On the structure of separable $\mathcal{L}_p$ spaces ($1 < p < \infty$)

by

D. ALSPACH* (Columbus, Ohio), P. ENFLO** (Stanford, Calif.)

and

E. ODELL* (Columbus, Ohio)

Abstract. It is shown that $\mathcal{L}_p$, $(\sum X_n)_p$, $B_p$ and $(\sum I_n)_p$ ($1 < p < \infty$) are primary. The proof for $\mathcal{L}_p$ is then extended to a class of rearrangement invariant function spaces. Also, if $\mathcal{X}$ is a subspace of $(\sum I_n)_p = Z_p$ ($1 < p < \infty$) which contains a subspace $\mathcal{Y}$ isomorphic to $Z_p$ and $e > 0$, then there is a subspace $\mathcal{Z} \subseteq \mathcal{X}$ with $d(\mathcal{Z}, Z_p) < 1 + e$ and a projection $P$ of $\mathcal{Z}$ onto $\mathcal{Z}$ with $\|P\| < 1 + e$.

Introduction. A Banach space $\mathcal{X}$ is said to be primary if whenever $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$ then either $\mathcal{Y}$ or $\mathcal{Z}$ is isomorphic to $\mathcal{X}$. It is known that $c_0$, $l_p$ ($1 < p < \infty$) and $C[0, 1]$ are primary (see [13] and [9]). In the first part of Section 1 of this paper we show that $\mathcal{L}_p$ ($1 < p < \infty$) is primary. The main technique in the proof is a result of Casazza and Lin (Lemma 1.1 of Section 1). In the latter part of Section 1, we employ a similar argument to show that certain other $\mathcal{L}_p$ spaces (namely, $(\sum X_n)_p$, $(\sum I_n)_p$ and $B_p$ (see [14] for the definitions)) are primary.

In Section 2 we turn to the study of the isomorphic structure of subspaces of $(\sum I_n)_p = Z_p$. In particular, we show that if $\mathcal{X}$ is a subspace of $Z_p$ which contains an isomorph of $Z_p$, then for all $e > 0$ there is a subspace $\mathcal{Y}$ of $\mathcal{X}$ with $d(\mathcal{Y}, Z_p) < 1 + e$ and such that there is a projection $P$ of $\mathcal{Z}$ onto $\mathcal{Z}$ with $\|P\| < 1 + e$.

We use standard Banach space notation throughout as may be found, for example, in the book of Lindenstrauss and Tzafriri [10]. By subspace we mean closed linear subspace. If $A \subseteq \mathcal{X}$, by $[A]$ we mean the smallest subspace containing $A$. $\mathcal{X} \sim \mathcal{Y}$ means that $\mathcal{X}$ is isomorphic to $\mathcal{Y}$.

We wish to thank Professor W. B. Johnson for many useful discussions regarding the material contained herein.

* The contribution of the first and third named authors is part of their respective Ph. D. dissertations being prepared at The Ohio State University and The Massachusetts Institute of Technology under the supervision of Professor W. B. Johnson.

** The second named author was supported in part by NSF Grant MT8 74-07230 A01.
1. In this section we will show that \( L_p(\sum X \| X \|_p) \text{ and } B_p \) are primary. The basic technique we use is essentially due to Casazza and Lin [4]. We wish to thank Professor W. B. Johnson for pointing out to us that the following lemma follows easily from the arguments in [4].

**Lemma 1.1.** Let \( (a_n) \) be a bounded unconditional basis for \( X \) with biorthogonal functionals \( (b_n) \). Assume that \( T \) is an operator on \( X \) such that \( (T a_n) \) is a block basis of \( (b_n) \) for some subsequence \( (a_n) \) and \( \| T x \|_X \geq \varepsilon \) for all \( \varepsilon \) and some fixed \( \varepsilon > 0 \). Then the basic sequence \( (X a_n) \) is equivalent to \( (a_n) \) and \( (T x) \) is unconditionally equivalent in \( X \).

Our next lemma follows immediately from a theorem of Gamlen and Gaudet [6].

**Lemma 1.2.** If \( (a_n) \) is the Haar basis for \( L_p \) \( (1 < p < \infty) \) and \( (b_n) \) is a block basis of \( (b_n) \), then either \( [(a_n)] \sim L_p \) or \( [(b_n)] \sim L_p \).

