Normalized weakly null sequence
with no unconditional subsequence*

by

B. MAUREY (Paris) and H. P. ROSENTHAL (Urbana, Ill.)

Abstract. Examples are given of sequences of norm-one elements of certain
Banach spaces which tend weakly to zero, yet have no unconditional subsequence.

§ 1. Introduction. Let \((b_n)\) be a finite or infinite sequence of non-zero
elements of a real Banach space \(B\) and \(K\) a positive number. \(b_n\) is said
to be normalized if \(\|b_n\| = 1\) for all \(n\); weakly null if \((b_n)\) is an infinite se-
quence which tends weakly to zero as \(n\) tends to infinity; \(K\)-uncondi-
tional if for all \(n, F \subseteq \{1, \ldots, n\}\), and scalars \(a_1, \ldots, a_n\), \(\sum_{j=1}^{n} a_j b_j\|
\leq K\|F\|\sum_{j=1}^{n} a_j\|b_j\|\); unconditional if it is \(K\)-unconditional for some \(K < \infty\).

It is a famous open question if every infinite dimensional Banach
space contains an infinite unconditional basic sequence. We show that
the following related question, stated in 1968 in [2], has a negative
answer: Does every normalized weakly null sequence in a Banach space have
an infinite unconditional subsequence?

Here are some related positive results. As shown in [3], if a Banach
space has an unconditional basis (i.e. an unconditional sequence with
dense linear span), then every normalized weakly null sequence in the
space does have an unconditional subsequence. It is proved in [9], using
Ramsey's theorem, that if \((b_n)\) is a normalized weakly null sequence
in a Banach space and \(\epsilon > 0\) and \(k\) a positive integer are given, then
there is a subsequence \((b_{n_n})\) of \((b_n)\) so that for all \(m, E \subseteq \{1, \ldots, m\}\) with
\(|E| \leq k\|E\|\) the cardinality of \(E\) and scalars \(a_1, \ldots, a_m\), \(\sum_{j=1}^{m} a_j b_j\|
\leq (1 + \epsilon)\|\sum_{j=1}^{m} a_j b_j\|\).

In particular, every subsequence of \((b_{n_n})\) with \(k\)-elements is \(1 + \epsilon\)-un-
conditional, a previously known result which can also be deduced from
the results of [9]. It is shown in [9] that if \(S\) is a set and \((A_n)\) is a sequence
of non-empty subsets of \(S\) with \((\chi_{A_n})\) weakly null in \(l_1(S)\), then \((\chi_{A_n})\)
has a \(1\)-unconditional subsequence. \((l_1(S)\) denotes the Banach space of

* The research for this paper was partially supported by NSF Grant GP-38677.
all bounded real-valued functions on $S$ under the sup norm; $\mathcal{C}_A$ denotes the characteristic function of the set $A$. We prove in Theorem 3.4 that every normalized weakly null sequence in $C(\omega^\omega+1)$ has an unconditional subsequence. (An ordinal $\alpha$ is identified with the set of ordinals preceding it; $C(\kappa)$ denotes the Banach space of real-valued continuous functions on the space $\kappa$.) (It is known that $C(\omega^\omega+1)$ has no unconditional basis.)

Finally, we note that if an infinite-dimensional Banach space has no normalized weakly null sequence, then it contains an infinite unconditional basic sequence, in fact it contains a subspace isomorphic to $l_2$, the Banach space of all absolutely converging series (this is an immediate consequence of the results in [7]).

It is convenient to introduce the following terminology in order to describe some of the features of our example.

Let $\mathcal{B}$ denote the Banach space of all converging series of real numbers. For any $(a_n)$ with $\sum a_n$ convergent, put $\|a_n\| = \sup_k \left| \sum_{j=1}^k a_j \right|$. Let $e_1, e_2, \ldots$ be the unit vector-basis for $\mathcal{B}$, i.e. $(e_j)_n = \delta_{jk}$ for all $j$ and $k$. We call $(e_j)$ the summing basis. It is evident and well known that $(e_j)$ is not unconditional. Indeed, by considering the expansion $e_j = 2e_k + 2e_k + \ldots + (-1)^{n-1}e_k$, one sees that the unconditional constant of $(e_j, \ldots, e_j)$ equals $n$ for all $n$. (The unconditional constant of a sequence is by definition the smallest $K$ so that it is $K$-unconditional.)

Given $K$, a positive number, and sequences $(x_n)$, $(y_n)$ in Banach spaces $X$ and $Y$, respectively, we say that $(x_n)$ is $K$-block-represented in $(y_n)$ if $(x_n)$ is $K$-equivalent to a block-basis of $(y_n)$; that is, there exists a sequence $(z_j)$ in $Y$ a sequence $F_1, F_2, \ldots$ of finite subsets of $N$, the positive integers, and scalars $(s_j)$ so that for all $n$, max $F_n < \min F_{n+1}$,

$$\left\| \sum_{j=1}^n s_j z_j \right\| \leq K \left\| \sum_{j=1}^n s_j z_j \right\|$$

for all scalars $s_1, \ldots, s_n$. We say that $(x_n)$ is $K$-finitely block-represented in $(y_n)$ if $(x_n)$ is infinite and every finite subsequence of $(x_n)$ is $K$-block-represented in $(y_n)$.

Finally, we say that $(x_n)$ is block-represented (resp. finitely block-represented) in $(y_n)$ if $(x_n)$ is $K$-block-represented (resp. $K$-finitely block-represented) in $(y_n)$ for some $K < \infty$. It is trivial that $(x_n)$ is block-represented in $(y_n)$ if $(x_n)$ is finitely block-represented in $(y_n)$; it is easily seen that:

If $(y_n)$ is an unconditional sequence and $(x_n)$ is finitely block-represented in $(y_n)$, then $(x_n)$ is an unconditional sequence.

In Example 1 of §2 we construct a normalized weakly null sequence $(b_n)$ in a Banach space $B$ so that the summing basis is block-represented in every subsequence of $(b_n)$. (These properties are proved in Theorem 1.) We note that $B^*$ is necessarily unseparable. [A normalized weakly null sequence in a space with a separable dual has a shrinking subsequence (see e.g. [6]) and every block-basis of a shrinking sequence is shrinking; the summing basis is not shrinking.]

In Example 2, we construct a normalized weakly null sequence $(b_n)$ in a Banach space $B$ which isometrically embeds in $C(\omega^\omega+1)$ so that the summing basis is finitely block-represented in every subsequence of $(b_n)$. (The proof that $B$ embeds in $C(\omega^\omega+1)$ is given in Theorem 3.2.) Thus $B^*$ is separable.

In Example 3 we show that for every $K$ there exists a Banach space $\mathcal{B}$ isomorphic to Hilbert space (i.e. $l_2$) and a sequence $(b_n)$ in $B$ equivalent to the usual basis for Hilbert space so that every subsequence of $(b_n)$ has unconditional constant at least $K$. (This example is due to W. B. Johnson.)

Finally, in Example 4, we construct a normalized weakly null sequence in a uniformly convex Banach space $B$ which has no unconditional subsequence. It follows that $B$ does not embed in $C(\omega^\omega+1)$, and that $B^*$ is not separable. [A normalized weakly null sequence in a space with a separable dual has a shrinking subsequence (see e.g. [6]) and every block-basis of a shrinking sequence is shrinking; the summing basis is not shrinking.]

The construction of this example uses the technique of interpolation due to Lions--Peetre and a theorem of Beurling (see [1]).

Our work suggests the following open questions:

1. Is there an infinite-dimensional reflexive Banach space $B$ such that the summing basis is block-represented in every normalized weakly null sequence in $B^*$? (Such a space would have no infinite unconditional sequence.)

2. Is there a normalized weakly null sequence in some Banach space so that no constant-coefficient block basis of a subsequence is unconditional?

3. Does every normalized weakly null sequence $(a_n)$ in $l^1$ have a unconditional subsequence? (By the results of [8], the answer is affirmative if the closed linear space of the $a_n$’s is reflexive.)

§2. Motivation and construction of the examples. We shall first give a pictorial description of the construction of the examples. Later we shall pass to a careful analytical description, less intuitive but easier to work with.

We may think of any Banach space as a subspace of $l^\infty(S)$ for some set $S$; a normalized weakly null sequence may then be realized as a se-
sequence of functions \( f_n \) in \( l^p(S) \) with \( \|f_n\|_p = 1 \) for all \( n \) and the \( f_n \)'s satisfying a stronger condition than \( f_n \to 0 \) pointwise on \( S \). (Precisely, the condition is that if \( \delta > 0 \) and \( n_1 < n_2 < \ldots \) is any increasing sequence of indices, then there is a \( k \) so that \( \sum_{n=k}^{\infty} |f_n(s)| > \delta \) = \( \Omega \); see [9]).

