Quotients of $L_p(0, 1)$ for $0 < p < 1$

by

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Abstract. One of the main results of this paper is a lifting theorem for operators from $L_p$, $0 < p < 1$, into a quotient space $L_p/N$. (The theorem is developed separately for $L_0$ and for $L_p$, $0 < p < 1$; the hypotheses on $N$ are different in the two cases.) A corollary is that if $N$ is a non-trivial finite dimensional subspace of $L_p$, $0 < p < 1$, then $L_p/N$ is not isomorphic to $L_p$. Several similar results are obtained; at the end of the paper, the idea of a $K$-space ($K_p$-space) is introduced and studied in connection with the lifting theorems.

1. Introduction. Let $L_p = L_p[0, 1]$ be the space of all real (or complex) measurable functions on $[0, 1]$ with the topology of convergence in measure. A Pełczyński has asked whether the quotient of $L_p$ by a non-trivial finite-dimensional subspace is isomorphic to $L_p$. In this paper we prove a lifting theorem for operators on $L_p$; using this theorem, we can show that if $B$ is a non-trivial closed subspace of $L_p$ which is either locally bounded or which admits a continuous linear functional, then $L_p/B \cong L_2$. Parallel results are developed for the spaces $L_p$ ($0 < p < 1$), where again we have that the quotient of $L_p$ by a non-trivial finite-dimensional subspace cannot be isomorphic to $L_2$ (contrasting of course with the case $p = 1$).

In Section 2, we show from certain general considerations that for $0 < p < 1$, $L_p/V \cong L_p/W$ whenever $\dim V = \dim W < \infty$. This enables us to define $L_p/n$ to be the (unique) space obtained by forming the quotient of $L_p$ by a subspace of dimension $n$. In Section 3 we prove our main lifting theorems and in Section 4 we apply them to show that $(L_p/n) \cong (L_p/m)$ if and only if $m = n$. We conclude Section 4 by giving an example of two isomorphic locally bounded subspaces of $L_p$, $B_1$ and $B_2$, such that $L_p/B_1 \cong L_p/B_2$.

In Section 5 we develop the idea of a $K$-space; this is an $F$-space $X$ such that every short exact sequence of $F$-spaces $0 \to Y \to X \to 0$ splits. Using this notion we show that $L_0[N \cong L_0$ implies that $N$ has no non-zero continuous linear functionals. Similar ideas for $p$-Banach spaces are also developed.
Throughout this paper an $F$-space will complete a metric complete topological vector space. An $F$-norm $\| \cdot \|$ on a space $X$ is a mapping from $X$ to $\mathbb{R}_+$ such that

(a) $\|x+y\| \leq \|x\| + \|y\|$ if $x, y \in X$,
(b) $\|ax\| = |a| \|x\|$ if $|a| \leq 1$ and if $x \in X$,
(c) $\|ax\| \to 0$, as $a \to 0$, for each $x \in X$,
(d) $\|x\| = 0$ if and only if $x = 0$.

For $0 < p \leq 1$, a $p$-Banach space is an $F$-space with an $F$-norm $\| \cdot \|$ such that

(e) $\|a\| \leq |a| \|x\|$ for all $\lambda$ and $x \in X$.

If $X$ and $Y$ are $p$-Banach spaces and if $S: X \to Y$ is a continuous linear operator, we define

$$\|S\| = \sup \{ \|Sx\| : \|x\| \leq 1 \}.$$  

We denote by $\mathcal{L}(X)$ the space of all linear operators on $X$. If $X$ is a $p$-Banach space, then so is $\mathcal{L}(X)$; if $X$ is an $F$-space, then $\mathcal{L}(X)$ has, in general, no convenient $F$-norm topology. Unless otherwise stated, "linear map" and "linear operator" always refer to continuous maps.

