Method of orthogonal projections and approximation of the spectrum of a bounded operator

by

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Abstract. For a given bounded operator \( A \) and a sequence \( (P_n) \) of orthogonal projections converging strongly to the identity operator on a complex Hilbert space \( H \) we can define operators

\[ A_n = P_n A |_{P_n H} : P_n H \rightarrow P_n H. \]

These operators are compressions of \( A \) and approximate it in some way. In this work the asymptotical behaviour of spectra of operators \( A_n \) is studied.

Notation. In the following \( H \) will denote a complex Hilbert space with a scalar product \( \langle \cdot , \cdot \rangle \), \( L(H) \) denotes the space of linear bounded operators on \( H \), \( F(H) \), \( L^0(H) \) denote the sets of finite-dimensional and compact linear operators on \( H \). By a projection (not necessary orthogonal) is meant an operator \( P \in L(H) \) with \( P^2 = P \). Two projections are said to be ordered in their natural order \( P < Q \) if \( PQ = QP = P \). \( P \in F(H) \), \( P = P^* \).

The spectrum, resolvent set of an operator \( A \) are denoted by \( \Sigma(A) \), \( \rho(A) \) respectively. \( \partial \Omega \) means a boundary of a set \( \Omega \), \( \Omega_\epsilon(\lambda) \) means the \( \epsilon \)-neighbourhood of the set \( \Omega \).

If \( \lambda \in C, \Omega \subset C \) (\( C \) — the complex plane) then

\[ \delta(\lambda, \Omega) = \inf_{\omega \in \Omega} |\omega - \lambda|, \]

\[ \text{dist}(\Omega', \Omega'', \epsilon) \] denotes the Hausdorff distance between the sets \( \Omega', \Omega'' \).

We define (following [1], VII) a spectral set for the operator \( A \) to be any set \( \Omega \subset C \) for which \( \Omega \cap \Sigma(A) \) is open and closed in \( \Sigma(A) \). For each spectral set \( \Omega \) the projection \( B(\Omega, A) \) is defined by the formula

\[ B(\Omega, A) = \frac{1}{2\pi i} \int B(\lambda, A) d\lambda, \]

where \( B(\lambda, A) = (A - \lambda)^{-1} \), and \( \Gamma \in \rho(A) \) is any rectifiable Jordan curve containing \( \Omega \cap \Sigma(A) \) but no other points of \( \Sigma(A) \) in its interior. It is known
that
\[ E(\Omega, A)A = AE(\Omega, A), \]
\[ E(\Omega, A)E(\Omega', A) = E(\Omega \cap \Omega', A), \]
\[ E[I|_{E(\Omega, A)}] = A \cap A, \]
\[ E[I|_{E(\Omega, A)}] = 1, \text{ and if } \Omega \subset A \text{ then } E(\Omega, A) = 0. \]

**Basic lemmas.** Two following lemmas are well known.

**Lemma 1.** ([3], Lemma 3.1, p. 151). If \( \{P_n\} \) is a sequence of projections in \( H \), \( P_n \rightarrow 1 \) strongly, then \( P_n \rightarrow 0 \) strongly.

**Lemma 2.** ([5], p. 131, [2], VII.6, p. 585). If \( A, B \in L(H) \), \( \lambda \in \sigma(A) \), \( \|A - B\|\|E[A, A]\| < 1 \) then \( \lambda \in \sigma(B) \) and the following inequalities hold
\[ \|R(\lambda, B)\| < 2\|R(\lambda, A)\|, \]
\[ \|R(\lambda, B) - R(\lambda, A)\| < 2\|A - B\|\|E[A, A]\|. \]

It is also known that \( \|R(\lambda, A)\| > \|E[A, A]\|^{-1} \) for \( \lambda \in \sigma(A) \).