In order to obtain such an \( f_n \) with the summing basis finitely block-represented in every subsequence of \( f_n \) it is useful to have the following sufficient criterion for a sequence \( \{g_n\} \) in \( l^p(S) \) to be isometrically equivalent to the summing basis:

**The Summing-Basis Criterion.** Let \( \{g_n\} \) be a finite or infinite sequence in \( l^p(S) \) so that for all \( j \) for which \( g_j \) exists and \( s \in S \), \( 0 \leq g_j(s) < 1 \), \( \{s \in S : g_j(s) = 0\} \) is non-empty and \( \{s \in S : g_j(s) \neq 0\} \) is finite if \( j \) is odd. Then for all \( n \), \( g_1, \ldots, g_n \) is isometrically equivalent to the first \( n \) terms of the summing basis (if \( g_n \) exists).

We note that if each \( g_j \) is zero-or-one valued, the criterion simply reduces to the assertion that \( \{g_j\} \) is a strictly decreasing sequence of sets, where \( A_j = \{s \in S : g_j(s) = 1\} \) for all \( j \).

To see the criterion, let \( n \) be fixed and scalars \( a_1, \ldots, a_n \) be given. For any \( k \leq n \) choose a \( j \) so that \( g_j(s) = 1 \) and \( g_{k+1}(s) = 0 \) if \( k < n \). Then \( g_j(s) = 1 \) for all \( j \leq k \) and \( g_j(s) = 0 \) all \( r > k \), hence \( |\sum_{j=1}^k a_j g_j(s)| = |\sum_{j=1}^k a_j| \).

Thus \( \|\sum_{j=1}^k a_j g_j\|_\infty \geq \min \{\sum_{j=1}^k a_j, \sum_{j=1}^k a_j\} \).

Now suppose \( s \in S \); choose \( k \leq n \) so that \( g_j(s) = 1 \) and \( g_{k+1}(s) \neq 1 \) if \( k < n \). Then \( g_j(s) = 1 \) for all \( j \leq k \) and \( g_j(s) = 0 \) all \( r > k+1 \). If \( k < n \), so of \( k = n \), \( |\sum_{j=1}^k a_j g_j(s)| = |\sum_{j=1}^k a_j| \).

while if \( k = n \),

\[
|\sum_{j=1}^n a_j g_j(s)| = |a_1 + \cdots + a_n + g_{k+1}(s) a_{k+1}|
\]

\[
= \left| \frac{1}{g_{k+1}(s)} (a_1 + \cdots + a_n) + g_{k+1}(s) (a_1 + \cdots + a_n) \right|
\]

\[
\leq \max \{a_1, \ldots, a_n\} |g_{k+1}(s)| + |a_1 + \cdots + a_n| \leq \sum_{j=1}^n |a_j| \leq \sum_{j=1}^n a_j.
\]

Now by standard arguments, if a sequence \( \{g_j\} \) is a small perturbation of a sequence \( \{g_j\} \) satisfying the criterion, then \( \{g_j\} \) is still equivalent to the summing basis. We shall first describe the construction of normalized weakly sequence \( \{f_n\} \) so that \( \{s_1, e_1\} \) is 1 + \( e \)-block-represented in every subsequence of \( \{f_n\} \) for every \( \epsilon > 0 \).

Let \( 2 \leq b_1 < b_2 < \ldots \) be a strictly increasing sequence of positive integers with \( \lim_{n \to \infty} \frac{b_n}{b_n} = 0 \). Let \( A_1, A_2, \ldots \) be a sequence of infinite disjoint subsets of a set \( S \). Each \( A_j \) shall again be decomposed into a sequence of disjoint subsets with a strange enumeration. For each \( j \), let \( F_j = \{F \subseteq N : |F| = b_j \} \) and \( j = \min F \). Now let \( \{A_{1,j} \in F \in \mathcal{A} \} \) be a family of infinite subsets of \( A_j \), with \( A_{1,j} \cap A_{1,j} = \emptyset \) if \( F \neq F' \). Fix \( F \), \( F \in \mathcal{A} \) and \( \{A_{1,j} \} \). We define \( f_n(s) \) as follows:

\[
f_n(s) = \begin{cases} 
|F|^{-1/2} & \text{if } n \in F, \\
1 & \text{if } n = j, \\
0 & \text{if } n \text{ is not as above.}
\end{cases}
\]

Now if \( \{x \in A_{1,j} \} \) is a finite or infinite sequence in \( l^p(S) \) so that for all \( j \) for which \( g_j \) exists and \( s \in S \), \( 0 \leq g_j(s) < 1 \), \( \{s \in S : g_j(s) = 0\} \) is non-empty and \( \{s \in S : g_j(s) \neq 0\} \) is finite if \( j \) is odd. Then for all \( n \), \( g_1, \ldots, g_n \) is isometrically equivalent to the first \( n \) terms of the summing basis (if \( g_n \) exists).

We note that if each \( g_j \) is zero-or-one valued, the criterion simply reduces to the assertion that \( \{g_j\} \) is a strictly decreasing sequence of sets, where \( A_j = \{s \in S : g_j(s) = 1\} \) for all \( j \).

To see the criterion, let \( n \) be fixed and scalars \( a_1, \ldots, a_n \) be given. For any \( k \leq n \) choose a \( j \) so that \( g_j(s) = 1 \) and \( g_{k+1}(s) = 0 \) if \( k < n \). Then \( g_j(s) = 1 \) for all \( j \leq k \) and \( g_j(s) = 0 \) all \( r > k \), hence \( |\sum_{j=1}^k a_j g_j(s)| = |\sum_{j=1}^k a_j| \).

Thus \( \|\sum_{j=1}^k a_j g_j\|_\infty \geq \min \{\sum_{j=1}^k a_j, \sum_{j=1}^k a_j\} \).

Now suppose \( s \in S \); choose \( k \leq n \) so that \( g_j(s) = 1 \) and \( g_{k+1}(s) \neq 1 \) if \( k < n \). Then \( g_j(s) = 1 \) for all \( j \leq k \) and \( g_j(s) = 0 \) all \( r > k+1 \). If \( k < n \), so of \( k = n \), \( |\sum_{j=1}^k a_j g_j(s)| = |\sum_{j=1}^k a_j| \).

while if \( k = n \),

\[
|\sum_{j=1}^n a_j g_j(s)| = |a_1 + \cdots + a_n + g_{k+1}(s) a_{k+1}|
\]

\[
= \left| \frac{1}{g_{k+1}(s)} (a_1 + \cdots + a_n) + g_{k+1}(s) (a_1 + \cdots + a_n) \right|
\]

\[
\leq \max \{a_1, \ldots, a_n\} |g_{k+1}(s)| + |a_1 + \cdots + a_n| \leq \sum_{j=1}^n |a_j| \leq \sum_{j=1}^n a_j.
\]

Now by standard arguments, if a sequence \( \{g_j\} \) is a small perturbation of a sequence \( \{g_j\} \) satisfying the criterion, then \( \{g_j\} \) is still equivalent to the summing basis. We shall first describe the construction of normalized weakly sequence \( \{f_n\} \) so that \( \{s_1, e_1\} \) is 1 + \( e \)-block-represented in every subsequence of \( \{f_n\} \) for every \( \epsilon > 0 \).

