On the inversion of pseudo-differential operators

by

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Abstract. Let $A$ be a properly supported pseudo-differential operator on a manifold $X$. An important problem is to know when $A$ has a pseudo-differential inverse. If this inverse exists, one deduces immediately that $A$ is elliptic and injective in $C^0_0(X)$. When $X$ is a compact manifold, these conditions are nearly sufficient. In fact, let $A$ be a self-adjoint elliptic pseudo-differential operator on a compact manifold $X$. Suppose that $A$ is injective in $C^0_0(X)$. Then $A$ has a two-sided self-adjoint pseudo-differential inverse (see [1]).

Here we consider the case of a non-compact manifold. We have a function $a$ with certain properties, and we ask for the existence of properly supported pseudo-differential operators $A$ and $B$, such that $A \circ B$ is the identity operator $I$ and $a$ is a principal symbol of $A$. We state the main result in 2. The rest of the paper is devoted to prove it.

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1. Notation and basic definitions. Let $X$ be a $C^\omega$ paracompact manifold of dimension $N$. If $\xi$ is an element in the cotangent bundle $T^*(X)$, the length $|\xi|$ of $\xi$ can be defined in terms of a riemannian metric on $X$. We shall also assume that there is given a volume element on $X$, which in any local coordinate system can be expressed as $f dx_1 \cdots dx_n$, with $f \in C^\omega$ and $f > 0$.

A function $a = a(x, \xi) \in C^\omega(T^*(X))$, is in the class $S^m_{\rho, \delta}$, or is a symbol of order $m \in \mathbb{R}$ and type $\rho, \delta$, $0 \leq 1 - \rho \leq \delta \leq \rho \leq 1$ if, in local coordinates, it satisfies

$$
\left| \frac{\partial}{\partial x} \right|^{|\rho|} \left| \frac{\partial}{\partial \xi} \right|^{|\delta|} a(x, \xi) \leq C_{\rho, \delta}(1 + |\xi|)^{-|\rho| - |\delta|(|\rho| + |\delta|)}
$$

for all $a, \beta$.

This class is well-defined on $T^*(X)$.

A linear and continuous operator $A: C^\omega_0(X) \to C^\omega(X)$ belongs to the class $S^m_{\rho, \delta}$, or is a properly supported pseudo-differential operator of order $m$ and type $\rho, \delta$ if for a given local coordinate system defined in an open set $U$ there exists $a(x, \xi) \in S^m_{\rho, \delta}$ such that for $f$ with support in $U$ and $x \in U$

$$
Af(x) = \int e^{i\xi \cdot x} a(x, \xi) f(\xi) \, d\xi
$$
and the distribution kernel $K_A$ of $A$ vanishes outside a neighborhood of the diagonal. Therefore, we can write

$$Af(\xi) = \lim_{\epsilon \to 0} \epsilon^{2n-2} a(x, y, \xi) \eta(\epsilon x) f(y) \, dy,$$

where $a(x, y, \xi) \in S^{\alpha}_{2\lambda}$, $a(x, y, \xi) = 0$ for $|x - y| > M$ and $\eta \in C_0^\infty$ and equals 1 for $|\xi| \leq 1$.

$a(x, y, \xi)$ is called an amplitude. The class of $a(x, \xi)$ modulo $S^{\alpha}_{2\lambda}$ is well defined on the cotangent bundle $T^*(X)$. It will be called the principal symbol of $A$, $\sigma_p(A)$. If $a, b$ belong to $\sigma_p(A)$, we shall write $a \sim b$.

A function on $T^*(X)$ which belongs to $\sigma_p(A)$ will also be called a principal symbol of $A$. We shall say that an operator $A$ in $L^2_0$ is elliptic if it has a principal symbol $a$ such that $|a(x, \xi)| \geq C |\xi|^{2\lambda}$ for $|\xi| \geq R$ and $(x, \xi) \in T^*(X)$. The operator $A$ belongs to $I^{-m}$ if it is in $L^2_0$, for all $m \in R$. For the properties of pseudo-differential operators that we shall use, see [1].