**Lemma 3.** Let \( Q, P_n, n = 1, 2, \ldots \), be projections in \( H \) such that \( Q \in F(H), P_n \rightarrow 1 \) strongly, \( P_n \rightarrow 1 \) strongly, then there exists a sequence \( \{Q_n\} \) of projections in \( H \) such that
\[ Q_n < P_n, \quad \|Q - Q_n\| \rightarrow 0. \]

**Proof.** Let \( B_n = QP_n + (1 - Q) \), note that
\[ \|B_n - 1\| \leq \|QB_n - Q\| + \|B_nQ - Q\|. \]

It follows from Lemmas 1 and 2 that for a large enough there exist operators \( B_n^{-1} \) and
\[ \|1 - B_n^{-1}\| < 2\|P_nQ - Q\|. \]

Note that
\[ 1 - B_n^{-1}B_n = B_n^{-1}QBP_nB_n^{-1}QBP_n(1 - Q) = QBP_nB_n^{-1}QBP_n(1 - Q) = 0. \]

Multiplying this identity by \( (1 - Q) \) we obtain that
\[ 1 - Q = (1 - Q)B_n^{-1}. \]

We shall show that the operators \( Q_n - P_nB_n^{-1}QBP_n \) satisfy the thesis of our lemma:
\[ Q_n - Q_n^2 = P_n(1 - B_n^{-1}QBP_n)B_n^{-1}QBP_n = P_nB_n^{-1}(1 - Q)B_n^{-1}QBP_n 
- P_nB_n^{-1}(1 - Q)QP_n = 0. \]

Thus \( Q_n \) is a projection and obviously \( Q_n < P_n \).

The convergence \( \|Q_n - Q\| \rightarrow 0 \) follows from Lemma 1, (1) and the computation below
\[ \|Q - Q_n\| = \|Q - P_nB_n^{-1}QBP_n\| \]
\[ < \|Q - P_n\| + \|QBP_n - P_nB_n^{-1}QBP_n\| \]
\[ < \|Q - Q_n\| + \|QBP_n - P_nB_n^{-1}QBP_n\| \]
\[ < \|Q - Q_n\| + \|P_nB_n^{-1}QBP_n - P_n\| \]
\[ < \|Q - Q_n\| + 2\|P_nB_n^{-1}QBP_n\| \rightarrow 0. \]

The next lemma shows that the assumptions of Lemma 3 are necessary.

**Lemma 4.** Let \( \{P_n\} \) be a sequence of projections in \( H \) such that \( \|P_n\| \rightarrow 1, n = 1, 2, 3, \ldots \). If for any projection \( Q \in F(H) \) there exists a sequence \( \{Q_n\} \) of projections in \( H \) such that \( Q_n < P_n, \|Q_n - Q\| \rightarrow 0 \) then
\[ P_n \rightarrow 1 \text{ strongly and } P_n \rightarrow 1 \text{ strongly}. \]

**Proof.** For a given \( \lambda \in \sigma(A) \) take \( y \in H \) such that \( \langle y, y \rangle = 1 \). Define the projection \( Q \) by the formula \( Qz = \langle z, y \rangle y \) then
\[ Q^*z = \langle z, y \rangle y \] and \( Q^*y = y \).

Let \( \{Q_n\} \) be a sequence of projections such that \( \|Q_n - Q\| \rightarrow 0 \), \( Q_n < P_n \). Then
\[ \|P_n^* - P_n\| \leq \|P_n^*Q_n - Q_n^* - P_nQ_n - Q_n\| \leq \|Q_n^* - Q\| \]
\[ \leq \|Q_n^* - Q\| < \|Q - Q_n\| \rightarrow 0 \text{ as } n \rightarrow 0. \]

This shows that \( P_n \rightarrow 1 \) strongly. Proof of the convergence \( P_n \rightarrow 1 \) strongly is alike as we omit it.

Lemmas 1–4 are also valid in any Banach space.

**Spectra and numerical ranges.** The set \( \Sigma(A) \) of all those \( \lambda \in \sigma(A) \) such that \( \lambda \) is an isolated point of \( \sigma(A) \) and \( B(\lambda, A) = B(\lambda, I) \in F(H) \) is called the discrete spectrum of the operator \( A \).

The set \( \Sigma(A) \) is called (Browder) **essential spectrum**.

N. Salinas proved in [6]:

**Lemma 5.**
\[ \Sigma(A) = \bigcap_{1 \neq P \in L(H)} \sigma(PA). \]

The numerical range \( W(A) \) of an \( A \in L(H) \) is defined as
\[ W(A) = \{\langle Ax, x \rangle; \|x\| = 1\}, \]

the essential numerical range is given by the formula
\[ W_e(A) = \bigcap_{1 \neq P \in L(H)} W(PA). \]
It is known that $W(A), W_+ (A)$ are convex sets, that $\Sigma (A) \subset W(\bar{A})$ and
\begin{align}
E (\lambda, A) = \{ d (\lambda, W (\bar{A})) \}^{-1} \quad \text{for} \quad \lambda \notin W (\bar{A}) \tag{2}
\end{align}

(3), Th. 3.2). Then next lemma gives a useful characterization of $W_+ (A)$.