Let \( 2 \leq b_1 < b_2 < \ldots \) be a strictly increasing sequence of positive integers with \( \lim_{n \to \infty} \frac{b_n}{b_n} = 0 \). Let \( A_1, A_2, \ldots \) be a sequence of infinite disjoint subsets of a set \( S \). Each \( A_j \) shall again be decomposed into a sequence of disjoint subsets with a strange enumeration. For each \( j \), let \( F_j = \{F \subseteq N : |F| = b_j \} \) and \( j = \min F \). Now let \( \{A_{1,j} \in F \in \mathcal{A} \} \) be a family of infinite subsets of \( A_j \), with \( A_{1,j} \cap A_{1,j} = \emptyset \) if \( F \neq F' \). Fix \( F \), \( F \in \mathcal{A} \) and \( \{A_{1,j} \} \). We define \( f_n(s) \) as follows:

\[
f_n(s) = \begin{cases} 
|F|^{-1/2} & \text{if } n \in F, \\
1 & \text{if } n = j, \\
0 & \text{if } n \text{ is not as above.}
\end{cases}
\]
enumeration of $M$, it is sufficient to assume that $y_1/y_{j_0} = 0$. Now assume that $(b_1, b_2, \ldots; j = 1, 2, \ldots$ and $F \in \mathcal{A})$ is a set of distinct elements of $M$. For each $j$ and $F \in \mathcal{A}$, let $x_{b,j} \equiv \{G < N : |G| = \text{b}_j \text{ and max } \text{F} < \text{min } \text{G}\}$. Let $\{A_{j,F,G} : G \in \mathcal{A}\}$ be a family of infinite subsets of $A_{j,F,G}$. Define a new sequence $(f_n)$ on $N$ as follows: Fix $j$, $F \subseteq A_j$, $A_{j,F,G}$, and $x_{A_{j,F,G}}$:

$$f_n(x) = \begin{cases} 
(G^{-1,2}/2) & \text{if } n \in G, \\
|F|^{-1,2} & \text{if } n \in F, \\
1 & \text{if } n = j, \\
0 & \text{if } n \text{ is not as above.}
\end{cases}$$

It may again be verified that if $i > 0$ is given and $M'$ is an infinite subset of $N$, then for $j < M'$, $j$ large enough; choosing $E \subseteq M'$ with $E \subseteq F_j$ and then $G \subseteq M'$ with $G \in \mathcal{A}$, that $(g_1, g_2, g_3, g_4)$ is a $1 + \varepsilon$-equivalent to the first three terms of the summing basis, where $|g_1| = f_j$, $g_2 = |F|^{-1,2} \sum_{F \in \mathcal{A}} f_n$, and $g_3 = |G|^{-1,2} \sum_{G \in \mathcal{A}} f_n$. In fact, if $g_1 = g_1, g_2 = g_2, g_3 = g_3, A_{j,F,G}$, and $g_4 = g_4, A_{j,F,G}$, then $(g_1, g_2, g_3, g_4)$ satisfies the Summing-Basis Criterion and is a small perturbation of $(g_1, g_2, g_3, g_4)$ for $j$ large enough.

To obtain a weakly null normalized sequence so that the summing basis is finitely block-represented in every subsequence, we consider sets of the form $A_{j,F_{r-1},A_{j,F_r}}$ where $1 < k < j$, $F_{r-1} = \min F_r$, for all $r$, and $A_{j,F_{r-1},A_{j,F_r}}$ is a family of disjoint subsets of $A_{j,F_{r-1},A_{j,F_r}}$ with cardinalities equal to a certain function of $(j, F_{r-1}, A_{j,F_r})$. As long as these cardinalities are sufficiently lacunary and disjoint for $(j, F_{r-1}, A_{j,F_r}) \neq (j, F_{r-1}, A_{j,F_r})$, the functions $(f_n)$ defined by $f_n(x) = |F|^{-1,2}$ if $x \subseteq A_{j,F_{r-1},A_{j,F_r}}$ and $\in A_{j,F_{r-1},A_{j,F_r}}$, have the desired properties (where $F_j = \{G_j\}$ by definition).

We wish now to give purely analytical expressions for the examples. From now on, we let $(f_n)$ denote the usual coordinate functions on the positive integers; that is $f_n(m) = 1$ if $m = m_0$, $f_n(m) = 0$ if $m \neq m_0$. Let $\lambda(m)$ denote the linear span of the $f_n$'s, i.e. the space of all functions on the positive integers with finite support.

For $f_n$ on $\lambda(m)$ and $g$ any real-valued function on $\lambda(m)$, let $\langle f, g \rangle$ denote the usual inner product of $f$ and $g$.

$$\langle f, g \rangle = \sum_{n=1}^{\infty} f_n(m)g(n).$$

Now the first example that we described, as a Banach space, is isometric to the completion of $\lambda(m)$ under the norm $\|f\| = \sum \langle f_n, x \rangle + \|F|^{-1,2} \sum_{x \in \mathcal{A}} f_n$, the supremum taken over all $j$ and sets $F \in \mathcal{A}$ with $|F| = b_j$ and $\min F > j$. Put another way, the map which sends the $u_{b,j}$'s of the example to our coordinate $f_n$'s is a linear isometry. We wish to point out in passing that certain features of this example are similar to those of an old example of Schreier [10]. Schreier’s space has the property that there is a weak null sequence $(b_n)$ in it so that for no subsequence $(b_{k_1}, b_{k_2}, \ldots)$ does one have $\lim \frac{1}{n} \sum_{k=1}^{n} b_{k} = 0$. Schreier’s space is essentially the completion of $\lambda(m)$ under the norm $\|f\| = \sup \langle f, x \rangle$ (where $\|F\|$ ranges over finite subsets of $N$).

**Example 1.** Let $\mathcal{G}$ be the family of all finite sequences of finite subsets of $N$.

$$\mathcal{G} = \{(F_1, \ldots, F_k) : F_i \subseteq N, \ |F_i| < \infty, \ 1 \leq i \leq k, \ k = 1, 2, \ldots \}.$$ 

Let $M = \{4^j : j = 1, 2, \ldots\}$. Let $\psi : \mathcal{G} \to M$ be a one-one map. Let $\mathcal{B}$ be the family of all infinite sequences $(F_1, F_2, \ldots)$ of finite subsets of $N$ so that for all $k > 1$,

(a) $|F_k| = 1$;

(b) $\max F_{k-1} < \min F_k$;

(c) $|F_k| = \psi(F_1, \ldots, F_{k-1})$.

Now define $\| \cdot \|$ on $\lambda(m)$ by

$$\|f\| = \sup \left( \frac{1}{n} \sum_{n=1}^{\infty} \left| \sum_{j=1}^{n} f_j \right| \right).$$

(While for $(F_j) \in \mathcal{B}$, $\sup \sum_{j=1}^{\infty} |F_j|^{-1,2} \sum_{x \in \mathcal{A}} f_n$ denotes the function $g$ so that $g(n) = |F_j|^{-1,2}$ if $n \in F_j$ and $g(n) = 0$ if $n \notin \bigcup_{j} F_j$.)

**Theorem 2.1.** Let $B$ denote the completion of $\lambda(m)$ under the above norm.

Then $(f_n)$ is a normalized weak null sequence in $B$ so that the summing basis is block-represented in every subsequence of $(f_n)$.

It is convenient to summarize the properties of $M$ and $\mathcal{B}$ that we use. **Proposition 2.2.** Let $h(j, k) = \min \{k \cdot \text{gcd}(k, k') : j$ and $k$ positive integers. Then setting $\lambda = \sqrt{\frac{6}{\pi^2}}$,

$$\sum_{\text{gcd}(m_i, m_j) = 0} h(m_i, m_j) = e < 1.$$

For any $(F_k) \in \mathcal{B}$,

(1) for all $k$, (a) and (b) above hold and $|F_{k+1}| \leq M$ (by (c));

(2) for all $k$ and $n$, there exists a $(G_k)$ in $\mathcal{B}$ so that $F_j = G_j$ for all $j \leq k$

and $n < \min G_{k+1}$.
for any \((G_j)\) in \(G\) and any \(j\) and \(k\), if \(|F_{j+i}| = |G_{k+i}|\) then \(j = k\) and \(F_i = G_i\) for all \(i \leq k\);

(4) for any infinite subset \(N'\) of \(N\) and \(j \in N'\), there exists an \((F_j)_j \in G\) with \((j) = F_1\) and \(F_n \in N'\) for all \(n\).

Proof. We leave the simple verification of all but (4) to the reader.

(4) is really also evident; given \(N'\) and \(j \in N'\) define \(F_1 = (j)\); having defined \(F_1, \ldots, F_{i-1}\), let \(F_i\) be any finite subset of \(N'\) with \(|F_{i+1}| = \psi(F_1, \ldots, F_i)\) and min \(|F_{i+1}| = \max F_i\). The sequence \((F_n)\) thus defined by induction is the desired member of \(G\).

The following simple estimate will be useful in proving Theorem 2.1.

(1) \[
\langle |F|^{-1} \chi_{F}, |G|^{-1} \chi_{G} \rangle = \langle |F|^{-1} |G|^{-1} \chi_{F \cap G} \rangle \\
\leq \langle |F|^{-1} |G|^{-1} \min (|F|, |G|) \rangle = \epsilon (|F|, |G|)
\]

for any finite non-empty subsets \(F\) and \(G\) of \(N\) (where \(\epsilon\) is defined in Proposition 2.2).\]

Lemma 2.3. For any \((F_j)_j \in G\), any \(n\), and any scalars \(c_1, \ldots, c_n\),

\[
\sum_{k=1}^{n} \langle |F|^{-1} \chi_{F}, |G|^{-1} \chi_{G} \rangle \leq (1 + \epsilon) \sum_{k=1}^{n} |c_k|.
\]

It follows that the summing basis is block-represented in every subsequence of \((f_n)\). Let \((F_j)_j \in G\) be a subsequence. By (4) of Proposition 2.2, we may choose \((F_j)_j \in G\) with \(F_n \in N'\) for all \(n\). The lemma asserts that the block basis \((|F|^{-1} \chi_{F})_{F_n \in N'}\) is equivalent to the summing basis.

