A note on singular integrals

by

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Abstract. The purpose of the paper is to further investigate relationships between various conditions on singular kernels $K$ which imply continuity of the corresponding operator.

1. In the study of the existence and properties of singular integrals

$$
\int_{\mathbb{R}^n} f(y)K(x-y)\,dy
$$

various hypotheses about the kernel $K$ can be made, in addition to the basic properties that $K(x)$ is homogeneous of degree $-\alpha$ ($\alpha$ the dimension of the space) and that the mean value of $K$ over the surface $\Sigma$ of the unit sphere is 0.

One of the earliest assumptions used was (see e.g., [2]) that the kernel $K$ satisfies the Dini condition on $\Sigma$, that is to say that the modulus of continuity $\omega(t)$ of $K$ on $\Sigma$ be such that

$$(1.1) \quad \int_{\frac{1}{t}}^{\infty} \frac{\omega(t)}{t} \,dt < \infty.$$ 

This implicitly presupposes the continuity of $K$ on $\Sigma$. If this holds then the transformation

$$
\hat{f}(a) = Tf(a) = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} f(y)K(x-y)\,dy = \text{P.V.} \int_{\mathbb{R}^n} f(y)K(x-y)\,dy
$$

is of type $(p, p)$ for $1 < p < \infty$, and of weak type $(1, 1)$ (see [2]).

It may also be noted that condition (1.1) was merely used to show that

$$(1.2) \quad \int_{|x| < \varepsilon} |K(x-y) - K(x)|\,dx \leq C \quad (y \neq 0)$$

from which the properties of $T$ just stated were derived (see also [1], [5]).

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In view of the importance of singular integrals any weakening or significant modification of assumptions about the kernel $K$ may be of interest. For example, the theorem just stated about the operation $\mathcal{F} = TF$ holds if the modulus of continuity \( \omega(t) \) is replaced by the integral modulus of continuity \( \omega(t) \) (see below), that is, if
\[
\int_0^1 \frac{\omega(t)}{t} dt < \infty
\]

(1.3)

because, as was shown in [4], (1.3) implies (1.2). In that paper, it was also shown that (1.3) implies
\[
\int \log^+ |K(x)| dx < \infty
\]

(1.4)

which had been previously known to guarantee that $T$ is of type $(p, p)$, $1 < p < \infty$ (see [3]).

2. In this paper we want to establish some additional relations between the conditions (1.2), (1.3) and (1.4). As we said, (1.3) implies both (1.2) and (1.4). Here we shall show that, conversely, (1.2) implies (1.3) and (1.4). We recall the definition of \( \omega(t) \) (see [4]). Let $q$ be a proper rotation of $\mathbb{R}^n$ about the origin and let
\[
|q| = \sup_{|x| = 1} |x - qx|.
\]

Then
\[
\omega_{q}(t) = \sup_{|x| = 1} \int |K(x) - K(qx)| dx,
\]

where $dx$ denotes the surface area element of $\Sigma = \{ |x| = 1 \}$. We shall also consider two more moduli of continuity of the kernel $K(x)$, namely
\[
\omega_1(t) = \omega_{q}(t, a, b, qy) = \int_{|qy| = 1} |K(x - qy) - K(x)| dx,
\]

(2.1)

\[
\omega_2(t) = \omega_{q}(t, a, b) = \sup \int_{|qy| = 1} |K(x - qy) - K(x)| dx,
\]

(2.2)

where $0 < a < b$ and $|qy| = 1$. Let $a > 1$. Setting
\[
I_a(y) = \int_{|x| = 1} |K(x) - K(y)| dx,
\]

(2.3)

\[ J_a(y) = \sup_{y' \sim y} I_a(y'), \quad J_a(K) = \sup_y J_a(y) \]

(notice that $I_a(y)$ is a homogeneous function of degree zero of $y$), our main result can be formulated as follows:

**Theorem.** Let $K(x)$ be positively homogeneous of degree $-n$ and locally integrable in $|x| \neq 0$. Then, if $1 < a < b$,