2. The main result.

THEOREM 1. Let $a = a(x, \xi)$ be a real function in $C^{\infty}(T^*(X))$, positively homogeneous of degree $m$ in $\xi$ for $|\xi| \geq 1$ and such that $a(x, \xi) > 0$ for all $(x, \xi) \in T^*(X)$. Then

(i) If $m = 0$, there exist operators $A, B \in L^2_0$ with

$$A \circ B = I, \quad \sigma_p(A) \sim a.$$

(ii) If $m \neq 0$, there exist operators $A, B$ satisfying

$$A \in L^2_0, \quad B \in L^2_0, \quad \text{for all } 0 < \delta < 1,$$

$$A \circ B = I, \quad \sigma_p(A) \sim a.$$

3. The local version of the main result.

We shall first show the existence of an adequate covering of the manifold $X$.

LEMMA 1. There exists a covering $\{U^*_n\}$ of $X$ with relatively compact open sets such that, for each fixed $n$, $U^*_n \cap U^*_k = \emptyset$ if $n \neq k$.

Proof. When $X$ is a Euclidean space, there exists a family of cubes $\{Q^*_n\}$ satisfying those conditions.

Then, let us consider the general case. The Whitney immersion theorem asserts that there exists a differentiable mapping $f: X \to R^{2n+1}$ such that $X$ is homeomorphic to $f(X)$, with the topology induced by $R^{2n+1}$ and $f(X)$ is closed in $R^{2n+1}$ (see [2]).

If $\{Q^*_n\}$ is the covering of $R^{2n+1}$ with cubes as above, the sets $U^*_n = f^{-1}(f(X) \cap Q^*_n)$ yield the desired covering of $X$.

Let $\{Q^*_n\}$ be a partition of unity subordinated to the covering $\{U^*_n\}$. Consider, for each $\alpha$, $n$, $a^\alpha_n = \varphi^\alpha_n = 0^\alpha_n$.

It is easy to see that:

(i) If $m = 0$, $a^\alpha_n \in S^\alpha_{1, \infty}$.

(ii) If $m > 0$, $a^\alpha_n \in S^\alpha_{m, \infty}$ for all $0 < \delta < 1$.

(iii) If $m < 0$, $a^\alpha_n \in S^{\alpha}_{1, \infty}$ for all $0 < \delta < 1$.

In all three cases, $a^\alpha_n(\xi, \xi) = 1$, when $a \in X \cap K^\alpha$, for a compact set $K^\alpha \subset X$.

We shall suppose, without loss of generality, that $m$ is non-negative.

In this case, $1/a^\alpha_n \in S^\alpha_{1, \infty}$ when $m = 0$, and $1/a^\alpha_n \in S^\alpha_{1, \infty}$ for all $0 < \delta < 1$ when $m > 0$.

We are now in position to give the localized version of Theorem 1:

THEOREM 2. There are operators $A = A^\alpha_n, B = B^\alpha_n$ such that

(i) If $m = 0$, then $A, B \in L^2$, and

$$A \circ B = I, \quad \sigma_p(A) \sim a^\alpha_n,$$

$$A = I + \hat{A}, \quad B = I + \hat{B}, \quad \text{with } \hat{A}, \hat{B} \in L^2, \quad \text{and the corresponding distribution kernels have compact support contained in } U^*_n \times U^*_n.$$

(ii) If $m > 0$, then $A \in L^2_0, \quad B \in L^2_0, \quad \text{for all } 0 < \delta < 1$.

Furthermore, these operators satisfy the conditions in (i), except that now $\hat{A} \in L^2_0$ and $\hat{B} \in L^2_0$, for all $0 < \delta < 1$, instead of $L^2_0$.