To prove the lemma, let \(k \leq n\) and choose by (2) of Proposition 2.2 a \((G_j)\) in \(G\) satisfying its statement. Then

\[
\left\| \sum_{j=1}^{k} c_j |F_j|^{-1} \chi_{F_j} \right\| \geq \left| \sum_{j=1}^{k} c_j |F_j|^{-1} \chi_{F_j} \sum_{j=k+1}^{n} |G_j|^{-1} \chi_{G_j} \right| = \left| \sum_{j=1}^{k} c_j \right|.
\]

This proves the left inequality of Lemma 2.3.

Now let \((G_j)\) be an arbitrary element of \(G\) and let \(s\) be the greatest integer less than or equal to \(n\) such that \(|F| = |G_s|\). It follows from (3) of Proposition 2.2 that then \(F_j = G_j\) for all \(1 \leq j < s\) and moreover if for any \(j < s\) and \(k\) arbitrary one has \(|F_j| = |G_k|\) then \(j = k \leq s\). (Also for \(j \geq s\), \(k > s\), the \(|F_j|\)'s and \(|G_k|\)'s are distinct members of \(M\).) Hence

\[
\sum_{k=1}^{s} \min (|F_k|, |G_k|) \leq \epsilon/2
\]

by (1) of Proposition 2.2, since the sum in (2) is a partial sum of the series (\(\epsilon\) in Proposition 2.2. (Note that each term \(h(m_j, m_i)\) appears twice in (\(\epsilon\)) and once in (2.)) Now

\[
\left| \sum_{j=1}^{n} c_j |F_j|^{-1} \chi_{F_j} \sum_{j=1}^{n} |G_j|^{-1} \chi_{G_j} \right| \\
\leq \sum_{j=1}^{n} |c_j| |F_j|^{-1} |G_j|^{-1} \chi_{F_j} \chi_{G_j} \sum_{j=1}^{n} |G_j|^{-1} \chi_{G_j} \sum_{j=1}^{n} |G_j|^{-1} \chi_{G_j} \sum_{j=1}^{n} |G_j|^{-1} \chi_{G_j} \\
\leq (1 + \epsilon) \sum_{j=1}^{n} |c_j|
\]

by (1), (2), and a simple inequality in our earlier analysis of the summing basis. This completes the proof of Lemma 2.3.

It remains to show that \((f_n)\) is a normalized weak null sequence. Now its trivial that \(||f|| \leq 1\) for any \(j\). Fixing \(j\) and choosing \((F_n)\) in \(G\) (with \(F_1 = (j)\) by (4) of Proposition 2.2), we have that \(|\sum_{j=1}^{n} |F_n|^{-1} \chi_{F_n}|| = 1\), so \((f_n)\) is normalized. The fact that \((f_n)\) is weakly null is an immediate consequence of the following result:

**Lemma 2.4.** For any finite set \(E \subseteq N\),

\[
|\langle |E|^{-1} \chi_{E}, |G|^{-1} \chi_{G} \rangle| \leq (1 + \epsilon) |E|^{-1}.
\]

Proof. Let \(k = \langle |E|, (G_j) \rangle \in G\), \(j_1, \ldots, j_k\) the integers \(j\) so that \(G_j \cap E \neq \emptyset\), \(a_j = |G_j|^{-1} \chi_{G_j}\) and \(a_k = |G_k|^{-1} \chi_{G_k}\) for all \(1 \leq j \leq k\). We thus have that the \(a_j\)'s are distinct members of \(M\). Hence

\[
\sum_{j=1}^{k} |E|^{-1} \chi_{E} \sum_{j=1}^{k} |G_j|^{-1} \chi_{G_j} = \sum_{j=1}^{k} a_j^2 = \sum_{j=1}^{k} a_j^2 \leq (1 + \epsilon) |E|^{-1}
\]

by the definition of \(A\). If \(\{j_1, \ldots, j_k\} \neq \emptyset\) is non-empty, then

\[
\sum_{j=1}^{k} a_j^2 \leq |E|^{-1} \sum_{j=1}^{k} a_j^2 \leq |E|^{-1} \leq \frac{1}{\sqrt{\epsilon}} \sum_{j=1}^{k} a_j^2 \leq \frac{1}{\sqrt{\epsilon}} \sum_{j=1}^{k} a_j^2
\]

Hence \(\sum_{j=1}^{k} a_j^2 \leq (1 + \epsilon) |E|^{-1}\). This completes the proof of Lemma 2.4, and thus of Theorem 2.1. (We are indebted to L. Dor for an illuminating discussion concerning the last lemma. Our original proof that \((f_n)\) is weakly null did not contain the more precise information, due to Dor, con-
tained in the lemma. In fact, Der obtained a more refined analysis which shows that
\[
\lim_{|T| \to \infty} \|F^{(n)}|_{Z_B}\| = 1.
\]

\textbf{Example 2.} Let $B$ be as in Example 1. Let $F$ be the family of all finite sequences $(F_n)$ of finite subsets of $N$ such that $(F_n) = (F_1, \ldots, F_j)$ where $f = F_i$ and there exists an $E_{F_i}$-block $F_i$ with $F_i = F_i$ for all $1 \leq i \leq j$.

Define a norm $||| \cdot |||$ on $c_0$ by $|||f||| = sup \left( \sum_{F_i} \left| f_i \right| \right)$.

Let $B$ denote the completion of $c_0$ under the norm $||| \cdot |||$. Then our proof of Theorem 2.1 yields that $(f_j)$ is a normalized weakly null sequence in $B$ so that the summing basis is finitely block-represented in every subsequence of $(f_j)$. Indeed, Lemma 2.4 holds for the norm $||| \cdot |||$. Now, each $F_i$ is a finite subset of $N$ and $k$ is a positive integer, choose $k < j$ and $(F_1, \ldots, F_j)$ such that $F_i \subset F_i$ for all $1 \leq i \leq j$. The proof of Lemma 2.2 then shows that $(F_n)^{\infty}_{n=1}$ is a block basis of $(f_j)_{n=1}$ which is $1+\epsilon$-equivalent to the first $j$ terms of the summing basis. It can be demonstrated that $B$ is isometric to a subspace of $C(\omega^{\omega}+1)$.

\textbf{Example 2k.} Fix $k$ an integer larger than one and let $B_k = (E_1, \ldots, E_k)$. Define $||| \cdot |||$ on $c_0$ by
\[
||f|| = \sup_{(E_1, \ldots, E_k) \in B_k} \left( \sum_{E_i} \left| f_i \right| \right).
\]

Let $B_k$ be the completion of $c_0$ under $|| \cdot ||$. Then $(f_j)$ is a normalized weakly null sequence in $B_k$ such that the first $k$-terms of the summing basis are $1+\epsilon$-block-represented in every subsequence of $(f_j)$ for every $\epsilon > 0$. Let $B_k$ be the completion of $c_0$ under $|| \cdot ||$. Then $(f_j)$ is a normalized weakly null sequence in $B_k$ such that the first $k$-terms of the summing basis are $1+\epsilon$-block-represented in every subsequence of $(f_j)$ for every $\epsilon > 0$. We shall prove in Theorem 3.2 that $B_k$ is isometrically imbeds in $C(\omega^{\omega}+1)$.

We note that the particular form of the representability of the first two terms of the summing basis shows that the bounded constant of any subsequence of $(f_j)$ in the $\ell_2$-norm is equal to 2. That is, for any infinite subset $N'$ of $N$,
\[
\sup_{n \in c_0(N') \in \ell_2} \frac{\|f_n\|_{\ell_2}^2}{|||f|||} = 2.
\]

\textbf{Example 2'.} Define $|| \cdot ||'$ on $c_0$ by $||f||' = \sup (\sum_{F_i} \left| f_i \right|)$ and let $B'$ be the completion of $c_0$ under $|| \cdot ||'$. We shall prove in Theorem 3.2 that $B'$ is isometrically imbeds in $C(\omega^{\omega}+1)$ and that $(f_j)$ is a normalized weakly null sequence in $B'$ such that the summing basis is finitely block-represented in every subsequence of $(f_j)$.

