A generalization of Wiener's criteria for the continuity of a Borel measure

by

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Abstract. An identity is derived for the discrete part of a bounded complex-valued finitely additive set function defined on the Borel sets of an Abelian locally compact Hausdorff topological group. This allows us to establish a generalization of Wiener's necessary and sufficient condition for the continuity of a complex-valued bounded regular measure [16].

1. Introduction. Let $T = \{x \in C : |x| = 1\}$. Then $T$ with the multiplication operation and the topology induced by the usual topology on $C$ is a compact Abelian topological group. Let $\mathcal{B}(T)$ be the $\sigma$-algebra of Borel sets in $T$. Let $M(T) = (\mu, \mathcal{B}(T))$ ($\mu$ is a bounded regular measure). The Fourier coefficients of a measure $\mu \in M(T)$ are $\hat{\mu}(n) = \int x^n d\mu(x)$ for all $n \in \mathbb{Z}$. Recall that a measure $\mu \in M(T)$ is continuous if $\hat{\mu}(n) = 0$ for any point $x$ in $T$. A classical result of Wiener ([16]; [17], Theorem 9.6, p. 108; [35], Corollary, p. 42) states:

1.1. Theorem. Let $\mu \in M(T)$. Then

$$\sum_{x \in T} |\mu((x))|^2 = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\hat{\mu}(n)|^2.$$ 

In particular, $\mu$ is continuous if and only if

$$\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\hat{\mu}(n)|^2 = 0.$$ 

In this paper, it is shown that this theorem follows from a general result for bounded complex-valued finitely additive set functions defined on the Borel sets $\mathcal{B}(G)$ of an arbitrary locally compact Abelian Hausdorff

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topological group $G$. This result also generalizes that of W.F. Eberlein ([3], Theorem 1, p. 310) in the case of Radon measures on $B(G)$.

2. Preliminaries. In this section we introduce a slight generalization of a theorem of Sinclair ([14], p. 363) which is an essential tool for obtaining our main result. Let $\mathcal{A}$ be an algebra of subsets of a set $X$. A charge on $\mathcal{A}$ is a complex-valued bounded finitely additive set function defined on $\mathcal{A}$. For every bounded $\mathcal{A}$-measurable complex-valued function on $X$, we can define the integral $\int f\ dm$ by the usual Moore-Smith method ([12], pp. 185–191; [15], pp. 401–404) or, equivalently ([9]), by the Dunford–Schwartz method ([22], pp. 101–125). A function $f: X \to C$ is called $\mathcal{A}$-continuous if, for every $\varepsilon > 0$, there exists a finite partition $\langle E_i \rangle_{i < n}$ of $X$ such that $E_i \in \mathcal{A}$ and $\sup_{x \in E_i} |f(x) - f(y)| < \varepsilon$ for all $i = 1, 2, \ldots, n$. It is clear that if $X$ is a topological space and if $f: X \to C$ is bounded and continuous, then $f$ is $\mathcal{A}(X)$-continuous.

Let $X$ and $Y$ be arbitrary sets and let $f: X \times Y \to C$. We say that $f$ satisfies the double limit condition or $f$ is a DLC function if, whenever $(x_i), (y_j)$ are sequences in $X$ and $Y$, respectively, such that the iterated limits

$$a = \lim_{i \to \infty} \lim_{j \to \infty} f(x_i, y_j)$$

and

$$b = \lim_{j \to \infty} \lim_{i \to \infty} f(x_i, y_j)$$

exist, then $a = b$. This notion was introduced by Bannach ([11], p. 222) to give a criterion for the weak convergence to $0$ of a sequence in a Banach space, and used extensively by Grothendieck ([3], pp. 185–186) in his search for more general weak convergence criteria. In the case where $X$ and $Y$ are completely regular spaces and $f$ a real-valued bounded separately continuous function on $X \times Y$, this notion was used by Pták ([11], p. 573) to obtain extension criteria for $f$.