We shall prove this theorem in 5. If we accept it for a moment, we derive from it Theorem 1 in the following way:

For $1 \leq a \leq k$ fixed, the compositions

$$A^\alpha_n \circ \ldots \circ A^1 \circ \sigma_p, \quad B^\alpha_n \circ \ldots \circ B^1 \circ \sigma_p,$$

have a meaning as operators on $C^\alpha_{1, \infty}(X)$ and they define properly supported pseudo-differential operators. In fact, given a compact subset $K$ of $X$ and $f$ with support in $X$, since the $U^*_n$ are disjoint, there exists $N(K)$ such that

$$A^\alpha_n \circ \ldots \circ A^1(f) = A^\alpha_n \circ \ldots \circ A^1(f) \quad \text{for } n \geq N(K).$$

Furthermore, there exists also $N_1(K)$ such that

$$B^\alpha_n(f) = f \quad \text{for } n \geq N_1(K).$$

Then, the operators

$$A = A^\alpha_n \circ \ldots \circ A^1 \quad B = B^\alpha_n \circ \ldots \circ B^1$$

satisfy Theorem 1.

Remark. In general, given two operators $A \in L^2_0, \quad B \in L^2_0$, $C = A \circ B \in L^2_0$. But in certain cases the order of $C$ can be less than $m_1 + m_2$, as happens, for example, in this case.
4. Preliminary results.

**Lemma 2.** Let $U$ be a relatively compact open subset of $X$ and let 
$s = s(\xi, \xi) \in S_{p,q}^m$, $m \geq 0$, a real symbol such that, for $0 < \varepsilon < \varepsilon_1$ and a compact set $\mathcal{K} \subseteq U$,

\[
\begin{align*}
    s(x, \xi) &> \varepsilon_1 \quad \text{for all } (x, \xi) \in T^*X, \\
    s(x, \xi) &< \varepsilon_1 \quad \text{if } x \in X \setminus \mathcal{K}, \\
    \forall \sigma - \varepsilon_1 \in S_{p,q}^m.
\end{align*}
\]

Then there exists a self-adjoint operator $A \in \Gamma_{p,q}^m$ such that

\[
s_{\sigma}(A) \sim \sigma, \quad A \text{ is injective in } C^0_c(\mathcal{K}),
\]

$A = a_0 I + C$, $C \in \Gamma_{p,q}^m$, and the support of the distribution kernel $K_C$ of $C$ is a compact subset of $U \times U$.

**Proof.** It suffices to define

\[
A = a_0 I + (\tilde{A} + V \varepsilon_0 - \varepsilon_1) \circ (\tilde{A} + V \varepsilon_0 - \varepsilon_1 I)
\]

where the principal symbol of $\tilde{A}$ coincides with $V \varepsilon_0 - \varepsilon_1$ and its kernel $K_{\tilde{A}}$ vanishes outside an adequate neighborhood of the diagonal in $X \times X$.

**Lemma 3.** Let $U$ be a relatively compact open subset of $X$ and $s = s(\xi, \xi) \in S_{p,q}^m$ a real symbol such that

\[
s(\xi, \xi) > 0 \quad \text{for all } (\xi, \xi) \in T^*X, \\
\forall \sigma \in S_{p,q}^m,
\]

$1/\sigma \in S_{p,q}^m$, for $\varepsilon > 0$ and satisfies the hypothesis of Lemma 2.

Then there exists a self-adjoint operator $B \in \Gamma_{p,q}^m$ with

\[
s_{\sigma}(B) \sim \sigma, \quad B \text{ is injective in } C^0_c(X),
\]

$B = eI + C$, for some $c > 0$ and $C \in \Gamma_{p,q}^m$, such that the support of the distribution kernel $K_C$ is a compact subset of $U \times U$.