\textbf{Example 3.} Let $B_k$ be the completion of $c_0$ under $|| \cdot ||'$.

\textbf{Proposition 2.5.} Let $B_k$ be the completion of $c_0$ in the norm $|| | \cdot ||$. Then $B_k$ is isomorphic to a Hilbert space, $(f_j)$ is a normalized weakly null sequence in $B_k$, and $f_i$ is $V$-equivalent to the usual $V$-basis; every subsequence $(f_i')$ of $(f_j)$ has unconditional constant at least as large as $V/k$.

Indeed, for any $f \in c_0$, $(f_k') 
\sum_{k} \left| f_k \right| \left| f_k' \right| = V/k$.

\textbf{Example 4.} For a norm $|| \cdot ||$ on $c_0$ we let $|| \cdot ||^*$ be the dual norm on $c_0$ defined by $||f||^* = \sup \langle f, x \rangle$, the supremum taken over all $x \in c_0$ with $||x|| \leq 1$. Now let $|| | \cdot ||$ be the norm on $c_0$ given as $|| | \cdot ||$ in Example 1. Then $|| | \cdot ||^*$ is the usual $\ell^1$-norm.

Then let $|| | \cdot ||$ be a uniformly convex norm on $c_0$ satisfying
\[
||f|| \leq (||f||^* ||f||)^{1/2} \quad \text{and} \quad ||f||^* \leq (||f|| \cdot ||f||)^{1/2}
\]

for all $f \in c_0$. (Such a norm is obtained using Lions–Peetre interpolation between the norms $|| \cdot ||$ and $|| | \cdot ||$.) We have that $f$ is uniformly convex in $c_0$.

\textbf{Theorem 2.6.} Let $U$ be the completion of $c_0$ under the norm $|| | \cdot ||$ of Example 4. Then $U$ is a uniformly convex Banach space and $(f_j)$ is a normalized weakly null sequence in $U$ having no unconditional subsequence.
§ 3. Normalized weakly null sequences in $\mathcal{O}(a)$ for ordinal $a$. We show in this section that certain of our examples imbed in $\mathcal{O}(a^{*}+1)$ while in $\mathcal{O}(a^{*}+1)$ every normalized weakly null sequence has an unconditional subsequence.

We need the following technical result:

**Lemma 3.1.** Let $1 < n < \infty$.

(a) There exists a sequence $(A_j)$ of clopen (closed and open) subsets of $\omega^a+1$ so that

(i) any $n$ of the $A_j$'s intersect in a singleton; i.e. for all $F$ with $|F| = n$, 
\[ \bigcap_{j \in F} A_j = \{x\} \]

(ii) no $n+1$ of the $A_j$'s have a common point;

(iii) for every isolated point $x$ of $\omega^a+1$, there exists an $F$ with $|F| = n$ and \( \bigcap_{j \in F} A_j = \{x\} \).

(b) For any sequence $(A_j)$ of non-empty clopen subsets of $\omega^a+1$ with \( \bigcap_{j=1}^n A_j = \emptyset \) weakly in $\mathcal{O}(a^{*}+1)$ there exists a subsequence $(A'_k)$ of $(A_j)$ and an $1 \leq k \leq n$ so that any $k$ of the $A'_j$'s have a common point, yet no $k+1$ of the $A'_j$'s have a common point.

**Proof.** As we have already noted in the introduction, the case $n = 1$ is trivial. Suppose (a) is proved for $n$. Let $B_1, B_2, \ldots$ be a sequence of disjoint compact open subsets of $\omega^{a+1}$ so that $B_j$ is homeomorphic to $\omega^a+1$ for all $j$ and $\omega^{a+1} = \bigcup_{j=1}^\infty B_j$. For each $m$, let $(A_m^j)_{j=m}^{m+1}$ be clopen subsets of $B_m$ satisfying (i)-(iii) for $A'_j = A_m^j$ and $\omega^{a+1} = B_m$.

For each $m = 1, 2, \ldots$, put $A_m = B_m \cup \bigcup_{j=m}^\infty A_m^j$. We claim that $(A_m)$ satisfies (i)-(iii) for $n+1$. It is evident that the $A_m$'s are clopen subsets of $\omega^{a+1}$. Let $|F| = n+1$ with $F = \{m_1, \ldots, m_{n+1}\}$ and $m_1 < m_2 < \cdots < m_{n+1}$. Then

\[ \left| \bigcap_{j=1}^m A_m \cap B_m \right| = \left| m_1 \right| \leq \left| m_2 \right| = 1. \]

If $m > m_1$, $A_m \cap B_m = \emptyset$. If $m < m_1$, then

\[ \bigcap_{j=m+1}^n A_m \cap B_m = \bigcup_{j=m+1}^n A_m \cap B_m = \emptyset \]

by (ii). Hence $\bigcap_{m=1}^\infty A_m = \{x\}$. For any isolated point $x$ of $\omega^{a+1}$, we may choose $m_j$ with \( x \in B_{m_j} \). By (iii) we may choose $m_1, \ldots, m_{n+1}$ with $m_1 < m_2 < \cdots < m_{n+1}$ for $2 \leq j \leq n+1$ and $\bigcap_{j=m}^n A_m = \{x\}$. Thus $\bigcap_{m=1}^\infty A_m = \{x\}$ and (iii) is established. It is also easily seen that (ii) holds for $n+1$.  

\[ \text{lim sup}_{n \to \infty} n^{-1} \left| \bigcap_{F \subseteq \omega^a} F \right| = 0. \]
Remark. It is possible to introduce an ordering \( \leq_{\omega} \) on \( \omega^\omega \) (the set of all functions from \( \omega \) to \( \omega \)) so that \( \omega^\omega \) in the order topology is homeomorphic to the set of isolated points of \( \omega^{\omega+1} \) in the relative topology. We then put \( A_\omega = (\omega^\omega \setminus \omega F) \). It can be shown that with this ordering, the \( A_\omega \)'s carry over to an appropriate sequence of clopen subsets of \( \omega^{\omega+1} \) satisfying (i)-(iii).

Proof. (b) It suffices to prove (by induction) that there is a subsequence \( (A'_j) \) of the \( A_n \)'s so that \( n+1 \) of the \( A'_n \)'s have a common point. For then let \( k = \min \{ j \in \omega : \text{\( A'_j \) intersects \( \bigcap_{i < j} A'_n \)} \} \) and then \( A'_k \). Thus the \( A'_k \)'s have a common point \( n+1 \) of the \( A'_n \)'s are not equal to \( \omega \).

By Ramsey's theorem there exists an infinite sequence \( N' \) of \( \omega \) so that \( F \subset C' \) for all \( F \subset N' \) with \( |F| = k \). Now it is easy to see that the \( A'_n \)'s are compact open subsets of \( \omega^\omega \) with the property that \( \omega_{\omega, N} = 0 \) pointwise (hence \( \omega_{\omega, N} \) is weakly null in \( C(\omega^\omega + 1) \), yet \( \bigcup_{i \in \omega} A'_i \not= \emptyset \) for all \( i \). Indeed, it can be seen that for any infinite sequence \( j_i \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n |A_{j_i}| > 0.
\]

The closed linear span of the \( \omega_{\omega, N} \)’s is essentially the same space as the one of Schreier’s mentioned earlier; this combinatorial description motivated certain of the constructions which follow.

**Theorem 3.2.**

(a) Let \( n \geq 1 \). There exists a normalized weakly null sequence \( (f_j) \) in \( C(\omega^{\omega+1}) \) such that the first \( n+1 \) terms of the summing basis are 1 \(-\varepsilon\)-block represented in every subspace of \( (f_j) \) for every \( \varepsilon > 0 \).

(b) There exists a normalized weakly null sequence \( (f_j) \) in \( C(\omega^\omega + 1) \) so that summing basis is finitely block-represented in every subspace of \( (f_j) \).

Remark. Fix \( n \) and let \( (f_j) \) be the sequence from part (a). It follows that the unconditional constant of every subspace of \( (f_j) \) is at least equal to \( n+1 \). We show in Theorem 3.4 that this is the best possible result (up to an arbitrary \( \varepsilon > 0 \)) for \( n+1 \).

**Proof of Theorem 3.2.** (a) It suffices to show that the Banach space \( J_{n+1} \) constructed as Example 2 is essentially isometrically isomorphic to \( C(\omega^{\omega+1}) \).