(i) $I_a(y) \leq I_b(y) \leq \frac{b-1}{a-1} I_b(y)$;

(ii) $\frac{a}{b-a} \int_0^a \frac{\omega_2(t)}{t} dt \leq I_a(y) \leq \frac{b}{b-a} \int_0^b \frac{\omega_2(t)}{t} dt$, where $a$ and $b$ are as in (2.1) and $y = y'/|y|$;

(iii) $\frac{a}{b-a} \int_0^a \frac{\omega_2(t)}{t} dt \leq J_a(K) \leq \frac{b}{b-a} \int_0^b \frac{\omega_2(t)}{t} dt$, $\delta > 0$, where $\delta$ depends only on the dimension $n$, and $c$ depends on $n, a$ and $b$;

(iv) $\omega_1(t) \leq \omega_0(t), 0 < t < 2$, where $c$ depends on $a, b$ and $n$; and finally, if

\[
\lambda = J_a(K) + \int_{|a| < |q| < \infty} |K(a)| d\nu < \infty,
\]

then

\[
\int_{|a| < |q| < \infty} \frac{|K(a)|}{\lambda} \log \left(1 + \frac{|K(a)|}{\lambda}\right) d\nu \leq c \text{ where } c \text{ depends on } a, b, a \text{ and } n.
\]

3. We begin proving (i). Let $\beta - 1 \leq 2(a-1)$. Then
\[
J_a(K) = \int_{|x| > |y|} |K(x) - K(y)| dx\]

(3.1)

\[
\leq \int_{|x| > |y|} |K(x) - K(x - 2y)| dx + \int_{|x| > |y|} |K(x - 2y) - K(y)| dx.
\]

Setting $\Xi = x - 2y$ in the first integral on the right above and observing that $|\Xi - y| \geq a|y|$ implies $|\Xi| \geq (2a-1)\frac{|y|}{2} \geq \frac{3a}{2}$, we see that this integral is majorized by
\[
\int_{|x| > |y|} |K(x - 2y) - K(x)| dx.
\]

Now, because $\beta \leq 2a$, this also majorizes the second integral on the right of (3.1). Consequently,
\[
\int_{|x| > |y|} |K(x) - K(y)| dx \leq 2 \int_{|x| > |y|} |K(x - 2y) - K(y)| dx
\]

and
\[
I_a(y) \leq 2I_b(y/2) = 2I_b(y)
\]
which implies (i) for \( \beta = 1 \leq 2(a - 1) \). In particular, we have

\[ I_{(y)} \leq 2I_{(x)}, \quad \beta - 1 = 2(a - 1) \]

and from this we obtain

\[ I_{(y)} \leq 2^k I_{(x)}, \quad 2^{k-1}(a - 1) \leq \beta - 1 \leq 2^k - 1, \quad k = 1, 2, ... \]

whence (i) follows in the general case.

To prove (ii) we set \( x = \bar{x}, y = \bar{y}, |\bar{x}| = |\bar{y}| = 1 \) in the integral defining \( I_{(y)} \) in (2.3) and obtain

\[ I_{(y)} = \int_{|y| < \delta} |K(x - y) - K(x)|\,dy \int_{\partial \Sigma} |K(\bar{x} - s\bar{y}) - K(\bar{x})|\,ds \]

where \( \Sigma \) denotes the unit sphere \( |x| = 1 \) and \( d\sigma \) the surface area element.

We replace now \( t \) by the variable \( t = \bar{r} \), where \( \bar{r} \) is a constant for the moment, and, using the homogeneity of \( K(x) \), we find that

\[ I_{(y)} = \int_{|y| < \delta} \frac{dt}{t} \int_{\partial \Sigma} |K(\bar{x} - t\bar{y}) - K(\bar{x})|\,d\sigma. \]

If we integrate this equation with respect to \( \bar{r} \) over the interval \( (a, b) \), \( 0 < a < b \), and we write \( t \) for \( \bar{r} \), we obtain

\[ I_{(y)} = \frac{1}{b - a} \int_{a}^{b} dt \int_{0}^{\infty} e^{t-1} \bar{r} \int_{\partial \Sigma} |K(\bar{x} - t\bar{y}) - K(\bar{x})|\,d\sigma \]

where \( |\Sigma| \) denotes the surface area of \( \Sigma \).

