## Contents of volume LXV, number 1

<table>
<thead>
<tr>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>C. -K. Fong, On M-hypormal operators</td>
<td>1-6</td>
</tr>
<tr>
<td>J. Johnson and J. Wolfe, Norm attaining operators</td>
<td>7-10</td>
</tr>
<tr>
<td>A. Piotrowski, Method of orthogonal projections and approximation of the spectrum of a bounded operator</td>
<td>21-29</td>
</tr>
<tr>
<td>J. M. Ash, F. F. Ass, C. L. Pepperman, and R. L. Jones, Singular integral operators with complex homogeneity</td>
<td>31-60</td>
</tr>
<tr>
<td>R. Kaufman, On the sum of two Brownian paths</td>
<td>51-54</td>
</tr>
<tr>
<td>R. A. Macias and C. Segovia, Singular integrals on generalized Lipschitz and Hardy spaces</td>
<td>55-75</td>
</tr>
<tr>
<td>A. P. Calderón and A. Zygmund, A note on singular integrals</td>
<td>77-87</td>
</tr>
</tbody>
</table>

### STUDIA MATHEMATICA

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### STUDIA MATHEMATICA, T. LXV. (1979)

#### On M-hypormal operators

by

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Abstract. Direct integral decompositions of dominant (or M-hypormal) operators and spectral operators which are quasi-affine transforms of M-hypormal operators are considered.

According to Stampfli and Wadhwa [6], a (bounded) operator $T$ on a Hilbert space $H$ is said to be dominant if $\text{range}(T - z) \subseteq \text{range}(T - z)^*$ for all $z \in C$, and $T$ is said to be M-hypormal if

$$\left\langle (T - z)x, x \right\rangle \leq M \left\langle (T - z)x, x \right\rangle$$

for all $z \in C$ and $x \in H$. It is not hard to see that the following statements are each equivalent to each other:

1. $T$ is dominant.
2. For each $z \in C$, there is an operator $A_z$ such that $T - z = (T - z)^* A_z$.
3. For each $z \in C$, there is a positive number $M_z$ such that

$$\left\langle (T - z)x, x \right\rangle \leq M_z \left\langle (T - z)x, x \right\rangle \quad (x \in H),$$

i.e.,

$$\langle (T - z)(T - z)^* \leq M_z^2 (T - z)^* (T - z) \rangle.$$

This follows from [1]. Also [1] implies that $T$ is M-hypormal if and only if, for each $z \in C$, there is an operator $A_z$ such that $\|A_z\| \leq M$ and $T - z = (T - z)^* A_z$.

In this paper we present some variants of the results in [6]. First we record a lemma which appears in [3].

**Lemma 1.** Let $T$ be a spectral operator on a Hilbert space $H$ with the resolution of the identity $E$. Let $C$ be a closed set in $C$ and $x \in H$. If there exists a bounded function $g : C \to \mathbb{C}$ such that $(T - z)g(z) = x$ for all $z$, then $E(0)x = x$.

The next lemma is the basis of the subsequent results. The proof is a modification of [6].
Lemma 2. Let $T$ be an $M$-hyponormal operator. Suppose there exists an operator $W$ one-one with dense range and a spectral operator $S$ such that $TW = WS$. Then there exists a positive operator $P$, a normal operator $N$ and a quasi-nilpotent operator $Q$ such that $(T - N)P = PQ$ and $TN = NT$.

Proof. By the polar decomposition of $W$ and the fact that $S$ is spectral, we may replace $W$ by a positive operator $P$ and assume that the scalar part $N$ of $S$ is normal. Let $N = \{ zE_x \}$ be the spectral decomposition of $N$. Since $T$ is $M$-hyponormal, for each $x \in C$, there is an operator $A_x$ such that $\| A_x \| \leq M$ and $T - z = (T - z)^* A_x$. Let $K$ be a closed set in $C$ and $x \in E(K)$. Then there is an analytic function $f : K \to H$ such that $(S - z)f(x) = x$. Thus, for $x \notin K$,