**Lemma 6.** The following conditions are equivalent:

(i) $\lambda \in W_+ (A)$;

(ii) $(a_n, x_n) \to \lambda$ for some sequence of unit vectors such that $x_n \to 0$ weakly.

(iii) $\lambda \in \bigcap_{\rho \in \text{Re}(\rho)} \overline{W (P \Lambda \rho)}$.

For a proof we send reader to [2] and [5].

These lemmas imply that $\Sigma_2 (A) \subset W_+ (A)$ so
\begin{align}
\Sigma (A) \cap W_+ (A) \subset \Sigma_2 (A).
\end{align}

$W_+ (A)$ is a convex set, so it is obvious that the convex hull of $\Sigma_2 (A)$ is contained in $W_+ (A)$. N. Salinas proved in [4] that if $A \in L(H)$ is a hypo-norm operator then $\text{conv} \Sigma_2 (A) = W_+ (A)$.

**Projections.** For a given operator $A \in L(H)$, and a given sequence $\{P_n\}$ of projections in $H$, we define operators
\begin{align}
\overline{\lambda}_n = P_n \lambda P_n \in L(H_n) \quad \text{where} \quad H_n = P_n H.
\end{align}

The following theorem holds:

**Theorem 1.** If $A \in L(H)$, $\Omega$ is a subset of the complex plane such that $\Omega \cap \overline{W_+ (A)} = \emptyset$ and $\bigcap \Sigma_2 (A)$, $\{P_n\}$ is a sequence of orthogonal projections in $H$, $P_n \to 1$ strongly then
\begin{align}
\| E (\Omega, A) - E (\Omega, \overline{\lambda}_n) P_n \| \to 0 \quad \text{with} \quad n \to \infty.
\end{align}

It is known that if $P, Q$ are projections such that $P - Q < 1$ then $\text{dim} P H = \text{dim} Q H$ [(3), p. 33].

Using this result to projections $E (\Omega, A)$ and $E (\Omega, \overline{\lambda}_n) P_n$, we obtain under the assumptions of Theorem 1 the following

**Corollary 1.** (i) $\text{dist} \{ \Omega \cap \Sigma_2 (A), \Omega \cap \Sigma (A) \} > 0$.

(ii) If $\lambda \notin W_+ (A)$ then $\lambda \notin \Sigma (A)$ if and only if $d (\lambda, \Sigma (A_n)) > 0$.

**Proof of Theorem 1.**

1. We can choose a number $\varepsilon > 0$ in such a way that
\begin{align}
\inf \{ |w - y| : x, y \in W_+ (A) \} > \varepsilon \quad \text{and} \quad \Sigma_2 (A \cap \partial W_+ (A) \cap (\varepsilon)) = \emptyset.
\end{align}

Now put $G = W_+ (A) + (\varepsilon)$, note that the set $\Sigma (A \cap G)$ is a finite subset of $\Sigma_2 (A)$ thus the projection $Q - E (\Omega, G - A)$ is of a finite dimension.

Because $\Omega \cap G = \emptyset$ so $E (\Omega, A) < Q$, Lemma 3 implies that there exists a sequence $\{Q_n\}$ of projections in $H$ such that $Q_n \to P_n$, $\|Q_n - Q\| \to 0$.

Using the projections $Q_n, Q_n$ we define some new operators:
\begin{align}
C_A = A_{\Omega - \partial G} : (1 - Q) H \to (1 - Q) H < H,
\end{align}

$D_A = A_{\Omega - \partial G} : QH \to QH < H$,

$C_n = (1 - Q_n) A_{\Omega - \partial G} : (1 - Q_n) H \to (1 - Q_n) H < H$,

$D_n = Q_n A_{\Omega - \partial G} : Q_n H \to Q_n H < H$,

\begin{align}
C = Q_n A_{\Omega - \partial G} : (1 - Q_n) H \to (1 - Q_n) H < H,
\end{align}