Let now \( 0 < a < c < b < b_1, a_1 = a_2 = b = a + a_2 \), and set

\[ \theta(y) = \int_{c < |y| < b} |K(x - y) - K(x)|\,dx, \]

\[ \theta_1(y) = \int_{c < |y| < a_1} |K(x - y) - K(x)|\,dx. \]

Then, if \( |y| \leq \frac{a}{2}, |y| \leq a \), we have

\[ \theta(y_1 + y_2) \leq \frac{1}{c < |x| < b} \int_{c < |y| < b} |K(x - y_1) - K(x - y_2)|\,dx + \int_{c < |y| < a_1} |K(x - y_1) - K(x)|\,dx \leq \theta_1(y_1) + \theta_1(y_2). \]

If \( A_1 \) denotes the annulus \( \lambda \leq |x| \leq 2\lambda \) and

\[ a_1 = \frac{1}{|A_1|} \int_{|y| < a_1} \theta_1(y)\,dy, \quad E_1 = \{ y \in A_1, \theta_1(y) \leq 4a_1 \}, \]

then \( \theta_1(y) > 4a_1 \) on \( A_1 \setminus E_1 \), which clearly implies \( |A_1 - E_1| < \frac{1}{4} |A_1| \) and, consequently, \( |E_1| > \frac{3}{4} |A_1| \). Now, if \( y = y_1 + y_2, y_1, y_2 \in E_1 \), then according to (3.4) we have

\[ \theta(y) \leq \theta_1(y_1) + \theta_1(y_2) < 8a_1. \]
But Lemma 1 asserts that $E_1 + E_2$ contains the sphere $|y| < \delta_1$ and we find that the preceding inequality holds for $y \leq \delta_1$. Recalling the definition of $\omega_3(t)$, this shows that $\omega_3(t) < 8\delta_1$ for $t \leq \delta_1$. Thus we have

$$\int_{\delta_1}^{\delta} \frac{\omega_3(t)}{t} dt < 8\delta_1 \log 2 = 8 \log \frac{1}{|x|} \int_{\delta_1}^{\delta} \theta_1(y) dy < c \int_{\delta_1}^{\delta} \theta_1(y) y^{-\alpha} dy.$$

Setting $\lambda = 2^{-\alpha} a \alpha / a - 2^{-\alpha} a / 2a$, $h = 1, 2, \ldots$, and adding the corresponding inequalities we obtain

$$\int_{\delta}^{\delta_1} \frac{\omega_3(t)}{t} dt \leq c \int_{|y| < \delta} \theta_1(y) y^{-\alpha} dy,$$

which combined with (3.3) yields

$$\int_{\delta}^{\delta_1} \frac{\omega_3(t)}{t} dt \leq \omega_1(K),$$

where $c$ depends on $a$ and $b$. Thus the first half of (iii) is established. The second half follows from the second inequality in (i) by observing that $\omega_0(t) \leq \omega_1(t)$.

We pass now to the proof of (iv). Our argument depends on Lemma 6 in [4], which is also valid in the following slightly different situation.

**Lemma 2.** There exist positive constants $c, \eta$ depending only on the dimension $n$ such that if

$$A = \{ x \mid a < |x| < \delta \}, \quad A' = \{ x \mid a - \delta < |x| < b + \delta \},$$

$$A'' = \{ x \mid a - 2\delta < |x| < b + 2\delta \},$$

and $h(x) = [h_1(x), \ldots, h_n(x)]$ is a $C^1$ vector-valued function satisfying

(a) $|h(z)| \leq \delta \leq \delta_0, \quad \left| \frac{\partial h_i}{\partial y_j}(z) \right| \leq \eta$ for all $z \in A'$,

then

$$\int_{A'} \left| f(z + h(z)) - f(z) \right| dz \leq c \sup_{|y| < \delta} \int_{A'} \left| f(z + y) - f(z) \right| dz$$

for every function $f$ integrable in $A''$.