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2. Transitive $F$-spaces. In this section we show that if $V$ and $W$ are two subspaces of $L_p$ ($0 < p < 1$) of the same finite dimension, then $L_p/W \cong L_p$. We approach this result through some general results on transitive $F$-spaces. An $F$-space $X$ is said to be transitive if given $x, y \in X$ with $x \neq 0$, there exists $T \in \mathcal{L}(X)$ with $Tx = y$. We shall say that $X$ is strictly transitive if for any $k \in \mathbb{N}$, $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_k \in X$ such that $x_1, \ldots, x_k$ are linearly independent, there exists $T \in \mathcal{L}(X)$ with $Tx_i = y_i$.

We do not know whether a transitive $F$-space is strictly transitive; however, it is possible to generalize standard arguments in Banach algebra theory (cf. Rickart [6], pp. 60–62) to yield the following:

**Proposition 2.1.** Let $X$ be a transitive $F$-space; suppose that

(a) $X$ is separable,
(b) The centre of $\mathcal{L}(X)$ consists only of scalar multiples of the identity operator.

Then $X$ is strictly transitive.

If $X$ is a $p$-Banach space, condition (a) may be omitted; if $X$ is a complex $p$-Banach space, then conditions (a) and (b) may be omitted.

**Proof.** By [6], Lemma 2.4.3, it is enough to show that given two linearly independent elements $v, w \in X$, there exists $T \in \mathcal{L}(X)$ such that $Tv = 0$ and $Tw \neq 0$. If not, we may denote a (not necessarily continuous) operator $D$ on $X$ by $Du = Tw$ when $Tu = v$ (cf. [5], Theorem 2.4.6). Then $D$ commutes with each $T \in \mathcal{L}(X)$.

It is necessary to show that $D \in \mathcal{L}(X)$; at this point we require condition (a) in general. Consider $\mathcal{L}(X)$ with the topology of pointwise convergence. Then $\mathcal{L}(X)$ is a Borel space and by the Open Mapping Theorem, the map $s \in \mathcal{L}(X) \mapsto T_s$ defined by $T_s(T) = Ts$ is open. Since $D \circ T_s = T_w$, it follows that $D \in \mathcal{L}(X)$. If $X$ is a $p$-Banach space, then so is $\mathcal{L}(X)$ with its usual topology and the Open Mapping Theorem may be applied to this topology.

Now by condition (b), $D$ is a multiple of $I$ and we have a contradiction. If $X$ is a complex $p$-Banach space, then it may be shown that the centre of $\mathcal{L}(X)$ is a field and by Zelazko's extension of the Gelfand–Mazur theorem [11], condition (b) must hold.

It is easy to check that each of the spaces $L_p$ ($p < 0$) satisfies conditions (a) and (b) of the proposition and is transitive (use an argument similar to [7], pp. 253–234; see also [5]), and hence is strictly transitive.

**Proposition 2.2.** Suppose $X$ is a strictly transitive $F$-space and $X \cong X \oplus X$; then if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are two linearly independent sets in $X$, there exists an invertible $T \in \mathcal{L}(X)$ such that $T_{x_i} = y_i$, $1 \leq i \leq n$.

**Proof.** First we prove that there exists a projection $P \in \mathcal{L}(X)$ such that $P(X) \cong X$, $(I-P)(X) \cong X$ and $\dim(P) = \dim(X)$ is linearly independent. Let $P$ be the linear span of $(x_1, \ldots, x_n)$; then we may choose a projection $P$ so that $P(X) \cong X$, $(I-P)(X) \cong X$ and $\dim(P) = \dim(X)$ is maximal. Since $(I-P)(X) \cong X$, there exists a projection $Q \in \mathcal{L}(X)$ such that $PQ = GP = 0$ and $Q(X) \cong (I-P)(X) \cong X$. Then $P \cong (I-P)(X)$. Hence $P$ is one-to-one on $(P+Q)(P)$ and so if we have $\sum a_i P_{x_i} = 0$, then also $\sum a_i (P+Q) x_i = 0$. Similarly we have $\sum a_i (I-P) x_i = 0$; combining, $\sum a_i x_i = 0$ and $a_1 = a_2 = \ldots = a_n = 0$, i.e. $(P_{x_1}, \ldots, P_{x_n})$ is linearly independent.