**Proof.** If $s(\xi, \xi) = s(\xi, \xi) > 0$, in the complement of a compact subset $K$ of $U$, there exists $\tilde{B} \in \Gamma_{p,q}^m$ such that the principal symbol of $\tilde{B}$ coincides with $V \varepsilon_0 - \varepsilon_1$ and its kernel $K_{\tilde{B}}$ vanishes outside a neighborhood of the diagonal in $X \times X$. Then the operator $B_1 = (\tilde{B} + V \varepsilon_0) \circ (\tilde{B} + V \varepsilon_0 I)$ satisfies all conditions except, possibly, the injectivity. We shall modify $B_1$, in order to obtain also this condition.

There is an operator $A \in \Gamma_{p,q}^m$ satisfying Lemma 2 with respect to $1/\sigma$. Since $a_0(A \circ B_1) \sim 1$, we can write $A \circ B_1 = I + D$, where $D$ belongs to $\Gamma_{p,q}^m$ and the distribution kernel $K_D$ has compact support in $U \times U$.

Now, $D$ is a compact operator in $L^2(X)$ and so the null space of $I + D$ is finite dimensional. Furthermore, it consists of $C^0_c$ functions (see [1], p. 36). Then, the null space $M$ of $B_1$ as a bounded operator in $L^2(X)$ is also finite dimensional and consists of functions in $C^0_c(U)$.

Let $(f_1, \ldots, f_k)$ be an orthonormal basis for $M$ and let

\[
Bf = R_1 f + \sum_{i=1}^k (f, f_i)_{L^2} f_i, \quad f \in L^2(X).
\]

The operator $\sum_{i=1}^k (f, f_i)_{L^2} f_i$, is the orthogonal projection on $M$. Since the $f_i$ are in $C^0_c(U)$, it is an integral operator with $C^0_c$ kernel, of compact support contained in $U \times U$. Thus, $B$ satisfies the same conditions as $B_1$. Furthermore, $B$ is injective in $L^2(X)$. For if $Bf = 0$, since $B_1$ is self-adjoint, we have

\[
0 = (B_1 f, f)_H = - \sum_{i=1}^k (f, f_i)_{L^2} (f, f_i)_{L^2} = -(f, f)_H.
\]

Then, $B_1 f = 0$. Therefore, $f$ is in $M$ and since $f$ is orthogonal to $M$, we conclude that $f = 0$.

It will be necessary to know, in Theorem 2, whether the $L^2$-inverse of a certain pseudo-differential operator is also pseudo-differential (see [1]):

**Theorem 3.** Let $C \in \Gamma_{p,q}^m$, $m > 0$, and suppose that the distribution kernel $K_C$ of $C$ has compact support $\subset X \times X$. Suppose further that $I + C$ is injective in $C^0_c(X)$. Then $I + C$ has a two-sided inverse of the form $I \pm C', \quad C' \in \Gamma_{p,q}^m$.

To prove this, we need the following

**Lemma 4.** Let $\Phi: X \to L^2(X)$ be a strongly measurable function with compact support $K \subset X$. Then there exists a measurable function $\varphi: X \times X \rightarrow C$ the complex field, such that

\[
\Phi(x)(y) = \varphi(x, y) \quad \text{a.e. in } X.
\]

**Proof of Lemma 4.** For each $x \in X$, $\Phi(x)$ is a class of square integrable functions such that any two of them coincide almost everywhere. We want to show that it is possible to select an element $h$ in each class such that $\varphi(x, y) = h(x, y)$ is a measurable function on $X \times X$. The strong measurability of $\Phi$, implies that there exists a sequence $(\Phi_n)_{n \geq 1}$, such that

\[
\begin{align*}
(1) \quad & \Phi_n(x, y) = \sum_{h=1}^H \hat{h}(x, y) \chi_h^n(x, y), \quad \text{where } \hat{h}(x, y) \in L^2(X), \text{ and for each } n, \chi_h^n \text{ are the characteristic functions of measurable disjoint subsets of } X, \\
(2) \quad & \lim_{n \to \infty} \int_X (\Phi_n(x, y) - \Phi(x)(y))^2 \, dy = 0 \text{ a.e. in } X.
\end{align*}
\]