To prove the lemma we argue as follows. Choosing $\eta$ sufficiently small, the matrices with entries $\delta y_j + \partial h_i / \partial z_j$, which is the functional matrix of the change of variables $z = x + h(z)$, will have a determinant of absolute value larger than $1/2$. Consequently, for any function $g(x)$ we shall have

$$\int_{A''} \left| g(z + h(z)) \right| dz \leq 2 \int_{A'} |g(z)| dz,$$

as is readily seen by changing variables in the integral on the left. Let now $\varphi > 0$ be a function in $C_c^\infty$ with support in $|x| \leq 1$ and such that $\varphi \varphi dx = 1$, and let

$$\varphi_0(x) = \delta^{-\eta} \varphi \left( \frac{x}{\delta} \right), \quad f_0(x) = f * \varphi_0.$$

Let us also denote the supremum on the right-hand side of (3.5) by $\omega(\delta)$. Then, since $\varphi_0(x)$ has support in $|y| \leq \delta$ and $\int \varphi_0(x) dx = 1$, we have

$$\int_{A''} \left| f(z) - f_0(z) \right| dz = \int_{A'} \int \left| f(z) - f(z + y) \right| \varphi_0(y) |y| dy dz \leq \int \varphi_0(y) \int_{A'} \left| f(z) - f(z + y) \right| dz dy \leq \omega(\delta),$$

that is,

$$\int_{A'} \left| f(z) - f_0(z) \right| dz \leq \omega(\delta),$$

which combined with (3.6) gives

$$\int_{A''} \left| f(z + h(z)) - f_0(z + h(z)) \right| dz \leq 2\omega(\delta).$$

On the other hand, because

$$\int \frac{\partial}{\partial x_j} \varphi_0(y) dy = 0 \quad \text{and} \quad \int \frac{\partial}{\partial x_j} \varphi_0(y) dy \leq c \delta^{-1},$$

we have

$$\int_{A''} \left| \frac{\partial f_0}{\partial y_j}(z) \right| dz = \int_{A'} \int \left| f(z + y) - f(z) \right| \frac{\partial}{\partial y_j} \varphi_0(y) dy dz \leq \int_{A''} \int \left| f(z + y) - f(z) \right| \frac{\partial}{\partial y_j} \varphi_0(y) dy dz \leq c \delta^{-1} \omega(\delta).$$

From this and (3.6) which is also valid with $h(z)$, $0 \leq \theta \leq 1$, replacing $h(z)$, we obtain

$$\int_{A''} \left| f_0(z + h(z)) - f_0(z) \right| dz \leq \int_{A'} \left| \sum_{j=1}^n \frac{\partial f_0}{\partial y_j} (z + h(z)) h_j(z) dz \right| dz \leq \delta \sum_{j=1}^n \int_{A'} \left| \frac{\partial f_0}{\partial y_j} (z + h(z)) \right| dz \leq 2 \delta \int_{A''} \left| \frac{\partial f_0}{\partial y_j} (z) \right| dz \leq 2 c \omega(\delta).$$
that is,
\[ \int \left| f_1(\zeta + h) - f_2(\zeta) \right| d\zeta < 2\pi \omega(\zeta), \]
and this combined with (3.7) and (3.8) gives the desired result.