Now pick projections $P_1$ and $P_2 \in \mathcal{L}(X)$ so that $P_1(X) \cong P_2(X)$ and $P_1(X) \cong P_2(X)$, $(I-P_1)(X) \cong P_1(X)$ and $P_2(X)$ are linearly independent. Then there exists an invertible $T \in \mathcal{L}(X)$ such that $TP_1 = (I-P_2)T$. Since $X$ is strictly transitive, there exists $S : (I-P_2)(X) \to P_2(X)$ such that $S(I-P_2)(x)T = Ty$, for $1 \leq i \leq n$. Similarly, there exists $R : P_1(X) \to (I-P_2)(X)$ so that $R_x = (I-P_2)(y)$. Then $(I+R)(P_1)$ and $(I+R)(P_2)$ are invertible, since $(P \circ P)^* = (P \circ (I-P_2))^* = 0$ and $(I+R)(P_2)^* = (I+R)(P_1)^* = 0$.

We now have:

**Theorem 2.3.** If $0 < p < 1$ and $V$ and $W$ are two subspaces of $L_p$ with $\dim(V) = \dim(W) < \infty$, then $L_p/V \cong L_p/W$. 

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Proof. This is immediate, since there is an invertible operator $T$ on $L_p$ such that $T(V) = W$.

Let us denote now by $(L_p, m)$ the quotient of $L_p$ by an $n$-dimensional subspace of $(L_p, 0) = L_p$. Theorem 2.3 guarantees that $(L_p, m)$ is well-defined.

**Theorem 2.4.** For $0 < p < 1$, $L_p([m + n]) \cong (L_p/m) \otimes (L_p/n)$ when $m, n > 0$.

**Proof.** Let $V$ be a subspace of $L_p(0, 1)$ of dimension $m$ (embedded in $L_p$ in the obvious way) and $W$ a subspace of $L_p(1, 0)$ of dimension $n$. Then

$$L_p([m + n]) \cong (L_p(0, 1)/V) \otimes (L_p(1, 0)/W) \cong (L_p/m) \otimes (L_p/n).$$

3. **Lifting theorems.** Let $X$ be a $p$-Banach space $(0 < p \leq 1)$ and $N$ a closed subspace of $X$. It is easy to see that any linear operator $S : L_p \to X/N$ may be lifted to a linear operator $\hat{S} : L_p \to X$, so that $\hat{S} - S$ is $\pi : X \to X/N$ is the quotient map. In the case $p = 1$, a similar lifting property holds if $L_p$ is replaced by any $L_p$-space and $N$ is isomorphic to a complemented subspace of a dual space (this is effectively proved by Lindenstrauss [4]). Not surprisingly there is a corresponding result for the case $p < 1$. We say that a $p$-Banach space $Y$ is an $L_p$-space $(0 < p < 1)$, if there is an increasing net $(Y_\alpha : \alpha \in A)$ of finite-dimensional subspaces of $X$ such that $\bigcup_{\alpha \in A} \{Y_\alpha : \alpha \in A\}$ is dense in $X$ and there exist linear maps $S_\alpha : Y_\alpha \to L_p$ and $T_\alpha : L_p \to Y_\alpha$, where $s_\alpha = \dim Y_\alpha$, such that $\sup \|S_\alpha\| \|T_\alpha\| < \infty$ and $T_\alpha S_\alpha = I$ on $Y_\alpha$. Clearly $L_p$ is an $L_\infty$-space.

We shall also call a $p$-Banach space $Z$ pseudo-dual if there is a Hausdorff vector topology $q$ on $Z$ such that the unit ball is relatively compact for $q$. The space $L_p$ is not pseudo-dual (see [1]), but the spaces $l_p$ and $H_p$ $(0 < p < 1)$ are pseudo-dual.

**Theorem 3.1.** Let $Y$ be an $L_p$-space $(0 < p < 1)$ and $X$ a $p$-Banach space. Let $N$ be a closed subspace of $X$ and suppose $X$ is isomorphic to a complemented subspace of a pseudo-dual $p$-Banach space $Z$. Then any operator $S : X \to X/N$ may be lifted to an operator $\hat{S} : Y \to X$.

If $X = L_p$, then the lifting is unique.

