Abelian ergodic theorems for contraction semigroups

by

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Abstract. Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite measure space and \(L_p(\mu) = L_p(X, \Sigma, \mu)\), \(1 < p < \infty\), the usual Banach space. Let \(\{T(t) : t \geq 0\}\) be a strongly continuous semigroup of \(L_p(\mu)\) contractions for some \(1 < p < \infty\). Let \(R_\lambda\) be the resolvent of \((T(t))\). If \(p > 1\) and \(\{T(t)\}\) is a positive semigroup we show that \(\lim_{t \to \infty} R_\lambda f(x) = f(x)\) a.e. for \(f \in L_p(\mu)\). In case \(p = 1\), we show \(\lambda R_\lambda f(x) \to f(x)\) a.e. for \(f \in L_p(\mu)\) for an arbitrary semigroup of \(L_1(\mu)\) contractions.

Introduction. Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite measure space and \(L_p(\mu) = L_p(X, \Sigma, \mu)\), \(1 \leq p \leq \infty\), the usual Banach space of complex-valued functions. Let \(\{T(t) : t \geq 0\}\) be a strongly continuous semigroup of \(L_p(\mu)\) contractions for some \(1 \leq p < \infty\). This means that (i) \(\|T(t)\|_p = 1\), \(t \geq 0\); (ii) \(T(t+s) = T(t)T(s), s, t \geq 0\); (iii) \(\|T(t)f - T(t)g\| \to 0\) as \(s \to t\) for any \(f, g \in L_p(\mu)\). To simplify the notation we assume \(T(0) = I\); all results obtained hold with appropriate modification if \(T(0) \neq I\).

For \(\lambda > 0\), set

\[
R_\lambda f(x) = \int_0^\infty e^{-\lambda t} T(t)f(x) \, dt
\]

for \(f \in L_p(\mu)\). In case \(p > 1\) we show that

\[\lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x)\] a.e.

(*)

for \(f \in L_p(\mu)\) and \(\{T(t)\}\) a semigroup of positive \(L_p(\mu)\) contractions. This means: \(0 \leq f \in L_p(\mu) \Rightarrow T(t)f \geq 0\) for \(t \geq 0\). If \(p = 1\) we establish (*) for an arbitrary strongly continuous semigroup of \(L_1(\mu)\) contractions. This result extends a theorem in [2], p. 178. We remark that topological ergodic theorems for Abel means of operator semigroups have been studied in [5], [6], [10]. The question of pointwise convergence for Abel means has been considered in [2], [4], [9]. In [8] the author showed that for \(p > 1\) and \(\{T(t)\}\) a semigroup of positive \(L_p(\mu)\) contractions

\[\|f^n\| \leq \|f\|^{p - 1}\|f\|,\]

where \(n\) is a positive integer.
where $f^* = \sup_{\lambda > 0} \lambda R_\lambda f(a)$. This estimate was obtained by use of a dilation theorem appearing in [1]. The author used this estimate to show that $\lim R_\lambda f(a)$ exists and is finite a.e. for $f \in L_2(\mu)$.

Before proceeding further we will clarify the definition of $R_\lambda f(x)$. By Theorem III. 11.17 in [3], given $f \in L_2(\mu)$ there exists a scalar function $T(t) f(x)$, measurable with respect to the usual product measure on $(0, \infty) \times X$, such that (i) for $\lambda > 0$, $T(\lambda t)f(x) = \lambda T(t)f(x)$ and ii) there exists a $\mu$-null set $E(f)$, independent of $\mu$, such that $\mu E(f)$ implies $\int e^{-\lambda t} T(t) f(x) \, dt$, as a function of $x$, is in the equivalence class of $\int e^{-\lambda t} T(t) f(x) \, dt$. The scalar representation $T(t) f(x)$ is uniquely determined up to a set of product measure zero. Defining $R_\lambda f(x) = \int e^{-\lambda t} T(t) f(x) \, dt$, we see that $R_\lambda f(x)$ is in the equivalence class of $\int e^{-\lambda t} T(t) f(x) \, dt$ for every $\lambda > 0$. This justifies the definition of $R_\lambda f(x)$. We note that for $\frac{\lambda}{\eta} E(f)$, $R_\lambda f(x)$ is a continuous function of $\lambda > 0$.