Returning to the proof of (iv), let \( \Delta \) be the annulus \( \{ \zeta \mid 1 + \delta_1 < |\zeta| < b - \delta_1 \} \), where \( \delta_1 < a, \ 2\delta_1 < b - a \). Then, if \( \varphi \) denotes a rotation of \( \mathbb{R}^n \) about the origin, we have
\[ \int_{\mathbb{R}^n} |K(\varphi) - K(\zeta)| \, d\zeta = \frac{1}{\log \frac{b - a}{a + \delta_1}} \int |K(\varphi) - K(\zeta)| \, d\zeta. \]

This is readily seen if one takes into account the fact that \( K(\varphi) \) is homogeneous of degree \( -n \). Setting \( \varphi(x) = \varphi(a + x) = \varphi(x) \), and using the preceding lemma we find that
\[ \sup_{|\zeta| < \delta} \int |K(\varphi) - K(\zeta)| \, d\zeta \leq \frac{c}{\log \frac{b - a}{a + \delta}} \sup_{|\zeta| < \delta} \int |K(a + y) - K(\zeta)| \, dy, \]
provided that \( \delta \) is sufficiently small, say \( \delta \ll \varepsilon \), where \( \varepsilon \) depends only on the dimension \( n \). But according to the definitions of \( a_1 \) and \( a_2 \), this inequality is the same as
\[ a_1(\zeta) \leq a_2(\zeta), \quad t \leq \varepsilon. \]

In order to extend this inequality to the interval \( \varepsilon \leq t \leq 2 \), we observe that the group of proper rotations is compact and connected, and consequently, there exists a finite collection of rotations \( \varphi_0, \varphi_1, \ldots, \varphi_n \) such that for every \( \varphi \) there exists an element \( \varphi_i \) of this collection with the property that
\[ |\varphi - \varphi_i| \leq \varepsilon, \]
for all \( \varphi \) with \( |\varphi| = 1 \). Furthermore, there exist \( \varphi_0, \varphi_1, \ldots, \varphi_n \) such that \( I = \{ 0, 1, 2, \ldots, n \} \).

In other words, we have
\[ |\varphi - \varphi_1| \leq \varepsilon, \quad |\varphi_0 - \varphi_1| \leq \varepsilon, \quad |\varphi_i| < \varepsilon. \]

Now
\[ \int |K(\varphi) - K(\zeta)| \, d\zeta \leq \int |K(\varphi) - K(\varphi_1)| \, d\zeta + \]
\[ + \sum_{k=1}^{n} \int |K(\varphi_k) - K(\varphi_{k+1})| \, d\zeta + \int |K(\varphi_1) - K(\zeta)| \, d\zeta = \int |K(\varphi_1) - K(\zeta)| \, d\zeta + \]
\[ + \sum_{k=1}^{n} \int |K(\varphi_k) - K(\varphi_{k+1})| \, d\zeta \]
and since all rotations in this last expression have modulus less than \( \varepsilon \), we find that
\[ \int |K(\varphi) - K(\zeta)| \, d\zeta \leq \int |K(\varphi_1) - K(\zeta)| \, d\zeta \]
which implies that for \( t \geq \varepsilon \)
\[ a_1(\zeta) \leq (k + 1) a_1(\zeta) \leq \varepsilon(a(\zeta) + a(\zeta)\varepsilon). \]

Now it evidently depends only on the dimension \( n \). Thus (iv) is valid for all \( t \).

Now there only remains to prove (v). Clearly it will suffice to prove this inequality for the positive part \( K^+ \) of \( K \) (without loss of generality we may assume that \( K \) is real). Evidently we have
\[ |K^+(\varphi - y) - K^+(\zeta)| \leq |K(\varphi - y) - K(\zeta)|, \]
and, as account of (3.3),
\[ \int_{|y| < \varepsilon} |K^+(\varphi - y) - K^+(\zeta)| \, dy \leq |\zeta|^2 \int_{|y| > \varepsilon} |K(y)| \, dy. \]

Consider now the maximal function of \( K^+ \):
\[ \overline{K}(\varphi) = \sup_{|y| < \varepsilon} \int_{|y| < \varepsilon} |K^+(\varphi - y) - K^+(\zeta)| \, dy. \]