**Proof.** We observe that the unit ball of $Z$ may be supposed to be $q$-compact (by [1], Lemma 1). Then the argument is a straightforward imitation of the Lemmas of [4]. We omit the details.

In the case $X = L_p$, suppose $T$ is any other lifting. Then $T - \hat{S}$ maps $L_p$ into $N$ and there is a non-zero operator from $L_p$ into $Z$. The induced map into $(Z, q)$ is compact, contradicting the results of [2].

We now consider the case $p = 1$, which is rather different. Suppose $X$ is an $F$-space with $F$-norm $||$. For $a \in X$ we define $\sigma : X \to R \cup \{\infty\}$ by

$$\sigma(a) = \sup_{t \in X} \frac{a(t)}{1 + |a(t)|},$$

we have that $\sigma(a) = \mu(\{\sigma \geq a\})$.
In general, note that \( \sigma(ax) = \sigma(x) \) if \( a \neq 0 \) and that \( \sigma(x+y) \leq \sigma(x) + \sigma(y) \). If \( L \) is a linear subspace of \( X \), we define \( \sigma(L) = \sup(\sigma(x); x \in L) \).

We shall say that \( X \) admits \( L_\sigma \)-structure, if for any \( x > 0 \) there exist \( s = s(x) \) and subspaces \( X_1, \ldots, X_n \) of \( X \), such that \( X = X_1 \oplus \cdots \oplus X_n \) and \( \sigma(X_i) \leq s_i \). In addition to the obvious example of \( L_\sigma \) itself, any space of the type \( L_\sigma(Z) \) (all measurable functions from \([0,1]\) into an \( F \)-space \( Z \)) admits \( L_\sigma \)-structure.

The following proposition is trivial.

**Proposition 3.3.** Suppose \( X \) admits \( L_\sigma \)-structure and \( B \) is a locally bounded space. If \( T : X \to B \) is continuous, then \( T = 0 \).

We next prove two lemmas before giving the main lifting theorem.

**Lemma 3.4.** Suppose \( X \) is an \( F \)-space and \( B \) is a closed locally bounded subspace of \( X \); let \( \pi : X \to X/B \) be the quotient map. Let \( \delta \) be chosen so that the set \( \{ b \in B : ||b|| < \delta \} \) is bounded.

Then if \( x \in X/B \) and \( ||x|| \leq \delta \), there is a unique \( x' \in X \) such that \( \pi x = x' \) and \( ||x'|| \leq \delta \). For this \( x' \), \( \pi(x') = x' \).

**Proof.** We can find a sequence \( \{ x_n \} \in X \) such that \( x_n = x' \) and

\[
||x_n|| \leq 1 + n \cdot |x'|
\]

Let \( x_n = x_n - x' \) (\( n \in \mathbb{N} \)); then \( x_n \in B \) and if \( n \geq n \geq 2 \),

\[
||x_n|| \leq ||x_n - x'|| + ||x'||
\]

\[
\leq \left( 1 + \frac{1}{n} \right) ||x'|| + 1 + \frac{1}{n} ||x'|| \leq 2 + \frac{1}{n} + \frac{1}{m} \delta \leq \delta.
\]

By choice of \( \delta \), this implies that \( \{ x_n \} \) is a Cauchy sequence and hence so is \( \{ x_n \} \). If \( x = \lim x_n \), then \( \sigma(x) = \sigma(x') \leq \delta \). If \( y \) is any other point satisfying \( \pi y = x \) and \( \sigma(y) \leq \delta \), then \( x - y \in B \) and \( \sigma(x - y) \leq \frac{1}{2} \delta \); this implies \( x - y = 0 \).

**Lemma 3.5.** Under the assumptions of Lemma 3.4, let \( Y \) be a linear subspace of \( X/B \) with \( ||x'|| \leq \frac{1}{2} \delta \). Then there is a linear operator \( h : X \to X \) such that \( \pi h = h \) for all \( x' \in X \).