It is said that $\mathcal{A}$ separates points on $X$, or $\mathcal{A}$ is an SP algebra on $X$ if, whenever $x, y \in X$, $x \neq y$, there are disjoint sets $A, B \in \mathcal{A}$ such that $x \in A$ and $y \in B$. It is clear that if $X$ is a Hausdorff topological space, then $\mathcal{A}(X)$ is an SP algebra on $X$.

The following result is a trivial generalization of Theorem 4.4 of Sinclair ([14], p. 363):

2.1. THEOREM. Let $\mathcal{A}$ and $\mathcal{B}$ be SP algebras on the sets $X$ and $Y$, respectively. Let $\mu$ and $\nu$ be charges on $\mathcal{A}$ and $\mathcal{B}$, respectively. If $f: X \times Y \to C$ is bounded and satisfies the additional conditions:

(i) $f(\cdot, y)$ is $\mathcal{A}$-continuous for all $y \in Y$,

(ii) $f(x, \cdot)$ is $\mathcal{B}$-continuous for all $x \in X$, and

(iii) $f$ is a DLC function,

then

(1) $\int \int f(x, y) \, d\mu \, d\nu = \int \int f(x, y) \, d\nu \, d\mu$,

(2) $\int f(x, \cdot) \, d\mu$ is $\mathcal{A}$-continuous, and

(3) $\int f(\cdot, y) \, d\nu$ is $\mathcal{B}$-continuous.

3. Preparatory propositions. Henceforth $G$ will denote an Abelian Hausdorff locally compact group. When we need to give $G$ a topology different from the original, we will write $G_t$. In particular, we shall consider the discrete topology ($\tau_d$), the pointwise topology ($\tau_p$) and the topology of uniform convergence on compacta ($\tau_m$).

A character on $G$ is a continuous homomorphism on $G$ to $T$. The set of all characters on $G$ is an Abelian group under addition and, with the $\tau_m$ topology, is locally compact Hausdorff group ([100], p. 137). The topology group so obtained is called the dual group of $G$, denoted $G^\ast$. The value of an element $\sigma \in G^\ast$ at the point $g \in G$ will be denoted by $\langle \sigma, g \rangle$ and its complex conjugate by $\overline{\langle \sigma, g \rangle}$. For all $g \in G$ consider the function $u_g: G^\ast \to G^\ast$ given by $u_g(\varphi) = \langle \varphi, g \rangle$. Then $u_g$ is a character on $G$. The Pontryagin duality theorem ([77], p. 378) states that the mapping $\sigma \mapsto u_\sigma$ is a topological isomorphism of $G$ onto $G^\ast$. This result permits us to identify $G$ with its own second character group $G^\ast$. With $G$ can be associated its Bohr compactification ([115], p. 30) $G^\ast = \overline{\langle G \rangle}$ which is an Abelian Hausdorff compact topological group whose topology is $\tau_p$. It is well known that $G$ can be embedded into $G^\ast$ as a dense subgroup ([115], Theorem 1.3.2, p. 30). It is easy to show that the continuous function $\langle \sigma, \cdot \rangle: G^\ast \to \mathbb{C}$ can be extended to a continuous function on $G_d \times X^\ast$. We note that every element of the Bohr compactification of an Abelian Hausdorff locally compact topological group is a character on some discrete topological group.

3.1. LEMMA. The restriction to $G$ of an element of $G^{***}$ belongs to $G^\ast$.

Proof. We note the two trivial facts: (a) If $\sigma \in G^{***}$, then the domain of $\sigma$ contains the set $\tau_d$; (b) Let $K$ be a topological subgroup of an Abelian Hausdorff locally compact group $H$. If $\sigma \in H^\ast$, then $\sigma |_K \in K^\ast$. The lemma now follows by taking $E = G_t$ and $H = (G^\ast)_t$.

Remarks. (1) From fact (a) and the property $G^{***} = (G^\ast)^\ast$ it follows that $G_d$ is a topological subgroup of $G^{***}$.

(2) From Remark (1), it follows that $(G^\ast)_t$ is a topological subgroup of $G^{***}$.