$D = Q_n A_{\Omega - \partial G} : (1 - Q_n) H \to (1 - Q_n) H < H$,

This implies that
\begin{align}
\| C - B_n \| \leqslant \| Q_n - Q \| (1 + \| Q_n - Q \| + \| 1 - Q_n \| + \| 1 - Q_n \|)
\end{align}

so
\begin{align}
\| C - B_n \| \to 0.
\end{align}

This with the definitions of $\overline{\lambda}_n$ and $B_n$ implies
\begin{align}
\| \overline{\lambda}_n - B_n \| \leqslant \| A - B_n \| \to 0.
\end{align}

**Note that**

\begin{align}
\Sigma (G) = \Sigma (A \cap \partial G), \quad \Sigma (D) = \Sigma (A \cap \partial G).
\end{align}

This in part I shall show that for a large enough $\Sigma (\overline{\lambda}_n) \subset G$ and that there exists such number $M$ that for a large enough
\begin{align}
\| R (\lambda, \overline{\lambda}_n) \| \leqslant M \quad \text{for} \quad \lambda \notin \partial G.
\end{align}

If this is not true we could find sequences $a_n \in H_n$, $\lambda_n \in G$ such that:
\begin{align}
|a_n| = 1, \quad \lambda_n \notin \partial G, \quad \text{and zero is a cluster point of the sequence} \quad \| R (\lambda_n - \lambda_n) a_n \|.
\end{align}

The norms of the operators $\overline{\lambda}_n$ are bounded by a constant $\varepsilon$ independent of $n$, thus
\begin{align}
\| R (\lambda_n - \lambda_n) a_n \| \geqslant \| a_n \| - \| \overline{\lambda}_n \| \geqslant | \lambda | - r > 1 \quad \text{when} \quad | \lambda | > r + 1.
\end{align}

This shows that the sequence $\lambda_n$ is bounded. Choosing a subsequence and changing indices we can assume that $\lambda_n \to \lambda_n, a_n \to a_n$ weakly, $\| R (\lambda_n - \lambda_n) a_n \| \to 0$. Because $\lambda_n \notin \partial G$ so
\begin{align}
\lambda_n \notin \partial G.
\end{align}
An easy computation shows that
\[ \|Q_{n} - \lambda_{n}\|_{n} \to 0. \]
Because \( Q_{n} \to 0 \) so \( \|Q_{n}\| = \|Q - Q_{n}\| \to 0 \) and for any \( y \in H \)
\[ \langle Q_{n}y, y \rangle = \lim_{n} \langle Q_{n}y, y \rangle = 0; \]

hence

(7)
\[ Q \xi = 0. \]

The following identity holds:
\[ \langle Ax_{n}, x_{n} \rangle - \lambda_{n} = \langle Ax_{n}, x_{n} \rangle - \langle P_{n}(1 - Q_{n})Ax_{n}, x_{n} \rangle + \langle (Q_{n} - \lambda_{n})x_{n}, x_{n} \rangle \]
\[ = \langle Ax_{n}, x_{n} \rangle - \langle (1 - Q_{n})Ax_{n}, x_{n} \rangle + \langle (Q_{n} - \lambda_{n})x_{n}, x_{n} \rangle \]
\[ = \langle (Q_{n} - Q)Ax_{n}, x_{n} \rangle + \langle Ax_{n} - \lambda_{n}, x_{n} \rangle + \langle (Q_{n} - \lambda_{n})x_{n}, x_{n} \rangle. \]

To obtain the last equality we use the relations \( Q_{n}x_{n} = 0 \) and \( A = Q.A \).
This identity implies that
\[ \|Q_{n}x_{n} - \lambda_{n}\|_{n} \leq 2\|A\|\|Q - Q_{n}\| + \|(Q_{n} - \lambda_{n})x_{n}\| \to 0 \]
but \( \lambda_{n} \notin W_{n}(A) \), this and Lemma 6 imply that \( \lambda_{n} \neq 0 \).
For any \( y \in H \) the following identity holds:
\[ \langle (1 - Q)(A - \lambda_{n})x_{n}, y \rangle = \langle (1 - Q)(A - \lambda_{n})x_{n}, P_{n}y \rangle + \langle (Q_{n} - Q)(A - \lambda_{n})x_{n}, y \rangle \]
\[ = \|A - \lambda_{n}\|\|y\| + \|Q_{n} - Q\|\|A - \lambda_{n}\|\|y\| + \|1 - Q\|\|A - \lambda_{n}\|\|y\| - P_{n}y\|. \]

and because
\[ \|A - \lambda_{n}\| = \langle (Q_{n} - \lambda_{n})x_{n}, y \rangle \]
\[ \leq 2\|A\|\|Q - Q_{n}\| + \|Q_{n} - Q\|\|A - \lambda_{n}\|\|y\| - P_{n}y\|. \]