Choose \( (D_{j_k})_{k=1}^\omega \) a sequence of compact open subsets of \( \omega^\omega \) with \( \sup D_j < \text{inf } D_{j+1} \) and \( D_j \) homeomorphic to \( \omega^j + 1 \) for all \( j \), with \( \omega^\omega \)
\[ \bigcup_{j \in \mathbb{R}} D_j, \text{ where } K_j = \psi(f_j) \text{ for all } j \text{ and } \psi \text{ is defined in Example 1.} \]

For each \( k \), choose a sequence \( (A_{m}^{(k)})_{m=1}^{\infty} \) of clopen subsets of \( D_k \) satisfying (i)-(iii) of Lemma 3.1 for \( (a_j) = (A_{m}^{(k)}prev{,}A_{m}^{(k)}) = A_{m=1}^{(k)} \) and "\( a_j \)" = \( D_k \).

Now define functions \( f_m \) on \( a_j \) as follows: fix \( k \) and let \( x \in D_k \); put
\[
f_m(x) = \begin{cases} 
K_m^{(k)} & \text{if } x \in A_m^{(k)} \text{ where } m > k; \\
1 & \text{if } m = k; \\
0 & \text{otherwise.}
\end{cases}
\]

It is evident that the \( f_m \)'s thus defined are continuous functions on \( a_j \). Suppose that \( g = \sum_{m=1}^{\infty} q_m f_m \) for some \( m \) and scalars \( q_1, \ldots, q_m \) and \( \bar{x} \) is an isolated point of \( a_j \). Then there exists a \( k \) and a set \( F \) with \( |F| = K_k \) and \( k < \min F \) so that \( \bar{x} \in \bigcup_{j \in F} A_j \). Putting \( F_1 = \emptyset \) and \( F_2 = F \), it follows that \( g(\bar{x}) = \sum_{j \in F_2} q_j f_j = \sum_{j \in F_2} q_j f_j^{(k)} \). Since the norm of \( g \) equals its supremum on the set of isolated points of \( \bigcup_{j \in F_2} A_j \) is not empty-for any such \( k \) and \( F \), this proves that \( B_2 \) is isometric to the closed linear span of the \( f_m \)'s in \( C(a_j) \). Let now \( f_j = f_j^{(k)} \) for all \( j \).

Now suppose a sequence \( (f_j^{(k)})_{m=1}^{\infty} \) of continuous functions on \( a_j \) has been constructed so that there is a one-one correspondence \( T \) between a subset \( I_0 \) of the isolated points of \( a_j \) and \( B_2 \) satisfying the following conditions: for each \( g = \sum_{m=1}^{\infty} q_m f_m \) in the linear span of the \( f_m \)'s and \( \pi \in T \), \( g(\pi) = \sum_{m=1}^{\infty} q_m f_m^{(k)} \). Since \( C(I_0) \) isomorphic imbeds in \( C(a_j) \), it follows that \( B_2 \) also does. We note that our construction of the \( f_m \)'s satisfies these conditions for \( I_0 \) equal to the set of isolated points of \( a_j \).

Let \( \{D_j \mid j \in J\} \) be a family of compact open subsets of \( a_j \) so that for all \( y \neq y' \), \( D_y \cap D_{y'} = \emptyset \) and for each \( y \in I \), \( D_y \) is homeomorphic to \( a_j^{(k)} \).

For each \( y \), \( \{A_{m}^{(k)} \}_{m \in J \} \) be a sequence of clopen subsets of \( D_y \) satisfying (i)-(iii) of Lemma 3.1 for \( (a_j) = (A_{m}^{(k)}prev{,}A_{m}^{(k)}) = A_{m=1}^{(k)} \) and "\( a_j \)" = \( D_y \), where \( \pi = ((A_{m=1}^{(k)}), A_{m=1}^{(k)}) \).

Now we identify \( a_j^{(k)} \) with \( (a_j^{(k)} \times a_j^{(k)}) \) with the cartesian product endowed with the reverse-lexicographic order topology. We define functions \( f_m^{(k)} \), \( m = 1, 2, \ldots \) as follows: let \( (x, y) \in a_j^{(k)} \).\( \exists \) \( m \) so that \( f_m^{(k)}(x, y) = (\psi^{(a_j)}(x), y) \). If \( y \in I \), \( (A_{m=1}^{(k)}), A_{m=1}^{(k)}) \), \( \pi \in T \), and \( \pi : A_{m}^{(k)} \rightarrow a_j^{(k)} \) so that \( \pi = (\psi^{(a_j)}(x), y) \). If \( y \in I \) and \( m = 1 \) or \( y \in I \), let \( f_m^{(k)}((x, y)) = f_m^{(k)}(y) \).

The functions \( f_m \) are defined as follows: in general, find \( m \) and let
\[\|V_m - (y^{(a_j)} + m \text{ s.t. } m \in J \} \}

Then \( V_m \) is a finite set and \( A_{m=1}^{(k)} \times a_j^{(k)} \) is a clopen subset of \( a_j \times a_j \). An alternate description of \( f_m \) is that \( f_m^{(k)} = \sum_{j \in F_2} q_j f_j^{(k)} \) on \( \pi = (\psi^{(a_j)}(x), y) \). Since \( f_m^{(k)}((x, y)) = f_m^{(k)}(y) \) for all \( (x, y) \in a_j^{(k)} \times (\psi^{(a_j)}(y)) \).

We now let \( I_{n+1} = \{(x, y) \mid y \in I_n \} \), and we are an isolated point of \( D_y \).

Let \( (x, y) \in I_{n+1} \). Then \( f_m^{(k)}((x, y)) = f_m^{(k)}(y) \) unless \( j \in F_2 \), which \( (x, y) \) is \( f_m^{(k)}(y) \). We may choose an \( F_{n+1} \) depending uniquely on \( (x, y) \) with \( \pi = \psi^{(a_j)}(x), y) \) and \( f_m^{(k)}((x, y)) \) unless \( j \in F_2 \). Then \( f_m^{(k)}((x, y)) = \sum_{j \in F_2} m \text{ s.t. } m \in J \} \}

For any \( (A_{m=1}^{(k)}, A_{m=1}^{(k)}) \in T \) there is a unique \( (x, y) \in I_{n+1} \) with \( (x, y) \in I_{n+1} \) and \( \pi = (\psi^{(a_j)}(x), y) \) and \( (x, y) \in \bigcup_{j \in F_2} A_j \) so that \( \pi = (\psi^{(a_j)}(x), y) \). Since \( C(I_{n+1}) \) isomorphic imbeds in \( C(a_j^{(k)}) \), if \( \pi = (\psi^{(a_j)}(x), y) \) implies \( \pi : A_{m}^{(k)} \rightarrow a_j^{(k)} \), in which case \( f_m^{(k)}((x, y)) = \sum_{j \in F_2} m \text{ s.t. } m \in J \} \}

For any \( (x, y) \in I_{n+1} \), the norm \( \|f_m^{(k)}((x, y)) = f_m^{(k)}(y) \) is \( f_m^{(k)}(y) \). Hence
\[\|g((x, y)) = \sum_{j \in F_2} m \text{ s.t. } m \in J \} \}

for all \( (x, y) \in I_{n+1} \) and \( \pi = (\psi^{(a_j)}(x), y) \) and \( (x, y) \in \bigcup_{j \in F_2} A_j \) so that \( \pi = (\psi^{(a_j)}(x), y) \). It is trivial to show that the \( f_m^{(k)} \) are continuous functions on \( a_j^{(k)} \). Fix \( m \) and scalars \( (q_j) \) with \( q_j = 0 \) for all \( j > m \); put \( \sum_{j \in F_2} q_j f_j^{(k)} \) and \( g = \sum_{j \in F_2} q_j f_j^{(k)} \). For an arbitrary \( \bar{x} \),
\[\|g\| = \sum_{j \in F_2} m \text{ s.t. } m \in J \} \}

the latter equality holding by part (a). Since \( ||f|| = \sup_{y \in a_j} ||f(y)\|, \) we have established that the closed linear span of the \( f_m \)'s in \( C(a_j^{(k)}) \) is isometric to the space \( B' \) defined in Example 2 of \( \|2 \) 2.

We thus need only prove the properties of \( B' \).
basis for $B_{k}$ for all $k$; i.e. for all $\{g_{n}\}_{n=1}^{\infty}$, all $k$, and all $n$, $\|x_{[1,...,m]}^{g_{n}}\|^{\tilde{p}} \leq 2\|g_{n}\|^{\tilde{p}}$.

This fact is easily deduced from the following property of $g$, which is slightly stronger than (2) of Proposition 2.2: for all $k,n,m$ and $(F)\in \mathcal{B}$ there exist $n \in \mathcal{N}$ with $E_{k} = G_{n}$ for all $j \leq n$, $G_{n+1} \cap \{1, \ldots, m\} = E_{k+1} \cap \{1, \ldots, m\}$ and $n < \min G_{n+1} \cap \{m+1, m+2, \ldots\}$.