(3) From the lemma it follows that the restriction to $G^\ast$ of an element of $G^{***}$ belongs to $G^\ast$. 

3.3. Lemma. If \( \{x_i\} \) and \( \{y_i\} \) are sequences in \( G \) and \( G^* \), respectively, then there are subsequences \( \{x_{i_n}\} \) and \( \{y_{j_n}\} \) of \( \{x_i\} \) and \( \{y_i\} \), respectively, such that
\[
\lim_{n \to \infty} \left< x_{i_n}, y_{j_n} \right> = \lim_{n \to \infty} \left< x_{i_n}, y_{j_n} \right>.
\]

Proof. Since \( \hat{z} \in \hat{G} \) is an element of \( \hat{G}^* \), there exists a subsequence \( \{y_{j_n}\} \) of \( \{y_i\} \) converging to an element \( \hat{z} \) of \( \hat{G}^* \). Hence, by Remark (1),
\[
\lim_{i \to \infty} \left< x_i, y_{i_n} \right> = \langle x_i, \hat{z} \rangle \quad (i = 1, 2, 3, \ldots)
\]
exists. Also, since \( \hat{G} \subseteq \hat{G}^* \), there exists a subsequence \( \{y_{j_n}\} \) of \( \{y_i\} \) converging to an element \( \hat{z} \) of \( \hat{G}^* \). Therefore,
\[
\lim_{i \to \infty} \left< x_i, y_{j_n} \right> = \langle x_i, \hat{z} \rangle \quad \text{and, by duality, the fact that}
\]
the topology on \( \hat{G}^* \) is \( \tau_F \), we have
\[
\lim_{i \to \infty} \left< x_{i_n}, y_{j_n} \right> = \langle x_{i_n}, \hat{z} \rangle.
\]
So
\[
\lim_{i \to \infty} \left< x_{i_n}, y_{j_n} \right> = \langle x_{i_n}, \hat{z} \rangle.
\]
On the other hand, since \( \{x_{i_n}\} \subseteq \hat{G}^* \), by Remark (2), the continuity of \( \hat{z} \) implies
\[
\lim_{i \to \infty} \left< x_i, y_{j_n} \right> = \langle x_i, \hat{z} \rangle = \langle x_i, \hat{z} \rangle
\]
exists. Hence, by Remark (3), \( \hat{G} \subseteq \hat{G}^* \). Therefore, by the fact that the topology on \( G^* \) is \( \tau_F \), we have
\[
\lim_{i \to \infty} \left< x_{i_n}, y_{j_n} \right> = \langle x_{i_n}, \hat{z} \rangle.
\]
This completes the proof.

3.3. Proposition. The function \( f : G \times G^* \to T \) given by \( f(x, \hat{z}) = \langle x, \hat{z} \rangle \) satisfies the double limit condition.

Proof. This follows immediately from Lemma 3.2 and the continuity of \( f \).

Let \( m \) be the normalized Haar measure on the Borel set \( \mathcal{B}(G^*) \) of \( G^* \). For a given \( x \in G \) and \( t \in T \), denote by \( x^{-1}(t) \) the set \( \{ \hat{z} \in G^* : \langle x, \hat{z} \rangle = t \} \).

3.4. Proposition. Given \( x \in G \), there are at most a finite number of points \( t \in T \) for which \( x^{-1}(t) \) has positive Haar measure.

Proof. Let \( t \in T \) be a point of \( T \) with \( m(x^{-1}(t)) > 0 \) and \( m(x^{-1}(t')) > 0 \). Then \( x^{-1}(t) \) and \( x^{-1}(t') \) are not empty. Let\( u \) and \( \hat{u} \) be two points in \( T \), where \( \langle u, \hat{u} \rangle = \langle u, \hat{u} \rangle \).

Choose an arbitrary \( \hat{u} \in x^{-1}(t) \) and an arbitrary \( \hat{u} \in x^{-1}(t) \). Let \( u = \hat{u} - \hat{u} \). If we choose a \( \hat{u} \in x^{-1}(t) \) and an arbitrary \( \hat{u} \in x^{-1}(t) \), we have \( \langle x, \hat{u} \rangle = \langle x, \hat{u} \rangle \), since \( \langle x, \hat{u} \rangle \).