**Proof.** For \( x \in X \), define \( h(x) \) to be the unique \( x' \in X \) such that \( \pi x = x' \) and \( \sigma(x') = \sigma(x) \). If \( a, b \in B \) and \( \delta, \eta \in \mathbb{R} \), then

\[
\sigma(ab \xi + \beta \eta) \leq \sigma(a \xi) + \sigma(b \eta) \leq \delta \beta.
\]

Thus

\[
h(a \xi + \beta \eta) = ab \xi + \beta \eta,
\]

and \( h \) is linear.

Now suppose \( x_n \to x \) in \( X \); choose \( x_n, x \in X \) such that \( x_n - x \to 0 \) and \( ||x_n|| \leq \delta ||x|| \).

Then \( x_n - h(x_n) \to 0 \). If \( x_n - h(x_n) \to 0 \), we may assume, by passing to a subsequence, that for some \( \varepsilon > 0 \) we have

\[
||x|| - h(x_n) \leq 0
\]

(since the set \( \{ b \in B : ||b|| \leq \delta \} \) is bounded).

Then

\[
||x|| - h(x) \leq 0
\]

Thus we have \( x_n - h(x_n) \to 0 \) and \( h(x_n) \to 0 \).

**Theorem 3.6.** Suppose \( X \) admits \( L_\sigma \)-structure, \( Y \) is an \( F \)-space, and \( B \) is a closed locally bounded subspace of \( Y \). Then if \( \bar{S} : X \to Y \) is a linear operator, there is a unique linear operator \( S : X \to Y \) such that \( \pi S = \bar{S} \) for \( \pi : X \to X/B \) the quotient map.

**Proof.** Choose \( \delta > 0 \) so that \( \{ b \in B : ||b|| \leq \delta \} \) is bounded, and then \( \varepsilon > 0 \); so that \( ||x|| \leq \varepsilon \) (\( x \in X \)), then \( ||\beta|| \leq \delta \). Let \( X_1, \ldots, X_n \) be closed subspaces of \( X \) such that \( X = X_1 \oplus \cdots \oplus X_n \) and \( \sigma(X_i) \leq \varepsilon \).

Then there exist linear operators \( h_i : X_i \to \bar{X} \), \( \sigma(h_i) \leq \delta \), and there exist linear operators \( h_i : X_i \to Y_i \), \( \sigma(h_i) \leq \delta \). If \( \sum |h_i| \leq \delta \), then \( \sum \sigma(h_i) \leq \delta \).

Now suppose \( x_n \to x \). If \( \sum \sigma(h_i) \leq \delta \), then \( \sum \sigma(h_i) \to \delta \).

Thus we have \( \sum \sigma(h_i) \to \delta \).

Now suppose \( x_n \to x \) in \( X \); choose \( x_n, x \in X \) such that \( x_n - h(x_n) \to 0 \). If \( x_n - h(x_n) \to 0 \), we may assume, by passing to a subsequence, that for some \( \varepsilon > 0 \) we have

\[
||x|| - h(x_n) \leq 0
\]

(since the set \( \{ b \in B : ||b|| \leq \delta \} \) is bounded).

Then

\[
||x|| - h(x) \leq 0
\]

Thus we have \( x_n - h(x_n) \to 0 \) and \( h(x_n) \to 0 \).

4. **Quotient spaces of \( L_\sigma \) \((0,p,1)\).** In this section we treat the case \( p = 0 \) first and in more detail than the case \( p > 0 \); the arguments are analogous.

**Theorem 4.1.** Suppose \( X_1 \) and \( X_2 \) are two \( F \)-spaces with \( L_\sigma \)-structure. Suppose \( B_1 \) and \( B_2 \) are closed locally bounded subspaces of \( X_1 \) and \( X_2 \), respectively. Then \( X_1/B_1 \to X_2/B_2 \) if and only if there is an isomorphism \( V : X_1 \to X_2 \) mapping \( X_1 \) onto \( X_2 \) and such that \( V(B_1) = B_2 \).