Let $C_n = \{x \in X \mid \|\Phi_n(x)\|_{L^2} \leq 2\}$, $C_{n+1} \subseteq C_n$. Writing $\Phi_n = \sum_{h=1}^H \hat{h}(x, y) \chi_h^n(x, y)$, then $0 \leq \Phi(x)(y) \leq 2$, for all $(x, y) \in X \times X$.
the characteristic function of $C_n$, we have

$$\|\Phi_n(x,y)\|_2 \leq 2\|\Phi(x)\|_2, \quad x \in X, \quad n \geq 1.$$  

Let $K^{(n)}_\infty = \{x \in K : \|\Phi(x)\|_2 \leq m \}$, $m \geq 1$. According to the dominated convergence theorem we have

$$\lim_{n \to \infty} \int_{K^{(n)}_\infty \times X} \Phi_n(x,y) - \Phi(x,y) \, dx \, dy = 0.$$  

Since $K^{(n)}_\infty \subset K^{(n)}$, we can extract a subsequence $(\Phi_n(x,y))$ converging a.e. in $K_1 \times X$; its limit is the desired function $\varphi(x,y)$.

Proof of Theorem 3. First, observe that $I+C$ is elliptic and is a bounded operator in $L^1(X)$. If $f$ is in the null space of $I+C$, then $f \in C^\infty(X)$ and $f$ has support in $K$, since $f = -\partial x$ and $\partial x$ has support in $K$. Thus, $f \in C^\infty(X)$ and our hypothesis imply that $f = 0$. Therefore, $I+C$ is injective in $L^1(X)$ and since $C$ is a compact operator in $L^1(X)$, it follows that $I+C$ has a two-sided inverse $B_1$ as an operator in $L^1(X)$.

Since $B_1 \circ (I+C) f = f$,  

$$B_1 \circ (I+C) f = f$$  

given the assumed properties of $C$, it follows that $B_1 f = f$ if $\text{supp}(f)$ is disjoint from $K$, and $B_1 f = f$ outside $K$ for all $f$. Since $I+C$ is elliptic, it has a two-sided inverse $B_1$ modulo $I^\infty$. This inverse has the form $B = I + C + \varphi$, with $C \in T_{\varphi}(K)$. Now, $B$ can actually be taken so that if $K_1$ is a compact set containing $K$ in its interior, then $B f = f$ for supp$(f)$ disjoint from $K_1$, and $B f = f$ outside $K_1$, for all $f$. For, let $\varphi$ vanish outside $K_1$, and let $\varphi = 1$ on $K$. Then $I+\varphi$ has the desired property (See [1], p. 37.) Furthermore, it is also a two-sided inverse of $I+C$ modulo $I^\infty$.

Now, let

$$B \circ (I+C) = I + R_1, \quad (I+C) \circ B = I + R_2,$$

then $R_1$ and $R_2$ belong to $I^\infty$. Setting $S = B - R_1$, we will show that $S \in I^\infty$ which will imply that $B_1 \in T_{\varphi}(K)$, and our assertion will be established. Now, from the preceding identities and the fact that

$$B_1 \circ (I+C) = (I+C) \circ B_1 = I,$$

we obtain

$$S \circ (I+C) = R_1, \quad (I+C) \circ S = R_2,$$

and multiplying on the right and on the left by $B_1$ we obtain, respectively,

$$S + S \circ B_1 = R_1 \circ B_1, \quad S + R_2 \circ S = B = R_2,$$

and multiplying the first equation on the left by $B_1$ and subtracting from the second we obtain

$$S = R_1 \circ S \circ R_2 - R_1 \circ B + B \circ R_2.$$  

Since $B f = f$ and $B f = f$ if sup$(f)$ is disjoint from $K_1$, then $B f = 0$ if sup$(f)$ is disjoint from $K_1$. Furthermore, since $B f = f$ and $B f = f$ outside $K_1$, it follows that sup$(f) \in K_1$.