Thus \( x^{-1}(t) \) is a finite set. Similarly we can show that the inclusion \( \hat{x} + x^{-1}(t') \)
\[
\subseteq x^{-1}(t').
\]
Since \( m \) is translation invariant,
\[
m(x^{-1}(t')) = m(x^{-1}(t')) = m(x^{-1}(t'))
\]
thus the proposition now follows from \( m(G^*) = 1 \) and the disjointness of the sets \( x^{-1}(t) : t \in T \).

Let \( \hat{V} \) denote the interior of \( \hat{V} \) and \( \hat{V} \) its closure.

3.5. Proposition. Given \( z \in G \) and \( n \in \mathbb{N} \), there exists a partition \( \mathcal{R}_n \) of \( G^* \) into Borel subsets such that
\[
(\text{i}) \quad m(\hat{V}_n) = m(\hat{V}) \quad \text{whenever} \quad V \in \mathcal{R}_n,
\]
\[
(\text{ii}) \quad \text{given} \quad z \in G^* \quad \text{there exists a} \quad V \in \mathcal{R}_n \quad \text{for which} \quad (a) \quad z \in V \quad \text{and} \quad (b) \quad \langle z, \hat{z} \rangle < \frac{1}{n} \quad \text{for all} \quad \hat{z} \in \hat{V}_n.
\]

Proof. Let \( I_1, I_2, \ldots, I_n \) be disjoint open intervals of \( T \) of length \( 2\pi/n \) each. By the proposition, we can rotate these arcs along \( T \) so that they consist of one of the end points \( t \) has \( m(x^{-1}(t)) > 0 \). Then let \( \mathcal{R}_n = (z^{-1}(I_i)) : i = 1, 2, \ldots, n \). Thus if \( V = z^{-1}(I_i) \), \( i = 1, 2, \ldots, n \), then \( m(V) = m(V \cap I_i) \) and this vanishes since \( I_i \cap I_j \) consists of two end points each with Haar measure of their inverse under \( z \). This proves part (i).

Part (ii) follows by construction.

Let \( A = \{ x \in \mathcal{B}(G^*) : m(\hat{A}) = m(\hat{A}) \} \). By Proposition 3.5, \( A \) is not empty.

3.6. Proposition. \( A \) is a subalgebra of \( \mathcal{B}(G^*) \).

Proof. Let \( U \) be an element of \( A \). Then \( U \in A \) since \( \hat{U} \subseteq \hat{U} \) and \( m(U) = m(U) \).

Choose \( U, V \in A \), we have \( m(U \cap V) \leq m(U) \) and \( m(U \cup V) \leq m(U) \).

Let \( M = \{ A \cap G^* : A \in A \} \).

3.7. Proposition. \( M \) is an SP algebra on \( G^* \).

Proof. A compact Hausdorff space is normal. Hence, by Urysohn's lemma, there exists a continuous function \( f : G^* \to [0, 1] \) such that \( f = 0 \) on the closed set \( \{ \hat{z} \} \) and \( f = 1 \) on \( \{ \hat{z} \} \). By the Stone–Weierstrass theorem, \( f \) can be uniformly approximated by polynomials \( \sum_{k=0}^{n} \alpha_k \langle x_k, \hat{z} \rangle \) where \( x_k \in G \), \( k = 1, 2, \ldots, n \). If \( \langle x_k, \hat{z} \rangle = \langle x_k, \hat{z} \rangle \) for all \( x_k \in G \), then \( f(\hat{z}) = f(\hat{z}) \), which is a contradiction. Hence, for some \( x_k \in G \), \( \langle x_k, \hat{z} \rangle \neq \langle x_k, \hat{z} \rangle \).