$$(T - z)^* A_x P f(x) = (T - z) P f(x) = P(S - z) f(x) = P x .$$

Hence

$$(S - z)^* A_x P f(x) = P(T - z)^* A_x P f(x) = P x .$$

Let $C$ be an arbitrary closed set in $C$ containing $K^* = \{ x \in C : \|x\| \leq K \}$ and a neighborhood of the infinity. Then $g(x) = P A_x P f(x)$ is bounded on $C - C$ and $(S - z) g(x) = P x$. By Lemma 1, $P x \in E(K)$. (Note that $x \to E(K)$ is the spectral measure of $N^*$ which is the scalar part of $S^*$.) Therefore $P x \in E(K)$. We have shown that $E(K) H$ is an invariant subspace of $P^2$ for every closed set $K$ in $C$. Regularity of the spectral measure $E$ thus implies that $N$ commutes $P$.

Now the identity $TP = PS$ can be written $(T - N)P = PQ$. Furthermore,


Since the range of $P$ is dense, we have $TN = NT$.

Corollary 3. If a spectral operator is $M$-hyponormal, then it has a normal scalar part.

Proof. From the proof of Lemma 2, we see that if $W$ is invertible, then so is $P$. Hence there is a normal operator $N$ such that $TN = NT$ and $T - N$ is quasi-nilpotent. The conclusion follows from the uniqueness of the canonical reduction of a spectral operator (see Dunford and Schwartz [2], Theorem XV, 4.3).

The following corollary is a special case of [3]; [3] is based on a result of Putnam [4].

Corollary 4. If $TW = WS$, where $S$ is spectral, $T$ is hypnormal and $W$ has a dense range, then $T$ is a normal scalar operator and $S$ is similar to $T$.

Proof. From Lemma 2, we have $TN = NT$ and $(T - N)P = PQ$ where $N$ is a normal operator, $P$ is a positive operator with a dense range and $Q$ is similar to the radical part of $S$. Now it suffices to show that $T - N = 0$.

Since $N$ is normal and $TN = NT$, Fuglede's theorem yields $T^* N = N T^*$. Furthermore, since $T$ is hypnormal, we have, for each $w \in H$ and $z \in C$,

$$\| (T - z)^* N w \|^2 = \| (T - z) N w \|^2 - 2 \text{Re} ((T - z)^* w | N w) + |z|^2 \| N w \|^2$$

$$= \| (T - z) w \|^2 - 2 \text{Re} ((N w | (T - z) w) + |z|^2 \| w \|^2 = \| (T - N - z) w \|^2 .$$

Therefore $T - N$ is hypnormal.

Next, for a bounded operator $A$ and $k > 0$, we write $M(A; k)$ for the spectral manifold

$$\{ z \in H : \text{there is an analytic function } f : \{ z \in H : |z| > k \} \to H \text{ such that } (A - z) f(z) = z \text{ for all } z \} .$$

It follows from the Laurent expansion that this set is equal to

$$\{ z \in H : \limsup |(A - z)^n w|^2 |z|^{kn} < k \} .$$

From $(T - N)P = PQ$, we have $P \{ M(Q; k) \} \subseteq M(T - N; k)$. Note that $M(Q; k) = H$ for all $k > 0$ and $M(T - N; 1)$ is always closed. (In fact, $M(T - N; k) = \{ z \in H : \| (T - N)^k w \|^2 < k \| w \|^2 \}$ for all $k > 1$, since a hypnormal operator is paranormal.) Hence $M(T - N; k) = H$ for all $k > 0$. By Baire's category theorem, it is easy to show that $Sp(T - N) = \{ 0 \}$. Now $T - N$ is a quasi-nilpotent hypnormal operator. Hence $T - N = 0$.

Next we consider direct integral decompositions of $M$-hypnormal operators.

Lemma 5. Let $T = \int_{\lambda}^T \delta t \, d\lambda(t)$ be a direct integral decomposition of $T$.

(a) If $T$ is dominant, then $T(t)$ is dominant a.e. (t).

(b) $T$ is $M$-hypnormal if and only if $T(t)$ is $M$-hypnormal a.e. (t).