We also wish to recall that if \( (a_n) \) is an unconditional basic sequence in \( L_p \), then there is a constant \( k < \infty \) such that for all \( n \) the non-zero sequences in \( a_n \),

\[
k^{-1} \left( \int \left( \sum |a_n b_n(j)|^{2p} \right)^{1/p} \right)^{1/p} \leq \left( \sum |a_n|^p \right)^{1/p} \leq k \left( \int \left( \sum |a_n b_n(j)|^{2p} \right)^{1/p} \right)^{1/p}
\]

(see [7]).

**Theorem 1.3.** \( L_p \) \( (1 < p < \infty) \) is primary.

**Proof.** It is well known that \( L_p \sim L_p(x) \), where

\[
I_p(h) = \{f \in L_p : f \in L_p \text{ and } II_p(f) = \left( \int \left( \sum |f(j)|^2 \right)^{p/2} \right)^{1/p} < \infty \}.
\]

Let \( (a_n) \) be the Haar basis for \( L_p \). Then \( (a_n) \) is an unconditional basis for \( L_p \), where

\[
h_p = (0, 0, \ldots, 0, b_0, 0, \ldots)
\]

\((b_i)\) stands in the \( i \)th place. For these facts and some related results see [10].

Assume \( L_p (X) = X \oplus Y \) and let \( P_X \) (respectively, \( P_Y \)) denote the projection of \( L_p (X) \) onto \( X \) with kernel \( Y \) (respectively, the projection of \( L_p (Y) \) onto \( Y \) with kernel \( X \)). We shall show that either \( X \) or \( Y \) contains a complemented isomorph of \( B_p \). The fact that \( L_p \) is primary then follows from the well-known decomposition technique of Pelczynski [13].

Since \( h_0 = P_X h_0 + P_Y h_0 \) either \( h_p^* (P_X h_0) \ni h_j \) or \( h_p^* (P_Y h_0) \ni h_j \) for each \( j \) and \( h_0 \) (here \( h_0 \) are the functionals biorthogonal to \( h_0 \)).

Let

\[
I = \{i_0: h_p^* (P_X h_0) \ni h_j \text{ for an infinite number of } j\}
\]

and

\[
J = \{i_0: h_p^* (P_Y h_0) \ni h_j \text{ for an infinite number of } j\}.
\]

By Lemma 1.2, either \( [(h_0)_{i_0}] \sim L_p \) or \( [(h_0)_{j_0}] \sim L_p \). Without loss of generality we assume that \( [(h_0)_{i_0}] \sim L_p \) and enumerate \( I \) as \( (a_n) \),

Since \( (P_X h_0) \) is complemented in \( (P_Y h_0) \) converges weakly to \( 0 \), we may assume (by standard perturbation arguments) that there are integers \( j_0 \) such that \( (P_X h_0) \) is a block of the basis \( (h_n) \) and \( h_n^* (P_X h_0) \ni h_j \). By Lemma 1.1, \( (P_X h_0) \) is equivalent to \( (h_n) \) and \( (P_Y h_0) \) is complemented in \( L_p \). But by (1) and (2), \( (h_n) \) is equivalent to \( (h_n) \) and thus we have shown that \( X \) contains a complemented isomorph of \( L_p \).

The second named author presented a different proof of Theorem 1.3 at the conference "The Geometry of Banach Spaces" at Oberwolfach, 1973. A proof similar to that and an extension to the case \( p = 1 \) has been given by Maurey [11].

Our next theorem shows that certain other \( (\mathcal{L}) \) spaces with a "nice matrix form" are primary. In what follows \( X \) and \( B_p \) \( (1 < p < \infty) \), \( p \neq 2 \) are the \( (\mathcal{L}) \) spaces of Rosenthal [14].

**Theorem 1.4.** \( (\sum X) \) and \( \{X \} \) are primary \( (1 < p < \infty) \), \( p \neq 2 \).

(J. Lindenstrauss has independently obtained this result for \( (\sum) \).

**Proof.** It suffices by duality to prove the theorem for \( p > 2 \) and we shall consider only the case of \( (\sum X) \) (the proof for the other cases are similar and simpler).