Thus, there exist disjoint open intervals \( I_1 \) and \( I_2 \) on the unit circle containing \( \langle x_k, \hat{z} \rangle \) and \( \langle x_k, \hat{z} \rangle \), respectively, for which
\[
\langle x_k, \hat{z} \rangle \cap I_1 \cap I_2 \}
and
\[ e^* (I_x) = \{ \hat{e} \in G^* : \langle e, \hat{e} \rangle \in I_x \}, \]
are disjoint elements of \( M \) containing \( \hat{y} \) and \( \hat{y'} \), respectively.

3.8. PROPOSITION. For all \( z \in G \), the function \( \langle e, \hat{e} \rangle : G^* \to T \) is \( M \)-continuous.

Proof. This follows immediately from Proposition 3.5 (ii).

3.9. PROPOSITION. If for some \( A, B \in R \), \( A \cap G^* = B \cap G^* \), then \( m(A \Delta B) = 0 \).

Proof. Since \( A \Delta B \subseteq (A \Delta \hat{A}) \cup (\hat{A} \Delta B) \cup (B \Delta \hat{B}) \), then \( m(A \Delta B) \leq m(A \Delta \hat{A}) + m(\hat{A} \Delta B) + m(B \Delta \hat{B}) \). But \( A \Delta B \) is open and \( G^* \) is dense in \( G^* \) and so \( A \Delta B = \emptyset \); for otherwise \( (A \cap G^*) \cap (\hat{A} \cap G^*) \neq (A \Delta B) \cap G^* = B \cap G^* \). Hence \( m(A \Delta B) = 0 \). Similarly \( m(B \Delta \hat{B}) = 0 \). The proposition is proved.

Remark. It is clear from Proposition 3.9 that the set function \( \nu(A \cap G^*) = m(A) \), where \( A \in R \), is well defined. The following proposition is trivial.

3.10. PROPOSITION. The set function \( \nu \) is a non-negative charge on \( M \).

Proof. Since the product of two \( M \)-continuous functions is \( M \)-continuous, from Proposition 3.5 it follows that the integral \( \int e^* \langle z, \hat{e} \rangle dw(\hat{e}) \) exists for all \( z, w \in G \).

3.11. PROPOSITION.
\[ \int e^* \langle z, \hat{e} \rangle dw(\hat{e}) = \begin{cases} 0 & \text{if } z \neq w, \\ 1 & \text{if } z = w. \end{cases} \]

Proof. Note that \( \langle z, \hat{e} \rangle \langle w, \hat{e} \rangle = \langle z-w, \hat{e} \rangle \) is a character. Let \( n \) be a positive integer. As in the proof of Proposition 3.5, consider disjoint half open arcs \( I_{n,i} = 1, \ldots, n \) of equal length for which \( (z-w)^{-1}(I_{n,i}) = V_{n,i} \in R, z-w \in \mathbb{R} \) and points \( \hat{e}_{n,i} \in V_{n,i} \cap G^* \). Then
\[ \int e^* \langle z-w, \hat{e} \rangle dw(\hat{e}) = \lim_{n \to \infty} \sum_{i=1}^{n} \langle z-w, \hat{e}_{n,i} \rangle m(V_{n,i}) = \lim_{n \to \infty} \sum_{i=1}^{n} \langle z-w, \hat{e}_{n,i} \rangle m(V_{n,i} \cap G^*) = \int e^* \langle z-w, \hat{e} \rangle dw(\hat{e}). \]

Since \( ([8], \text{exercise 6, p. 193}) \),
\[ \int e^* \langle z-w, \hat{e} \rangle dw(\hat{e}) = \begin{cases} 0 & \text{if } z \neq w, \\ 1 & \text{if } z = w, \end{cases} \]
the result follows.