Proof. Since the proof of (b) is similar to and easier than (a), we only prove part (a). By hypothesis, for each $s \in C$, there exists a positive number $M_s$ such that the operator

$$D_s = M_s (T - s)^* (T - s) - (T - s)(T - s)^*$$

is positive. (For definiteness, we assume that $M_s$ is the smallest positive number making $D_s \geq 0$.) Hence $D_s(t) \geq 0$ a.e. (t) for each $s$. Let $P_s = \{ z \in C : M_s \leq |z| \}$. Then $\bigcup P_s = C$. Let $Q_s$ be a countable dense subset of $P_s$.

Let $X = \{ t \in X : D_t(t) \geq 0 \}$ for $s \in \bigcup Q_s$. Then $m(X - Y) = 0$. Now it is easy to check that $T(t)$ is dominant for $t \in Y$.
LEMMA 6. Let \( T \) be a dominant operator. Then
(a) \( \ker(T - z) = \ker(T - z) \subseteq \ker(T - z') \) for each \( z \in C \), and
\[
\ker(T - z) \cap \ker(T - z') = \{ 0 \} \quad \text{if} \quad z \neq z',
\]
(b) if \( T \) is algebraic or of finite rank, then \( T \) is normal.

Proof. Straightforward. \( \blacksquare \)

The following theorem follows immediately from the above two lemmas.

THEOREM 7 (see [6]). If \( T \) is dominant and either \( T \) is \( n \)-normal or there is a nonconstant polynomial \( p \) such that \( p(T) \) is normal, then \( T \) is normal.

As a result of Lemma 5, we obtain:

COROLLARY 8. If \( T \) is \( M \)-hyponormal, \( N \) is normal and \( TN - NT \), then \( T + N \) is \( M \)-hyponormal.

Remark 1. The above corollary fails if "\( M \)-hyponormal" is replaced by "dominant". Take any dominant operator \( S \) which is not \( M \)-hyponormal for every \( M > 0 \). (Such operator exists, see e.g. [6].) Let \( T \) be a direct sum of countably many copies of \( S \), say \( T = \bigoplus_{k=1}^{\infty} S_k \), with \( S_k \) is unitarily equivalent to \( S \) for each \( k \). We can choose \( \varepsilon_k \in C \) such that \( \lim_{k \to \infty} M_k = \infty \), where
\[
M_k = \inf \{ M > 0 : \| (S - \varepsilon_k)^n \| \leq M \| (S - \varepsilon_k)^n \| \text{ for all } n \}.
\]

Obviously \( \{ \varepsilon_k \} \) must be bounded. Let \( N = \bigoplus_{k=1}^{\infty} A_k \). Then there is no positive number \( M \) such that \( \| (T + N)^n \| \leq M \| (T + N)^n \| \) for each \( n \).

Remark 2. We give an alternative proof of Corollary 8, without using the direct integral technique as follows: Let \( N = \bigoplus_{(n)} E_n B_n \) be the spectral decomposition of \( N \). Take a partition \( B = \{ B_1, \ldots, B_t \} \) of \( B(n) \) into Borel sets of small diameter. Take some \( \varepsilon_k \) in \( B_k \) for each \( k \). Put \( N_k = \sum_{n \in B_k} E_n B_n \). Now each \( E_n B_n H \) reduces \( T \). Let \( T_k = T \bigoplus_{n \in B_k} E_n B_n H \).

Then obviously \( T_k = \bigoplus_{n \in B_k} T_n \), and each \( T_n \) is \( M \)-hyponormal. Hence, for each \( k \), there exists an operator \( A_k \) on \( E_n B_n H \) such that \( \| A_k \| \leq M \) and \( (T_k + \varepsilon_k) = (T_k + A_k) \times (T_k + A_k)^* = (T_k + \varepsilon_k) A_k \). Let \( A_n = \sum_k A_k \). Then \( (T_k + \varepsilon_k) A_k \) and \( \| A_k \| \leq M \). Note that the net \( (N - N_k): B \) tends to zero. Choose a subnet of \( \{ A_n : B \} \) which converges in the weak operator topology to some \( A \). Then \( (T + N)^* = (T + N) A \) and \( \| A \| \leq M \). Now it is clear that \( T + N \) is \( M \)-hyponormal.

Combining Corollary 3 and Corollary 8, we obtain:

THEOREM 9. A spectral operator is \( M \)-hyponormal if and only if its scalar part is normal and its radical part is \( M \)-hyponormal.