**Preliminary results.**

1. **Lemma.** Let $\{T(t)\}$ be a strongly continuous semigroup of $L_2(\mu)$ contractions for some $1 \leq p < \infty$. Set $\mathcal{A} = \{R_\lambda f(x) = \int e^{-\lambda t} T(t) f(x) \, dt \mid f \in L_2(\mu)\}$. Then $\mathcal{A}$ is dense in $L_2(\mu)$ and $\lim R_\lambda f(x) = f(x)$ a.e. for any $f \in \mathcal{A}$.

Proof. $\mathcal{A}$ is dense in $L_2(\mu)$ since $R_\lambda f(x) \to f(x)$ in norm ([5], p. 321). We now prove that (1) holds for functions in $\mathcal{A}$.

By the resolvent equation, we have

$$\lambda R_\lambda R_\eta f = \eta R_\eta - \lambda R_\lambda f.$$  

Thus $\lim_{\lambda \to \infty} R_\lambda f = 0$ a.e. by the Lebesgue dominated convergence theorem ([3], III. 6.16). Hence

$$\lim_{\lambda \to \infty} R_\lambda f(x) = 0$$  

Thus $\frac{\lambda}{\eta} R_\lambda f(x) \to 0$ as $\lambda \to \infty$. For fixed $\eta > 0$, we have $\lim \lambda R_\lambda R_\eta f(x) = \eta R_\eta f(x)$ a.e. for any $f \in L_2(\mu)$.  

The following lemma is used in proving (1) when $p = 1$. Before stating it we introduce some notation. By Theorem 1 in [7] given a semigroup $\{T(t)\}$ of $L_0(\mu)$ contractions there exists a semigroup $\{T(t)\}$ of positive $L_0(\mu)$ contractions such that $T(\lambda t) f \geq \lambda T(t) f$ a.e. for any $f \in L_2(\mu)$ and $t \geq 0$. We set $S(t) = e^{-\lambda t} T(t)$, $t \geq 0$.

2. **Lemma.** Let $\{T(t)\}$ be a strongly continuous semigroup of $L_2(\mu)$ contractions. For fixed $0 < \eta L_2(\mu)$ set $\lambda = \int S(\lambda t) g \, dt$ and define a measure $m$ by $m(A) = \int h \mu(A).$ Then $\{T(t)\}$ is a strongly continuous semigroup of $L_2(\mu)$ contractions such that $\|T(t) f\|_2 \leq \|f\|_2$ for $f \in L_2(\mu)$.

Proof. We note that $h \in L_2(\mu)$ and $\eta > 0$ a.e. on $X$ since $S(0) = I$ and $\{S(t)\}$ is positive and strongly continuous, hence $P(t) f(x)$ is finite a.e. Henceforth denote $L_2(\mu)$ by $L_2(\mu)$, $1 \leq p < \infty$. Clearly, $\{P(t)\}$ is a semigroup since $\{S(t)\}$ is. To see that $\|T(t) f\|_2 \leq \|f\|_2$, 1 pick $f \in L_2(\mu)$. Then

$$\int |T(t) f|^2 \, \mu(A) = \int |S(t) |f|^2 |f|^2 \, \mu(A) = \int |f|^2 \, \mu(A) = \int |f|^2 \, \mu(A).$$  

Since $\|T(t) f\|_2 = \|S(t) f\|_2 = \|f\|_2$ for $f \in L_2(\mu)$, we see that $\{P(t)\}$ is strongly continuous since $\{S(t)\}$ is. We have $\|T(t) f\|_2 = \|S(t) f\|_2 \leq \|f\|_2$ a.e. From the positivity of $S(t)$ it follows that $\|S(t) f\|_2 = \|f\|_2$ a.e. for $f \in L_2(\mu)$. Hence $\|T(t) f\|_2 = \|f\|_2$ a.e. for $f \in L_2(\mu)$.  