4. THE MAIN RESULT. Let \( \mu \) be a charge on \( \mathcal{B}(G) \). We define the Fourier transform \( \hat{\mu} \) of \( \mu \) by
\[ \hat{\mu}(\hat{e}) = \int \langle z, \hat{e} \rangle d\mu(z). \]

Then \( \hat{\mu} \) is a bounded complex-valued function on \( G^* \). Taking \( f(z, \hat{e}) = \langle z, \hat{e} \rangle \) it is clear, by Propositions 3.3, 3.7 and 3.8, that the hypotheses of Theorem 2.1 are verified. Then part (1) of that theorem assures that \( \hat{\mu} \) is \( M \)-continuous. We are now in a position to establish our principal results:

4.1. THEOREM. There exist an algebra \( M \) of subsets of \( G^* \) and a non-negative charge \( \nu \) on \( M \) satisfying the following properties:

(1) For all \( z \in G \), the function \( \langle z, \hat{e} \rangle \) is \( M \)-continuous.

(2) For any charge \( \mu \) on \( \mathcal{B}(G) \) we have
\( \mu \langle z, \hat{e} \rangle \) (a) the Fourier transform \( \hat{\mu} \) of \( \mu \) is \( M \)-continuous, and
\( \hat{\mu} \) for all \( z \in G \), \( \mu(z) = \hat{\mu}(\hat{e}) \).

Proof. It remains to prove (b). Let \( \mu \) be a charge on \( \mathcal{B}(G) \) and let \( z \in G \). Then
\[ \int e^* \langle z, \hat{e} \rangle dw(\hat{e}) = \int \left( \int e^* \langle z, \hat{e} \rangle dw(\hat{e}) \right) \langle w, \hat{e} \rangle d\mu(w) \]
\[ = \int e^* \langle z, \hat{e} \rangle dw(\hat{e}). \]
Let \( f(w, \hat{e}) = \langle w, \hat{e} \rangle \langle z, \hat{e} \rangle = \langle z-w, \hat{e} \rangle \). Then, by Propositions 3.3 and 3.8, the hypotheses of Theorem 2.1 are verified. Then part (3) of that theorem allows us to write:
\[ \int e^* \langle z, \hat{e} \rangle dw(\hat{e}) = \int e^* \langle z, \hat{e} \rangle dw(\hat{e}) d\mu(w) = \mu(z) \] (Proposition 3.11).

4.2. COROLLARY. For any charge \( \mu \) on \( \mathcal{B}(G) \), we have
\[ \int_{\mathcal{B}(G)} |\mu(z)|^2 = \int_{\mathcal{B}(G)} |\mu(\hat{e})|^2 dw(\hat{e}). \]

In particular, \( \mu \) is continuous if and only if \( \int_{\mathcal{B}(G)} |\mu(\hat{e})|^2 dw(\hat{e}) = 0 \).

Proof. Applying Theorem 2.1 twice, we obtain
\[ \int_{\mathcal{B}(G)} |\mu(\hat{e})|^2 dw(\hat{e}) = \int_{\mathcal{B}(G)} \left( \int_{\mathcal{B}(G)} \langle w, \hat{e} \rangle d\mu(w) \right) \langle \hat{e}, \hat{e} \rangle dw(\hat{e}) \]
\[ = \int_{\mathcal{B}(G)} \left( \int_{\mathcal{B}(G)} \langle w, \hat{e} \rangle d\mu(w) \right) dw(\hat{e}) = \int_{\mathcal{B}(G)} |\mu(z)|^2 = \int_{\mathcal{B}(G)} |\mu(z)|^2 d\mu(z) = \sum_{z \in \mathcal{B}(G)} |\mu(z)|^2. \]

By the standard methods used in ([8], pp. 34–42), it is clear that the following corollary contains Theorem 1.1.
4.3. COROLLARY. Let \( \mu \) be a charge on \( \mathcal{B}(T) \). For all \( s \in T \),

\[
\mu((s)) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=1}^{N} \mu(n) s^n.
\]

Proof. By Theorem 4.1,

\[
\mu((s)) = \int_Z \hat{\mu}(n) s^n d\nu(n)
\]

for all \( s \in T \). Thus, it is sufficient to show that for any \( M \)-continuous complex-valued function \( f \) on \( Z \),

\[
\int_Z f(n) d\nu(n) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=1}^{N} f(n).
\]