**Proof.** The “if” part is clear. For the “only if” part, let \( S : X_1/B_1 \to X_2/B_2 \) be an isomorphism, and let \( \pi_1, \pi_2 \) be the quotient maps. Then
by Theorem 3.6, there exist lifts $V$, $U$ of $S\alpha_1$; $X_1 \to X_3$ and $S^{-1}\alpha_1$; $X_1 \to X_3/B_2$.

$$
\begin{array}{c}
X_1 \\
\downarrow \text{r} \\
X_3 \\
\downarrow \text{s} \\
X_1/B_2 \\
\downarrow \text{g} \\
X_3/B_2
\end{array}
$$

Then $UV; X_1 \to X_3$ is a lift of $\pi_1; X_1 \to X_1/B_1$. By the uniqueness, $UV = I_{X_1}$; similarly $VU = I_{X_3}$, so $V$ is an isomorphism of $X_1$ onto $X_3$.

Clearly $V(B_2) \subseteq B_2$ and $U(B_3) = V^{-1}(B_2) \subseteq B_2$; hence $V(B_2) = B_1$.

**Corollary 4.3.** If $X$ admits $L_0$-structure and $B \subseteq X$ is a locally bounded subspace, then $X/B$ admits $L_0$-structure if and only if $B = \{0\}$.

Theorem 2.3 and Theorem 4.3 give:

**Theorem 4.4.** Suppose $B_1$ and $B_2$ are locally bounded subspaces of $L_0$, then $L_0[B_1 \cong L_0[B_2]$ if and only if there is an invertible operator $T_0 : L_0 \to L_0$ such that $T_0(B_1) = B_2$.

In particular, $(L_0)[m] \cong (L_0)[n]$ if and only if $m = n$, and $(L_0)[1] \cong (L_0)^*$. This solves the problem of Pelczynski (see Introduction). Also in this section we shall illustrate this corollary by showing that $B_1 \cong B_2$ does not imply $L_0[B_1] \cong L_0[B_2]$. First however, we formulate the corresponding theorems for $p > 0$; the proofs are similar.

**Theorem 4.5.** If $B_1$, $B_2$ are two closed subspaces of $L_p$, each of which is isomorphic to a complemented subspace of a pseudo-$p$-Banach space or to a $q$-Banach space where $p < q < 1$.

Then $L_0[B_1] \cong L_0[B_2]$ if and only if there is an invertible operator $T_0 : L_0 \to L_0$ such that $T_0(B_1) = B_2$.

In particular, $(L_0)[m] \cong (L_0)[n]$ if and only if $m = n$, and $(L_0)[1] \not\subseteq L_0$.

**Theorem 4.6.** If $B \subseteq L_p$ is isomorphic to a complemented subspace of a pseudo-$p$-Banach space and $B \not\subseteq L_0$, then $(L_0)[B]$ is not an $L_p$-space.

**Proof.** If $(L_0)[B]$ is an $L_p$-space, then the identity map $I : L_0[B] \to (L_0)[B]$ may be lifted to a map $J : L_0[B] \to L_0$. Then on $L_0$, $I - Jx$ maps $L_0$ into $B$. Hence, by applying the results of [2] as in Theorem 3.1, $I = J\pi$ and so $B = \{0\}$.

**Example.** Let $B_1 \subseteq L_0$ be the closed linear span of the Rademacher functions $r_n$ on $[0,1]$, $(r_n(t) = \text{sgn} (\sin 2^n \pi t))$, and let $B_2$ be the closed linear span of a sequence of independent random variables normally distributed with mean zero and variance one. Then $B_1 \cong L_0 \cong L_2$; we shall show, however, that $L_0[B_1 ] \not\subseteq L_0[B_2]$.

For suppose $L_0[B_1 ] \cong L_0[B_2]$; then there is an invertible linear operator $T : L_0 \to L_0$ such that $T(B_1) = B_2$. By Kwapieñ’s Representation Theorem [3], $T$ takes the form

$$
T_\alpha(t) = \sum_{n=1}^\infty \phi_n(t) \#(\phi_n t) \#(\alpha n)
$$

where

(i) $\phi_n \in L_0$, $n \gg 1$,

(ii) $m(\{t; \phi_n(t) \neq 0 \text{ for infinitely many } n\}) = 0$,

(iii) $\phi_n \to 0$ in $(0,1)$; if $A$ is measurable, then $\phi_n(A)$ is measurable; if $m(A) = 0$, then $m(\phi_n(A) \cap \text{Supp} \phi_n) = 0$.