Hence in the limit we obtain that
\[ \|A - \lambda_{n}\|\|y\| - P_{n}y\| \to 0, \]
but \( y \in H \) is arbitrary, \( x_{n} \in (1 - Q)H \) so \( C_{n} = \lambda_{n}x_{n} \). But \( x_{n} \to 0 \) so \( \lambda_{n} \in \Sigma(G) \)
can not be in \( \partial G \), this contradicts (6) and proves the statement of part II.

III. For a large enough \( \Sigma(C_{n}) \subseteq G \). This statement may be proved like part II, but note that in the special case when \( P_{n} = 1 \), the projections \( Q_{n} \) may be defined to be the previous ones. Then \( C_{n} = C_{0} \), so this part is a simple corollary from part II.

IV. It is enough to prove the theorem in the case when the boundary \( \partial G \) of the set \( G \) is a regular Jordan curve with a finite length \( \|\partial G\| \).

Let \( M_{1} = \sup \|R(\lambda, A)\| \). (4) implies that for a large enough
\[ \|A - B\| \|R(\lambda, A)\| \leq \|A - B\| M_{1} < \frac{1}{\lambda - \partial G}. \]
Using Lemma 2 we see that for such \( \lambda \in \partial G \) and for \( \lambda \in \partial G \)
\[ \|R(\lambda, B)\| \leq 2M_{1}, \|R(\lambda, A) - R(\lambda, B)\| \leq M_{1}\|A - B\|. \]
Hence

(8)
\[ \|E(\Omega, A) - E(\Omega, B)\| \leq \frac{1}{2\pi} \int_{\partial G} \|R(\lambda, A) - R(\lambda, B)\| d\lambda \]
\[ \leq \frac{\|\partial G\|}{2\pi} M_{1}\|A - B\|. \]

Because \( R(\lambda, B) = E(\lambda, C_{n})(1 - Q_{n}) + E(\lambda, D_{n})Q_{n} \)
so
\[ E(\lambda, B) = E(\lambda, C_{n})(1 - Q_{n}) + E(\lambda, D_{n})Q_{n}. \]

We have shown in part III that for a large enough \( C \subset G \subset G \) then \( E(\lambda, C_{n}) = 0 \), this with (8) and (9) imply that for a sufficiently large

(10)
\[ \|E(\lambda, A) - E(\lambda, D_{n})Q_{n}\| \leq \frac{\|\partial G\|}{\pi} M_{1}\|A - B\|. \]

Because for a large enough \( \partial G \subset \Sigma(G_{n}) \implies \Sigma(C_{n}) \subseteq G \implies \partial G \subset \Sigma(C_{n}) \subset \Sigma(G) \subset \partial G \).
Hence
\[ (11) \ E(\lambda, B) = E(\lambda, C_{n})(1 - Q_{n}) + E(\lambda, D_{n})Q_{n}. \]

The identity \( R(\lambda, B) = R(\lambda, C_{n})(1 - Q_{n}) + R(\lambda, D_{n})Q_{n} \) together with part II of this proof implies that for \( \lambda \in \partial G \) and sufficiently large \( n \)
\[ \|R(\lambda, B)\| \leq \|Q_{n}\|\|R(\lambda, C_{n})\| + \|E(\lambda, D_{n})Q_{n}\| \leq K \|A + B\| M_{1} \]
\[ \leq \|A + B\| M_{1} \]
where \( K = \sup \|Q_{n}\| \), because \( \|E(\lambda, D_{n})\| \leq \|R(\lambda, B)\| \leq M_{1} \). So for a large enough
\[ \|A_{n} - B_{n}\| \|R(\lambda, B)\| \leq \|A - B\| M_{1} \]
and from Lemma 2 the following relations hold:
\[ \partial G \subset \Sigma(\lambda_{n}), \]
\[ \|R(\lambda, A_{n}) - R(\lambda, B_{n})\| \leq \|A - B\| M_{1} \]
\[ \|E(\lambda, A_{n}) - E(\lambda, B_{n})\| \leq \frac{\|\partial G\|}{\pi} M_{1}\|A - B\|. \]