Then
\[ \overline{K}(\varphi) \leq \sup_{|y| < \varepsilon} \int_{|y| < \varepsilon} |K^+(\varphi - y) - K^+(\zeta)| \, dy \leq \Omega K^+(\varphi), \]
where \( \Omega \) is the measure of \( \{ |y| < 1 \} \), and
\[ \overline{K}(\zeta) \leq \sum_{|y| < \varepsilon} \int_{|y| < \varepsilon} |K^+(\varphi - y) - K^+(\zeta)| \, dy \leq \Omega K^+(\zeta). \]

Integrating with respect to \( \varphi \) and using (3.10) we find that
\[ \overline{K}(\varphi) \, d\varphi \leq |\zeta|^2 \int_{|y| > \varepsilon} \frac{b - a}{a} J_{b} + \Omega \int |K^+(\zeta) \, d\zeta. \]
But a theorem of E. M. Stein (see [6]) asserts that if the maximal function \( K \) of \( K^+ \log(1 + K^+) \), and since the same argument applies to the negative \( K^- \) of \( K \), we conclude that

\[
\int_{a \leq |x| \leq b} |K(a)| \log(1 + |K(a)|) \, dx < \infty
\]

for every function \( K(a) \) which is homogeneous of degree \(-n\), is locally integrable in \( |x| > 0 \) and for which \( J_a(K) < \infty \) for some \( a, \alpha > 1 \). But this implies the inequality in (v). To prove this implication consider the convex function \( \Phi(t) = \log(1 + t) \), \( t > 0 \), and the space \( L_\alpha \) of functions \( F \) in \( a \leq |x| \leq b \) with the property that \( \Phi(|F|) \) is integrable, and define a norm in \( L_\alpha \) by (see [7], Chapter IV, Section 10)

\[
\int_{a \leq |x| \leq b} \Phi\left( \frac{|F|}{\|F\|_\alpha} \right) \, dx = 1.
\]

On the other hand, consider also the space \( B \) of functions \( K(a) \) which are homogeneous of degree \(-n\) and for which \( J_a(K) < \infty \), \( a, \alpha > 1 \), with the norm

\[
\|K\|_\alpha = J_a(K) + \int_{a \leq |x| \leq b} |K(a)| \, dx.
\]

As is readily verified, \( B \) is a Banach space and its embedding in \( L^1(a \leq |x| \leq b) \) is continuous.

Now, what we have shown above is that \( B \subset L_\alpha \). Consequently, we have

\[
B = L_\alpha \subset L^1(a \leq |x| \leq b).
\]

But the embedding of \( B \) in \( L^1(a \leq |x| \leq b) \) is continuous and, as is readily seen, so is that of \( L_\alpha \). Thus, according to the closed graph theorem, the embedding of \( B \) in \( L_\alpha \) is also continuous, that is, there exists a constant \( c \) such that

\[
\|K\|_\alpha \leq c \left( J_a(K) + \int_{a \leq |x| \leq b} |K(a)| \, dx \right).
\]

Clearly, we may assume that \( c > 1 \). Now, as is readily verified, the function \( \Phi(t)/t^2 \) is a decreasing function of \( t \) and therefore, since \( c > 1 \), we have

\[
\Phi(t/c)/\left( t/c \right)^2 \geq \Phi(t)/t^2,
\]

that is,

\[
\Phi(t/c) \geq \frac{1}{c^2} \Phi(t), \quad c > 1.
\]

Thus, setting

\[
\lambda = \int_{a \leq |x| < b} |K(a)| \, dx,
\]

(3.11) becomes \( \|K\|_\alpha \leq \alpha \lambda \), and we obtain

\[
1 = \int_{a \leq |x| < b} \Phi\left( \frac{|K|}{\|K\|_\alpha} \right) \, dx \geq \int_{a \leq |x| < b} \Phi\left( \frac{|K|}{\alpha \lambda} \right) \, dx \geq \frac{1}{\alpha^2} \int_{a \leq |x| < b} \Phi\left( \frac{K}{\lambda} \right) \, dx,
\]

that is

\[
\int_{a \leq |x| < b} \Phi\left( \frac{K}{\lambda} \right) \, dx \leq \alpha \lambda,
\]

which is the inequality (v). This concludes the proof of our theorem.

References.


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