Thus for almost every $t \in [0,1]$, the sequence $(T_\alpha(t))$ assumes only finitely many values. Hence for some $j_1, k_1$ with $j \neq k$, we must have $m(\{t; T_\alpha(t) = T_{k_1}(t)\}) > 0$. However $T_{j_1} - T_{k_1}$ is normally distributed and hence $T_{j_1} - T_{k_1}$. Thus $T$ is not injective, and we have a contradiction.

**Remarks.** For $p > 0$, let $n \in L_0$ be non-zero and let $V$ be the linear span of $n$. Let $\mathcal{S}(L_0)$ and $\mathcal{S}(L_0)$ be the $p$-Banach algebras of all bounded linear operators on $L_0$ and $L_0$, respectively. If $S \in \mathcal{S}(L_0)$, then $S : L_0 \to L_0$ be the unique lift of $S \circ \pi$. Then the map $S \to \tilde{S}$ is an algebra homomorphism, and in fact an embedding of $\mathcal{S}(L_0)$ into $\mathcal{S}(L_0)$. Thus $\mathcal{S}(L_0)$ is isomorphic to the closed subalgebra of $\mathcal{S}(L_0)$ consisting of all $T \in \mathcal{S}(L_0)$ such that $Tn \in V$. We may define a multiplicative linear functional $\varphi$ on $\mathcal{S}(L_0)$ by

$$
\varphi(S) = \tilde{S}n.
$$

5. $K$-spaces. In this section, we abstract a particular property of the spaces $L_p$ and consider it in more generality. We restrict to the real case, but the complex case is identical.

If $X$ is an $F$-space, we shall say that $X$ is a $K^*$-space if every short exact sequence $0 \to R \to X \to Y \to 0$, with $Y$ an $F$-space, splits. Alternatively, if $X = K^*$-space if every short exact sequence $0 \to R \to X \to Y \to 0$, with $Y$ a $p$-Banach space, splits.

**Theorem 5.1.** An $F$-space [p-Banach space] $X$ is a $K^*$-space if and only if whenever $Y$ and $Z$ are $F$-spaces [p-Banach spaces] and $S : Y \to Z$ is a surjective operator with $\dim S^{-1}(0) = 1$, then each linear operator $T : X \to Z$ may be lifted to an operator $T : X \to Y$ such that $ST = T$.

**Proof.** We prove the statement for $K$-spaces. Suppose $X$ is a $K$-space. Let $V \subseteq X \oplus Y$ be the subspace of all $(x,y)$ such that $Tx = Sy$,
and define \( P : V \rightarrow X \) by \( P(a, y) = a \). Then \( P : V \rightarrow X \) is surjective, and \( \dim X^{-1}(a) = 1 \). Hence there exists a linear operator \( B : X \rightarrow V \) such that \( PB = I_x \). Then \( R_B = (a, R_{B}) \); clearly \( R^2_B = T \).

For the converse take \( Z = X \) and \( T \) to be the identity.

We remark now that if \( X \) admits \( L_p \)-structure, then \( X \) is a \( K \)-space (Theorem 3.6), and that an \( L_p \)-space is a \( K \)-space.

**Theorem 5.2.** If \( X \) is an \( F \)-space (p-Banach space) and \( N \) is a closed subspace of \( X \) such that \( X/N \) is a \( K \)-space \( (K_p \)-space), then \( X \) has the Hahn-Banach Extension Property in \( X \).

**Proof.** Again we restrict to the \( K \)-space case. Suppose \( \varphi \in \mathcal{K}^* \) is non-zero; let \( M = \varphi^{-1}(0) \in N \). Consider the natural quotient map \( \pi : X/M \rightarrow X/N \); then there is a map \( S : X/N \rightarrow X/M \) such that \( S\pi = I \) on \( X/N \). Then \( S(X/N) \) is a closed subspace of \( X/M \) and so there exists \( \varphi \in (X/M)^* \) such that \( \varphi \varphi = 0 \) and \( \varphi \circ S = 0 \). If \( \varphi \) is non-zero, \( S\pi \) is a quotient map, then \( \varphi \in (X/M)^* \). If \( \varphi \in \mathcal{K} \), then \( \varphi \varphi = 0 \) only if \( \varphi = 0 \). Hence \( \varphi \in \mathcal{K} \) and \( \varphi \varphi = 0 \).