Integrating this inequality along the curve \( \partial G \) we obtain
\[ \|E(\lambda, A_{n}) - E(\lambda, B_{n})\| \leq \frac{\|\partial G\|}{\pi} (M_{1} + M_{2})\|A - B\|. \]

This with (10) and (11) imply
\[ \|E(\lambda, A) - E(\lambda, B)\| \leq \frac{\|\partial G\|}{\pi} (M_{1} + M_{2})\|A - B\|. \]
LEMMA 7. If \( H \) is an infinite dimensional Hilbert space, \( A \in L(H) \), \( P_n \in P(H), (\lambda_n)_{n=1}^\infty \) is a sequence of complex numbers such that \( \lambda_n \in W_n(A) \), \( (\mu_n)_{n=1}^\infty \) is a sequence of positive numbers then there exists a sequence \( (P_n)_{n=1}^\infty \) of orthogonal projections such that:

(i) \( P_n x = P_n x + \langle z_n, z_{n+1} \rangle x_{n+1} \) where \( P_n x_{n+1} = 0 \), \( ||x_{n+1}|| \leq 1 \);

(ii) \( \Sigma(A_{n+1}) \subset \Sigma(A_n) \cup \lambda_n \) where \( A_n = P_n A P_n, ||A_n - \lambda_n|| \leq \mu_n \);

(iii) \( A_n x_n = \lambda_n x_n, 0 < m \leq n \);

(iv) if \( \lambda_n \) is an interior point of \( W_n(A) \) then \( \lambda_n = \lambda_n \).

Proof. Suppose the projection \( P_n \) is just defined. Then let \( Q_n \) be the orthogonal projection onto the subspace \( P_n H + A P_n H + \lambda^* P_n H \). Because \( \lambda_{n+1} \in W_n(A) \subset W[(1-Q)A] \) so there exists a unit vector \( z_{n+1} \) such that:

\[
\langle A z_{n+1}, z_{n+1} \rangle - \lambda_{n+1} \leq \mu_{n+1}, Q_n z_{n+1} = 0.
\]

In this way we define one by one the projections \( P_n \) by the formula \( P_{n+1} x = P_n x + \langle z_n, z_{n+1} \rangle x_{n+1} \) and the numbers \( \lambda_n = \langle A z_n, z_n \rangle \).

Note that if \( \lambda_n \) is an interior point of \( W_n(A) \) then for any \( Q \in P(H) \)

\[
\lambda_n \in W[(1-Q)A] \setminus \Sigma(A),
\]

hence in this case we may choose \( z_n \) in such a way that \( \lambda_n = \lambda_n = \langle A z_n, z_n \rangle \).

Note that if \( x = P_n \) then \( A x = x_{n+1} \) so

\[
A_{n+1} x = P_{n+1} A P_n x = P_n A P_n x + \langle z_n, z_{n+1} \rangle x_{n+1} = A_n x.
\]

By induction the thesis of the lemma.

COROLLARY 2. If \( P_n \) is a sequence of finite dimensional orthogonal projections in \( H \) converging strongly to \( 1_H \), \( S \) is any subset of \( W_n(A) \), then there exists a sequence \( Q_n \) such that \( P_n < Q_n < P(H) \) and

\[
\text{dist}(\Sigma(A) \cup S, \Sigma(A_n)) \rightarrow 0 \quad \text{with} \quad n \rightarrow \infty,
\]

where

\[
A_n = P_n A P_n , \quad \Sigma(A_n) = L(P_n H).
\]

Proof. For any \( \delta > 0 \) there exists a finite subset \( S \) of \( S \) such that \( \text{dist}(S, S) < \delta \). It follows from the lemma that there exists a projection \( Q_n > P_n \) such that \( \text{dist}(\Sigma(A_n) \cup S_{n+1}, \Sigma(A_n)) < \delta \) hence

\[
\text{dist}(\Sigma(A_n) \cup S, \Sigma(A_n)) < 2 \delta n.
\]

This corollary explains why in Theorem 1 and Corollary 1 the set \( W_n(A) \) cannot be substituted by any smaller set.