We regard \( X \) as \( (\sum X) \), where \( (a_n) \) is a sequence of independent symmetric 3-valued random variables in \( L_p (0, 1) \) such that

\[
\|a_n\|_3 = a_i \text{ for all } a_i > 0.
\]

Thus \( (\sum X) \) is given by \( (a_n) \), where for each \( n \) \( (a_n) \) is a sequence as in (3) above, and

\[
\sum_n \left( \sum_j \sum_k a_{n,j} X_{n,k} \right) = \left( \sum_n \sum_j \sum_k a_{n,j} X_{n,k} \right)^{1/p}.
\]

Let \( (\sum X) \) be \( (\sum X) \) and let \( P_X \) be the projection onto \( X \) with kernel \( Z \) and define \( F \) similarly. As we need only show that either \( X \) or \( Z \) contains a complemented isomorph of \( (\sum X) \),

Let \( (a_n) \) be the functionals biorthogonal to \( (a_n) \) for each \( n \) and set

\[
A_n = (a_n) \text{ for an infinite number of } j,
\]

\[
B_n = (a_n) \text{ for an infinite number of } j.
\]

Then for each \( n \) either

\[
\sum_{a_n} \omega_{a_n}^p = \infty \text{ or } \sum_{a_n} \omega_{a_n}^p = \infty.
\]
Thus, without loss of generality, we may assume that
\[ (5) \quad \sum_{a \in A_n} w_n a_{j}^{2} = \infty \text{ for all } m = I \text{ for some infinite set of integers } I. \]

Let \( \alpha : \mathcal{N} \to \{0, 1, 2, \ldots, n \} \) be a bijection. We claim that \( (P_{\alpha})_{\mathcal{N}} \) is a small perturbation of a block of the basis \((s_m)_{m \in \mathbb{N}}\) for some choice of the \( j \)'s. To see this, let \( m = N \) and set
\[ Q_m = \sum_{a \in A_n} \sum_{j = 1}^{m} a_{j} s_{a_j}. \]

Let \( \varepsilon \) be an arbitrary sequence of positive numbers decreasing to 0. Now for each \( n \) and \( i \), \( P_{\alpha} s_{a_j} \) converges weakly to 0 as \( j \to \infty \), thus, if we let \( s_{(0,0,0)} = s_{1,1,1} \), then there is an integer \( m \), such that \( \| (I - Q_m) P_{\alpha} s_{(0,0,0)} \| < \varepsilon \), and an \( s_{(0,0,0)} \) such that \( \| (Q_m - Q_n) P_{\alpha} s_{(0,0,0)} \| < \varepsilon \). Suppose \( (s_{(0,0,0)}) \) has been chosen. Then there is an integer \( m \) and a \( j = 1 \) such that \( \| (I - Q_m) P_{\alpha} s_{(0,0,0)} \| < \varepsilon \) and \( \| (Q_m - Q_n) P_{\alpha} s_{(0,0,0)} \| < \varepsilon \). Since by (5), \( P_{\alpha} s_{(0,0,0)} \) is bounded away from 0 in norm, a sufficiently small choice of the \( \varepsilon \)'s yields the claim.

The theorem follows by Lemma 1.1 once we observe that \( \| (s_{(0,0,0)})_{n=1}^{\infty} \| = (\sum_{n=0}^{\infty} x_{n})_{n=0}^{\infty} \). This in turn follows from (5), the definition of \( a \), and the following result of Rosenthal [14]: There is a \( K < \infty \) such that if \( (s_{a}) \) is a sequence of 3-valued symmetric independent random variables with
\[ \frac{w_{a_n}}{w_{a_n}^{1/2}} = w_n, w_n > 0 \quad \text{and} \quad \sum_{a \in A_n} w_n a_{j}^{2} = \infty, \]
then
\[ d\left( (s_{a}), X_{\alpha} \right) \leq K. \]

Remarks 1. By a similar argument it can be shown that \( (\sum_{a}^{\infty})_{p} \) is primary, \( 1 \leq r, p < \infty \).

2. We do not know if \( X_{p} \) itself is primary. A simpler version of the above proof yields that, if \( X_{p} = X \otimes Z \), then either \( Y \) or \( Z \) contains a complemented isomorphism of \( X_{p} \).