Now it is easily seen that $\|\Phi\|^{\tilde{p}} = 1$ for all $k$ and the proof of Lemma 2.4 shows that $\|\Phi\|^{\tilde{p}} \leq (2 + c)\|F\|^{\tilde{p}}$ for all finite subsets $F$ of $N$, whence $\|\Phi\|^{\tilde{p}} \leq (4 + 2c)\|F\|^{\tilde{p}}$ for all such $F$. Thus $(f_{\alpha})$ is a normalized weakly null sequence in $B'$.

We now need the following technical result:

**Lemma 3.3.** There is an absolute constant $K$ so that for all $n > 1$ and $(F)\in \mathcal{B}$,

$$\sum_{j=1}^{n} \langle a_{j}^{1/2} x_{j} \rangle_{\tilde{p}} \leq K.$$ 

**Proof.** Let $(F)\in \mathcal{B}$, fix $n$ and $l > n$, let $F = F_{l}$, put $k = |F_{l}|$, choose $i$ so that $k = m_{l} - 4^{i}$ and let $(G_{m})_{m=1}^{\infty}$ in $\mathcal{B}_{i}$. Now $k \in \mathcal{B}_{i}(\{m_{l}\})$. Thus adhering to the notation of the proof of Lemma 2.4, and observing that $a_{j}^{1/2} \leq m_{l}^{1/4}$ for $j \leq m_{l}$.

$$\sum_{j \leq m_{l}} \|a_{j}^{1/2} x_{j}\|_{\tilde{p}}^{\tilde{p}} \leq \sum_{j \leq m_{l}} \|a_{j}^{1/2} x_{j}\|_{\tilde{p}}^{\tilde{p}} \leq m_{l}^{1/4} \sum_{j \leq m_{l}} \|x_{j}\|_{\tilde{p}}^{\tilde{p}}.$$

Thus

$$\sum_{j \leq m_{l}} \|a_{j}^{1/2} x_{j}\|_{\tilde{p}}^{\tilde{p}} \leq (1 + \epsilon) \sum_{j \leq m_{l}} \|x_{j}\|_{\tilde{p}}^{\tilde{p}} \leq m_{l}^{1/2} m_{l}.$$ 

Since $\|F_{l}\| \neq \|F_{l}'\|$ if $l \neq l'$, the result follows from $(\ast)$ of Proposition 2.2 if we simply put $K = c^{2} + 2c$.

We are now prepared to complete the proof of Theorem 3.2. Let $N'$ be an infinite subset of $N$, $n$ an integer larger than one, and choose $(F)\in \mathcal{B}$ so that $F < N'$ for all $l$ and $l \geq n$ where $(F) = F_{l}$. It follows from the proof of Lemma 2.3 that $(F_{l}^{1/2}X_{F_{l}})$ is $1 + c$-equivalent to the first $n$ terms of the summing basis in the $|\cdot|^{\tilde{p}}$-norm for all $k \geq n$. Let now scalars $c_{1}, \ldots, c_{n}$ be given with $\sum_{j=1}^{n} c_{j} = 1$ and put $g_{n} = \sum_{j=1}^{n} c_{j} F_{j}^{1/2} X_{F_{j}}$.

It follows by our choice of $(F)\in \mathcal{B}$ that

$$\|g_{n}\|^{\tilde{p}} \leq m_{l}^{1/2} = m_{l}^{1/2}.$$ 

Using the monotonicity of the basis $(f_{\alpha})$ in $|\cdot|^{\tilde{p}}$ and the above-mentioned summing-basis-equivalence,

$$\|f_{\alpha}(\cdot)^{1/2} f_{\alpha}(\cdot)\|^{\tilde{p}} \leq 2\|g_{n}\|^{\tilde{p}} \leq 2 + 2c$$ 

for all $k \geq n$.

Finally let $1 < k < n$. Again using Lemma 2.3 and also Lemma 3.3,

$$\|x_{[1,...,m]}^{g_{n}}\|^{\tilde{p}} \leq 2(1 + c) \sup_{l \leq k} \|a_{l}^{1/2} x_{l}\|_{\tilde{p}} \leq (2 + c)\|F\|^{\tilde{p}} \leq 2(1 + c) + 2K.$$

Thus $|\cdot|^{\tilde{p}} \leq 2(1 + c) + 2K$, so $(F_{l}^{1/2} X_{F_{l}})$ is $2(1 + c) + 2K$-equivalent to the first $n$ terms of the summing basis in $B'$. This completes the proof of Theorem 3.2.

Our final result gives positive results for normalized weakly null sequences in $C(\alpha + 1)$ for small ordinals $\alpha$.

**Theorem 3.4.** Let $\alpha$ be an ordinal, let $(f_{\alpha})$ be a normalized weakly null sequence in $C(\alpha + 1)$ and set $s > 0$.

(a) If $\alpha < \alpha^{s}$, $(f_{\alpha})$ has a subsequence $(f_{\alpha})$ with unconditional constant at most $1 + s$.

(b) If $\alpha = \alpha^{s}$, $(f_{\alpha})$ has a subsequence $(f_{\alpha})$ with unconditional constant at most $2 + s$.

**Proof.** From now on we deal with infinite sequences. We require the following standard

**Perturbation result** (cf. [3]): Let $L < \infty$ and $(f_{\alpha})$ and $(g_{\alpha})$ be infinite sequences in a Banach space with $(f_{\alpha})$ semi-normalized, $(g_{\alpha})$ $K$-unconditional and $\sum_{\alpha < \beta} \|f_{\alpha} - g_{\alpha}\|_{\tilde{p}} < \infty$. Then for every $s > 0$ there exists an $m$ so that $(f_{\alpha}^{m})$ is $(K + s)$-unconditional.

Now let $(f_{\alpha})$ be as in Theorem 3.4. Since $\alpha + 1$ is a countable and $(f_{\alpha})$ tends to zero pointwise, there exists an increasing sequence $(\lambda_{n})$ of positive numbers with $\lambda_{n} \to \infty$ so that $(f_{\alpha}^{m})$ tends to zero pointwise. By passing to a subsequence if necessary we may assume that

$$\sum_{n} \lambda_{n}^{-1} < \infty.$$ 

Since $\alpha + 1$ is a totally disconnected space, we may choose clopen subsets $E_{n}$ of $\alpha + 1$ so that

$$\|f_{\alpha}^{m}(\cdot)\|_{\tilde{p}} \leq \|f_{\alpha}^{m}(\cdot)\|_{\tilde{p}} \leq \|f_{\alpha}^{m}(\cdot)\|_{\tilde{p}}$$ 

for all $n > 1$. Then there is no $\alpha$ in $\alpha + 1$ with $a \in E_{n}$ for infinitely many $\alpha$-$\nu$ since $\lambda_{n} a(\cdot) \to 0$ for such an $n$. Hence

$$X_{n} \to 0 \text{ weakly in } C(\alpha + 1).$$ 

Now set $g_{n} = f_{\alpha}^{m} X_{n}$ for all $n$. Then since $\alpha \notin E_{n}$ implies $|f_{\alpha}^{m}(\cdot)\|_{\tilde{p}} < \lambda_{n}^{-1}$, $\|f_{\alpha}^{m} - g_{n}\|_{\tilde{p}} < \lambda_{n}^{-1}$ for all $n > 1$, so by (1) we have

$$\sum_{n} \|f_{\alpha}^{m} - g_{n}\|_{\tilde{p}} < \infty.$$
Now assume (a) and choose \( M < \infty \) so that \( a \leq a^M \). Since \( a + 1 \) is a dense subset of \( a^M \), we may apply (b) of Lemma 3.1 to choose a subsequence \( (g_n^j) \) of the \( g_n^i \)'s so that no \( M+i+1 \) of the corresponding \( E_n^j \)'s have a common point. Now let \( \varepsilon > 0 \). We may choose continuous functions \( h_n \) on \( a+1 \) and a finite subset \( \mathcal{G} \) of the real numbers so that for all \( n \),

\[
(4) \quad h_n(a+1) \subseteq \mathcal{G}, \quad |h_n|^{-1}(0, \infty) \subseteq E_n^j
\]

and \( |h_n - g_n|^1 < \varepsilon M \).