There is also a converse to Theorem 5.2.

**Theorem 5.3.** If \( X \) is a \( K \)-space \( (K_p \)-space) and \( N \subset X \) is a closed subspace with HBEPP, then \( X/N \) is a \( K \)-space \( (K_p \)-space).

**Proof.** Suppose we have a short exact sequence

\[ 0 \rightarrow R \rightarrow Z \rightarrow X/N \rightarrow 0, \]

and let \( \pi : X \times X/N \) be the quotient map. Then there is a lifting of \( \pi, \xi : X \rightarrow Z \), so that \( \pi \delta = \pi \) (by Theorem 5.2). Suppose first \( \xi \) is not surjective; then \( S(X) \) has co-dimension one in \( Z \) and \( \varphi \in S(X) \) is one-one. Define \( R : X/N \rightarrow Z \) by \( R\delta = \sigma \) where \( \sigma \in S(X) \) and \( \sigma \pi = \xi \). If \( \xi \pi \rightarrow 0 \) in \( X/N \), then there exists a sequence \( (x_n) \) in \( X \) such that \( x_n \rightarrow 0 \) and \( \pi(x_n) = \pi \xi \). Since \( S\pi \rightarrow 0 \) and \( S\pi = R\delta \), \( R\delta \rightarrow 0 \). Hence \( S\pi \rightarrow 0 \) implies \( \pi \xi \rightarrow 0 \). Hence \( S\pi = S\xi = 0 \).

Now suppose \( \xi \) is surjective; then \( S^{-1}(0) \) has co-dimension one in \( N \). Let \( \varphi \in \mathcal{K}^* \) be a non-zero linear functional with kernel \( S^{-1}(0) \). Then \( \varphi \) may be extended to \( \varphi \in \mathcal{K} \). Now define \( \delta : X \rightarrow Z \) by \( \delta = \varphi \varphi \). Then \( \varphi \delta = \varphi \varphi \varphi = \varphi \). Hence \( \varphi \delta = \varphi \). Hence \( \delta \varphi = \varphi \delta \). Thus \( \delta \) is one-one on \( S(X) \) and \( S(X) \) has co-dimension one in \( Z \); we can apply the previous part of the proof.

**Corollary 5.4.** \( X \) is a \( K_\infty \)-space if and only if \( X \cong L_p(I)/N \) where \( I \) is some index set and \( N \subset L_p(I) \) has the HBEPP.

We remark that if \( L_p(I) \cong L_p \), then \( N \) has HBEPP and the extension is unique, since \( L_p = \{0\} \).

**Corollary 5.5.** (i) If \( N \) is a closed subspace of \( L_p \), then \( L_p/N \) is a \( K \)-space if and only if \( N \cong \{0\} \). In particular, if \( L_p \) has \( L_p \)-structure, then \( N \cong \{0\} \).

(ii) If \( N \) is a closed subspace of \( L_p \), then \( L_p/N \cong L_p \) if and only if \( N \cong \{0\} \). In particular, if \( L_p \) is an \( L_p \)-space, then \( N \cong \{0\} \).

Note here that if we take for \( N \) the closed linear span of a sequence of functions with disjoint supports in \( L_p \), then \( N \cong \{0\} \), and hence \( L_p/N \cong L_p \). However \( L_p/N \cong \omega(L_p/L) \) (a countable product of copies of \( L_p/L \); hence \( \omega(L_p/L) \cong L_p \).

**Problem.** Is \( L_p \) or \( L_p \) a \( K \)-space for any \( r < p \), or even a \( K \)-space? In particular, is \( L_p \) (or any other Banach space) a \( K \)-space for any \( r < p \)? This latter question is essentially the same as a problem of Stiles [8]; if \( L_p/N \) is locally convex, must \( N \) have the HBEPP?

**References**


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