**Main results.**

3. **Theorem.** Let $\{T(t)\}$ be a strongly continuous semigroup of $L_2(\mu)$ contractions for some $1 < p < \infty$. Then $\lim \lambda R_\lambda f(x) = f(x)$ a.e. for every $f \in L_2(\mu)$.

Proof. We have $\lim \lambda R_\lambda f(x) = f(x)$ for $f \in \mathcal{A}$ and $\mathcal{A}$ is dense in $L_2(\mu)$. Also $T(t) f \leq \lambda T(t) f$ a.e. for any $f \in L_2(\mu)$ (see [8]). Employing Banach's convergence theorem ([3], IV. 11.3), we conclude that $\lim \lambda R_\lambda f(x)$ exists and is finite a.e. assuming $\lambda \to \infty$ through some countable subset of $(0, \infty)$, say the set of positive rationals. Since $\lambda R_\lambda f(x)$ depends continuously on $\lambda$ a.e. $x$, we have $\lim \lambda R_\lambda f(x)$ exists and is finite a.e. Since $\lambda R_\lambda f(x) \to f$ in norm, we have $\lim \lambda R_\lambda f(x) = f(x)$ a.e.  

4. **Theorem.** Let $\{T(t)\}$ be a strongly continuous semigroup of $L_2(\mu)$ contractions. Then $\lim \lambda R_\lambda f(x) = f(x)$ a.e. for $f \in L_2(\mu)$.

Proof. By our Lemma 2 and Theorem 1 in [9], we have for $a > 0$ and $f \in L_2(\mu)$

$$m(E(a)) \leq \left(1/a^p \right) \int |f|^p \, \mu(A).$$  

where $f^* = \sup_{a > 0 \mu} \int e^{-\lambda t} T(t) f(x) \, dt$ and $E(a) = \{f^* > a\}$. Hence $f^* < \infty$ a.e. on $X$. Applying Banach's convergence theorem again we get
STUDIA MATHEMATICA, T. LXI. (1977)

Corrigendum and addendum to the paper
"In general, Bernoulli convolutions have independent powers"

Studia Math. 47 (1973), pp. 141-152

by

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Abstract. In this paper we point out an error in our earlier paper with this title and prove that with a slight modification of the definitions the results remain true. Explicitly, we show that for virtually all (in the sense of Baire category) sequences \( \{a_n\} \in \mathbb{Z}^\infty \) the infinito convolution

\[
\nu(x) = \sum_{n=1}^{\infty} \delta(-\pi_n + \delta(a_n))
\]

has the property that the \( \sigma(L^\infty(U), D[\nu]) \) closure of \( [0,1] \) contains all constants in \([-1,1]\).

1. Corrigendum. We are indebted to Professor S. Saks for pointing out to us that Remark 4 on p. 142 of [1] is false. In addition, we have subsequently found an error in the proof of the main theorem of [1]. The error arises in the final paragraph of the proof of Lemma 4 because the sets \( M^n U \) are not necessarily open in the relative topology of \( B \). Nevertheless the main theorem of [1] remains true as stated and an appropriate variant of Remark 4 is obtained when, for example, the set \( F \) is replaced by the set \( F' \) defined by

\[
F' = \{ \{b_n\} : \sum_{n=1}^{\infty} b_n \leq \xi, b_n > 0 \text{ (n = 1, 2, 3, ...)} \}
\]

where \( \xi \) is any irrational number in \([0,1]\).

Since generalizations of the theorems stated in [1] will appear with full proofs in the forthcoming paper of Lin and Saks [2], we refrain from giving the details of the corrections needed in our original arguments. Instead we wish to state and prove a variant of the main theorem of [1] which admits a simple direct proof and which yields a more natural interpretation of the title result of that paper.

2. Addendum. For any sequence \( \{a_n\}_{n=1}^{\infty} \) of real numbers consider the (formal) Bernoulli convolution

\[
\nu(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta(-\pi_n + \delta(a_n)),
\]

References