It follows that for all sequences of \( g \) with only finitely many non-zero terms and any \( \varepsilon \), in \( a+1 \), that

\[
\left| \sum_{j \in \mathcal{G}} q_j (h_j - g_j) (x) \right| \leq \sup_{|g_j|} |X| \leq \sup_{|g_j|} \varepsilon,
\]

where \( X = \{ u \in X : x \in E_n^j \} \). Thus we have

\[
(5) \quad \left| \sum_{j \in \mathcal{G}} q_j (h_j - g_j) (x) \right| \leq \sup_{|g_j|} \varepsilon.
\]

We now appeal to the results of [9], which yield that \( (h_j) \) has a \( (1+\varepsilon)^j \)-unconditional subsequence \( (h_j) \). It then follows from (5), (3) and the standard perturbation result that \( (f_n^j) \) has a \( (1+\varepsilon)^j \)-unconditional subsequence. Since \( (h_j) \) is a semi-normalized and weakly null, it follows from the results of [9] that there exists a subsequence \( (h_j) \) of \( (h_j) \) so that for all \( n \), scalars \( c_1, \ldots, c_n \), and finite set \( \mathcal{F} \) with \( |\mathcal{F}| \leq M \),

\[
(6) \quad \left( \sup_{x \in \mathcal{F}} \left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| \right) \leq (1 + \varepsilon) \left( \sup_{x \in \mathcal{F}} \left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| \right) +
\]

\[
(\mathcal{F})^* \text{ denotes the maximum of } f \text{ and } 0). \text{ But then } (h_j) \text{ is itself } (1+\varepsilon)-\text{unconditional. Indeed, fix } n, \text{ scalars } c_1, \ldots, c_n, \mathcal{F} \text{ a subset of } n \text{ and } a \in a+1 \text{. Then since } h_{n+1} (0, \infty) \subseteq E_n^j \text{ for all } n, \text{ letting } \mathcal{F} = \{ j : 1 \leq n \leq h_j (a) \neq 0 \}, \text{ we have}
\]

\[
\left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| \leq \left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| \leq (1 + \varepsilon) \left( \sup_{x \in \mathcal{F}} \left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| \right);
\]

the arbitrariness of \( x \) thus yields that

\[
\left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| \leq (1 + \varepsilon) \left( \sum_{j \in \mathcal{F}} q_j h_j (x) \right).
\]

We pass now to the proof of Theorem 3.4 (b). The \( E_n^j \)'s chosen in the first part of our argument have the property that \( a^\infty+1 \) belongs to at most finitely many of them. We may thus assume that \( E_n^j \) is a compact subset of \( a^\infty \) for all \( n \). Then we choose simple continuous \( h_n \)'s supported on the \( E_n^j \)'s so that \( \sum |h_n - f_n^j| \leq \infty \) with \( |h_n| \leq 1 \) for all \( n \). By the standard perturbation result, it suffices to show that the \( h_n \)'s have a \( (1+\varepsilon)^i \)-unconditional subsequence for any \( \varepsilon > 0 \). Thanks to the statement containing (6), our proof of Theorem 3.4 (a) yields that for any compact subset \( A \) of \( a^\infty \), any semi-normalized weakly null sequence \( (h_n) \) in \( C (A) \) and any \( \varepsilon > 0 \), there exists a subsequence \( (h_n^j) \) so that for all \( n, c_1, \ldots, c_n \) and \( \mathcal{F} = \{ 1, \ldots, n \} \),

\[
(7) \quad \left( \sup_{x \in \mathcal{F}} \left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| \right) \leq (1 + \varepsilon) \left( \sup_{x \in \mathcal{F}} \left( \sum_{j \in \mathcal{F}} q_j h_j (x) \right) \right).
\]

Now fix \( \varepsilon > 0 \). By a standard result of Besançon and Pečaký [9], we may assume without loss of generality that \( h_n \) is already a \( (1+\varepsilon) \)-monotone basic sequence.

We now choose infinite subsets \( N = M_1, M_2, \ldots \) of \( N \) so that setting \( m_i = \min M_i \) then for all \( i, m_i < m_{i+1} \), \( M_i = M_{i+1} \) and for any set \( A \) in the Boolean ring generated by \( \{ h_j^i : c \neq 0 \) and \( j \leq i \} \), either \( \{ h_j^i : j < i \} \) satisfies the statement containing (7) or \( \sum |h_n| A | A | < \varepsilon \).

Finally, set \( h_j^i = h_n \) for all \( j \). We shall show that the \( h_n \)'s are almost two-unconditional. Fix \( n, \) scalars \( c_1, \ldots, c_n \) and \( \mathcal{F} \) a subset of \( \{ 1, \ldots, n \} \). We may assume without loss of generality that there is an \( \varepsilon \) so that

\[
1 = \left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| = \left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right|.
\]

We shall show that

\[
(8) \quad (2 + 4 \pi + 2 \pi)^i \leq \left( \sum_{j \in \mathcal{F}} q_j h_j (x) \right),
\]

which is sufficient enough to complete the proof.

Let \( k \) be the least integer so that \( h_j (x) \neq 0 \) and set

\[
A = \{ y \in a^\infty : h_j (y) = 0 \text{ for all } j < k \text{ and } h_j (y) = h_k (x) \}.
\]

\( A \) is thus a compact subset of \( a^\infty \) containing \( a \). Now if

\[
(9) \quad \sum_{j \in \mathcal{F}} |h_n| A | A | < \varepsilon,
\]

then

\[
1 = \left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| \leq |h_k h_k (x)| = \sup_{j} q_j x \leq (2 + 4 \pi + 2 \pi) \left( \sum_{j \in \mathcal{F}} q_j h_j (x) \right).
\]

Now suppose (9) does not hold; let \( f = \sum_{j \in \mathcal{F}} q_j h_j \) and \( \mathcal{F} = \sum_{j \in \mathcal{F}} q_j h_j \). If \( k \notin \mathcal{F} \) then

\[
\left| \sum_{j \in \mathcal{F}} q_j h_j (x) \right| = |f (x)| \leq (1 + \varepsilon) |f| A | A |.
\]

\[\[\text{[1] Studia Mathematica 61.1}\]
(by the statement containing (7) and the construction of the \( (h_n') \))

\[
\leq (1 + e) \| g \| \leq 2 (1 + e) \sum_{j=1}^{+\infty} \| h_j \|
\]

by the \((1 + e)-\)monotonicity of the \( h_j \). (In reality, it is only in the very last inequality that the constant \( e \) enters in a crucial way.)

Finally, suppose \( x \in F \). Then if \( \xi h_n(x) \) and \( f(x) \) are of opposite sign, either \( \xi h_n(x) \) or \( f(x) \) is larger than \( \left\| \sum_{j=1}^{+\infty} \xi h_j(x) \right\| \). But

\[
\max \{|\xi h_n(x)|, |f(x)|\} \leq 2 (1 + e) \sum_{j=1}^{+\infty} \xi h_j
\]

by the above argument. Now suppose \( \xi h_n(x) \) and \( f(x) \) are of the same sign; we may suppose both are positive. Then by (7) there exists a \( y \in A \) with \( f(x) \leq (1 + e) g(y) \). By the definition of \( A \),

\[
\sum_{j=1}^{+\infty} \xi h_j(x) = \xi h_n(x) + f(x) \leq \xi h_n(x) + (1 + e) g(y)
\]

\[
= \xi h_n(y) + (1 + e) g(y) \leq (1 + e) (\xi h_n(y) + g(y))
\]

\[
= (1 + e) \sum_{j=1}^{+\infty} \xi h_j(y) \leq (1 + e) \sum_{j=1}^{+\infty} \xi h_j
\]

This completes the proof.

---

**References**


Received March 9, 1976

---

**Czesław Bessaga and Aleksander Pełczyński**

**SELECTED TOPICS**

**IN INFINITE-DIMENSIONAL TOPOLOGY**

**MONOGRAFIE MATEMATYCZNE, Vol. 38**

355 pp., cloth bound

Appearing for the first time in book form are the main results concerning homeomorphic aspects of infinite-dimensional topology, the theory related to general topology, the topology of manifolds, functional analysis and global analysis. Emphasis is placed on the problems of topological classification of linear metric spaces and the techniques of constructing homeomorphisms of concrete metric spaces onto a Hilbert space. The main results concerning topological manifolds modeled on infinite-dimensional linear metric spaces are presented.

The book is primarily addressed to topologists and to functional analysts and may serve as a starting point for research by the graduate student. The book presupposes a knowledge of elementary facts of general topology and functional analysis.


All volumes of MONOGRAFIE MATEMATYCZNE may be ordered at your bookseller or at ARS POLONA, Krakow Przedmieście 6, 00-068 Warszawa, Poland