 R_n.
\]
Thus, $F^E_{\varepsilon}$ is not bounded. If $F(x)$ maps $D$ onto the covering space of $G$, then $F(x)$ is not in BMO.

References