3. The proof of Theorem 1.3. can be extended to show that, if \( X \) is a reflexive rearrangement invariant function space on \([0,1]\) with indices \( a, \beta, 0 < \beta < a < 1 \), then \( X \) is primary (see [2] or [3] for definitions). Thus, in particular, every reflexive Orlicz space on \([0,1]\) is primary. This extension was the result of a conversation with A. Pełczyński, to whom we are grateful. We sketch the proof below.

From results of Boyd [2] one can easily obtain the following theorem:

**Theorem 1.5.** Let \( T \) be a bounded linear operator on \( L_{p} \) for all \( p, 1 < p < \infty \). If \( X \) is a rearrangement invariant function space with indices \( a, \beta, 0 < \beta < a < 1 \), then \( T \) is continuous on \( X \) (i.e., \( TX = X \) and \( \|T_{X}\| < \infty \)).

By arguments of Mitjagin ([12], pp. 65–91), it can be shown that Theorem 1.3 also holds with \( L_{p} \) replaced by \( L_{q}(l_{q}) \) and \( X \) replaced by \( X(l_{q}) \). Moreover, \( X \) can be shown to be isomorphic to \( X(l_{q}) \).

If we examine the proof of Theorem 1.6, we see that the following results are needed:

(i) If the Haar system, \( (h_{k})_{k=1}^{\infty} \), is an unconditional basis for \( X \) and the corresponding system \( (h_{l})_{l=1}^{\infty} \) is a unconditional basis for \( X(l_{q}) \);

(ii) If \( 1, 2, 3, \ldots = I \cup J \), then either \( h_{k} \) or \( h_{k} \) is isomorphic to \( X \);

(iii) If \( h_{k} \sim X \), then \( h_{k} \sim X \);

(iv) If \( X \sim X \otimes Z \) and \( Y \sim A \otimes B \), then \( X \sim Y \).

The first three of these can be obtained from the corresponding results for \( L_{p} \) and Theorem 1.5. Indeed, consider (ii). An examination of the proof for the result for \( L_{p} \) ([6]) shows that one can construct an operator \( T \) from \( L_{p}(l_{q}) \) to \( L_{q}(l_{q}) \) (say) which is an isomorphism for all \( p, 1 < p < \infty \).

By Theorem 1.5, \( T \) is continuous on \( X \). Let \( P \) be the basic projection from \( L_{p}(l_{q}) \) onto \( L_{q}(l_{q}) \). Then \( T^{*}P \) is continuous on \( X \), by Theorem 1.5, and hence \( T \) is an isomorphism from \( X \) onto \( L_{q}(l_{q}) \).

Finally, (iv) follows from arguments of Mitjagin ([12], p. 95). The techniques used here would have wider application if the following problem has an affirmative solution.

**Problem.** If \( Y \) is isomorphic to a complemented subspace of \( X \) and \( X \) is isomorphic to a complemented subspace of \( Y \), is \( X \) isomorphic to \( Y \)?

2. Let \( Z_{p} = (\sum_{k=1}^{\infty} (1 < p < \infty) \). Our aim in this section is to prove the following theorem.

**Theorem 2.1.** Let \( X \) be a subspace of \( Z_{p} \) which contains a subspace \( X \) isomorphic to \( Z_{p} \). Then for any \( \delta > 0 \) there is a subspace \( Z \subset X \) with \( d(Z, Z_{p}) < 1 + \delta \) and a projection \( P \) of \( Z_{p} \) onto \( Z \) with \( \|P\| \leq 1 + \delta \).

We note that a theorem of Pełczyński shows that \( L_{p} \) possesses a similar property ([33]).

We first introduce the basic notation we shall be using. Let the natural basis of \( Z_{p} \) be given by \( (e_{i})_{i=1}^{\infty} \), where
\[ \left\| \sum_{j} a_{j} e_{j} \right\| = \left( \sum_{j} \left| a_{j} \right|^{p} \right)^{1/p}. \]

Let \( Q_{n} \) be the natural projection onto the first \( n \) Hilbert spaces and for \( E \subset X \) (finite or infinite) let \( P_{E} \) be the projection onto those Hilbert spaces indexed by \( E \). Thus
\[ Q_{n} \left( \sum_{j} a_{j} e_{j} \right) = \sum_{j=1}^{n} a_{j} e_{j}. \]
and

\[ P_{\mathcal{E}} \left( \sum_{j=1}^{N} a_{\mathcal{E}} e_{j} \right) = \sum_{j=1}^{N} a_{\mathcal{E}} e_{j}. \]

Also let \( Q^n = I - Q_n \) be the natural projection onto those Hilbert spaces past the first \( n \).