Now, suppose we show that $B_1 \circ S \circ R_2$ is an integral operator with square integrable kernel. Then, since $R_1 \circ B$ and $B \circ R_2$ are in $I^\infty$, they are also integral operators with square integrable kernels, and $S$ itself is an integral operator with a square integrable kernel. Thus, on the account of the above properties of $S$, this kernel will have support in $K_1 \times K_1$. But then, as is readily seen, $B_1 \circ S \circ R_2$ is an integral operator with a compactly supported $C^0(X \times X)$ kernel and therefore $K_1 \circ S \circ R_2 \in I^\infty$. Thus, we will have that $S \in I^\infty$, and our theorem will be established.

Let $x \in X$. The linear mapping

$$L^1(X) \to C,$$

$$f \to R_1 \circ S \circ R_2 f(x)$$

is continuous and, therefore, there exists $h_2 \in L^1(X)$ such that

$$R_1 \circ S \circ R_2 f(x) = \{h_2, f\}_2.$$  

If we show that the mapping

$$K_1 \to L^1(X),$$

$$x \to h_2$$

is strongly measurable, on account of Lemma 4, we will deduce that there exists $\varphi$ such that $\varphi(x,y) = \Phi(x,y)$ a.e. in $X$.

It is sufficient to prove that $\Phi$ is weakly measurable. This is clear, for, given $f \in L^1(X)$, the function

$$K_1 \to C,$$

$$x \to R_1 \circ S \circ R_2 f(x) = \{h_2, f\}_2$$

is continuous.

Furthermore, $\varphi$ is a square integrable function. For, the mapping

$$L^1(X) \to C^0(K_1),$$

$$f \to R_1 \circ S \circ R_2 f$$

is continuous, where $C^0(K_1)$ indicates the class of continuous functions $f : X \to C \subset C^0$ with compact support in $K_1$, with the topology of uniform
convergence. Thus, there exists $M > 0$ such that
\[ \|f\|_{L^2} \leq 1 \quad \text{implies} \quad \sup_{x \in X} \int X \varphi(x, y) f(y) \, dy \leq 1/M. \]
For a fixed $x \in X$,
\[ \left( \int X \varphi(x, y)^2 \, dy \right)^{1/2} = \sup_{|f| \leq 1} |R_1 \circ S \circ R_2 f(x)| \]
\[ = \sup_{|f| \leq 1} \int X \varphi(x, y) f(y) \, dy \leq 1/M. \]
Therefore
\[ \int X \varphi(x, y)^2 \, dy \, dx \leq \frac{1}{M^2} \, \text{mes}(X). \]

5. Proof of Theorem 2. In order to simplify the notation, $U$ will be a fixed open set of the covering \{ $U_n^m$ \} and $\varphi$ will be the corresponding function in the subordinate partition of unity. Let also $\sigma = \sigma^a$.

It is clear that $\sigma_1$ satisfies the assumptions of Lemma 2, with $\sigma_1 = \varphi = 1$, and $\delta = 0$ if $m = 0 < 0 < \delta < 1$ if $m > 0$. On the other hand, $1/\sigma_2$ satisfies the assumptions of Lemma 3 with $\sigma = 1$, $m = m$ and $\delta = 0$ if $m = 0 < 0 < \delta < 1$ if $m > 0$. Thus, we can obtain operators $A$ and $B_1$ as in Lemmas 2 and 3, respectively. Therefore $A \circ B_1$ has the properties of Theorem 3 and then its $L^2$-inverse $\tilde{B}$ is a pseudo-differential operator. Therefore the operators $A$ and $B_1 \circ \tilde{B}$ verify Theorem 2 with respect to $U$.

Remark. When $m = 0$, both symbols $\sigma_1$ and $1/\sigma_2$ satisfy the assumptions of Lemma 2.

References

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