The result will then follow by taking \( f(n) = \hat{\mu}(n) s^n \). Since \( f \) can be uniformly approximated by \( M \)-measurable step functions, it is sufficient to take \( f(n) = \chi_n \) where \( \chi_n \) is the characteristic function for \( E \in M \). Choose \( F \subset \mathbb{Z} \) such that \( F \cap Z = \emptyset \). Let \( \epsilon > 0 \). By the regularity of \( \mu \), there exist a compact set \( K \) and an open set \( V \) in \( \mathbb{Z}^* \) such that \( K \subset F \subset F \subset V \subset \mathbb{Z}^* \), \( \mu(\mathbb{Z}^* \setminus K) < \epsilon \) and \( \mu(V \setminus \mathbb{Z}^*) < \epsilon \). By Urysohn's lemma, there exist two continuous real functions \( g_F, g_V \) on \( \mathbb{Z}^* \) such that \( g_F | K = 1, g_F | (\mathbb{Z}^* \setminus F) = 0 \), and \( g_V | (\mathbb{Z}^* \setminus V) = 0 \), then \( g_F \leq g_V \leq g_F \). Since the restriction to \( Z \) of a continuous function on \( \mathbb{Z}^* \) is almost periodic, we can write ([7, p. 256])

\[
\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=1}^{N} g_F(n) \leq \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=1}^{N} \chi_n(n) \leq \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=1}^{N} g_V(n).
\]

Thus

\[
\limsup_{N \to \infty} \frac{1}{2N+1} \sum_{n=1}^{N} \chi_n(n) = \liminf_{N \to \infty} \frac{1}{2N+1} \sum_{n=1}^{N} \chi_n(n) \leq \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=1}^{N} \chi_n(n).
\]

But, the last term yields ([7], p. 256 and [10], pp. 169, 170)

\[
\int_Z |(g_F(n) - g_V(n))| d\mu(n)
\]

References:

Approximate isometries on bounded sets with an application to measure theory*

by

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Abstract. Given a δ > 0, an S ⊆ ℜ^n of diameter 1, and a function g: S → ℜ^n which alters distances by no more than δ (i.e. for all s, s' ∈ S, ||g(s) − g(s')| − |s − s'|| < δ) we show how to alter g to obtain a true isometry f: S → ℜ^n with ||f − g||∞ < 27δ^10.

D. H. Hyers and S. M. Ulam proved a similar result, but starting with an approximate isometry g from ℜ^n onto ℜ^n.

We use our theorem and an idea of J. Mycielski's to show that two Borel subsets of the Hilbert cube [0, 1]^n which are isometric under one of the metrics δₐ,(a, y) ≡ (∑ₐ∈ℕ|aₐ − yₐ|^2)^1/₂ must have the same product measure, provided that the aₐ tend to 0 fast enough so that δₐ^2(aₐ, aₐ) → 0 as aₐ → 0.

§ 0. Introduction. In [2] S. M. Ulam and D. H. Hyers proved that if g: ℜ^n → ℜ^n is surjective and preserves distances to within δ > 0 (i.e. for all x, y ∈ ℜ^n, ||x − y|| − ||g(x) − g(y)|| ≤ δ), then there is an isometry f: ℜ^n → ℜ^n which differs from g (sup norm) by no more than 10δ.

Following Ulam and Hyers, several people have considered the problem of finding an isometry near to an approximate isometry in very general contexts (see [1] and references therein), but to our knowledge no one has yet considered the problem when the approximate isometry is not defined on a full Banach space.

In §2 we give a construction which alters an approximate isometry g defined on a bounded subset of ℜ^n to give an isometry f; in Theorem 2.2 we show that the constructed f is near to g.

In §1 we develop the methods for proving this result.

In §3 we apply Theorem 2.2 to partly prove the following conjecture of Ulam's: If any two Borel subsets of the Hilbert cube [0, 1]^n are isometric under one of the metrics

δₐ,(x, y) = (∑ₐ∈ℕ|aₐ − yₐ|^2)^1/₂

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