The idea of the proof of Theorem 2.1 is to construct a sequence of almost disjoint Hilbert subspaces in \( Y \). The following lemma will be very useful. We omit the proof which is quite standard.

**Lemma 2.3.** Let \( (y_\mathcal{E}) \) be a block basis of \( \{e_\mathcal{E}\}_{\mathcal{E}=1}^{\infty} \). Then, if \( (a_\mathcal{E}) \) is a finitely non-zero sequence of scalars, we have

1. If \( p > 2 \), \( \left( \sum_{\mathcal{E}=1}^{\infty} |a_\mathcal{E}|^p |y_\mathcal{E}|^p \right)^{1/p} \leq \left( \sum_{\mathcal{E}=1}^{\infty} |a_\mathcal{E}|^2 \right)^{1/2} \left( \sum_{\mathcal{E}=1}^{\infty} |y_\mathcal{E}|^2 \right)^{1/2}; \)
2. If \( p < 2 \), \( \left( \sum_{\mathcal{E}=1}^{\infty} |a_\mathcal{E}|^2 |y_\mathcal{E}|^2 \right)^{1/2} \leq \left( \sum_{\mathcal{E}=1}^{\infty} |a_\mathcal{E}|^p |y_\mathcal{E}|^p \right)^{1/p}. \)

**Lemma 2.3** provides a sufficient condition for a subspace of \( Z_\mathcal{E} \) to be isometric to \( l_2 \) and well complemented.

**Lemma 2.3.** Let \( (y_\mathcal{E}) \) be a normalized block basis of \( \{e_\mathcal{E}\}_{\mathcal{E}=1}^{\infty} \) such that \( \|P_\mathcal{E}y_\mathcal{E}\| = \lambda_{\mathcal{E}} \) for all \( \mathcal{E} \) and \( i \). Then \( \left( \sum_{\mathcal{E}=1}^{\infty} |a_\mathcal{E}|^2 \right)^{1/2} \) for all scalars \( (a_\mathcal{E}) \) and \( \{y_\mathcal{E}\} \) is norm-1 complemented in \( Z_\mathcal{E} \).

**Proof.** The first assertion was observed by Rosenthal (cf. p. 292 of [14]) so we shall confine ourselves to producing the sequence \( P_\mathcal{E} \).

Let \( P_\mathcal{E}y_\mathcal{E} = y_\mathcal{E} \) so that \( \|y_\mathcal{E}\| = \lambda_{\mathcal{E}} \). For each \( \mathcal{E} \) and \( i \), define \( f_\mathcal{E} \) by \( f_\mathcal{E}(y_\mathcal{E}) = 1 \). Then \( \|f_\mathcal{E}\| = 1 \) for \( \lambda_{\mathcal{E}} \).

**Remark.** It is possible using a slightly different argument to take the \( y_\mathcal{E} \) as blocks of constant coefficient and constant length. This can be accomplished by taking long averages in order to "kill the \( \mathcal{E} \) part" of the \( a_\mathcal{E} \).

Our next lemma asserts that every Hilbert subspace of \( Z_\mathcal{E} \) must contain a subspace which "dies off uniformly". We wish to thank E. Dor for connecting an error in the proof of this lemma. If \( Q : X \to Z \) and \( Y \) is a subspace of \( X \) by \( Q \), we mean the operator obtained by restricting \( Q \) to \( Y \).

**Lemma 2.5.** If \( X \) is a subspace of \( Z_\mathcal{E} \) \( (1 < p < \infty, p \neq 2) \) which is isometric to \( l_2 \), then there is a subspace \( Y \subseteq X \) which \( \lim \|Q^n y_i\| = 0 \).

**Proof.** If \( 1 < p < 2 \), then we may take \( Y = X \) (cf. [14]) so we restrict ourselves to the case \( p > 2 \).

**Claim.** For every \( \delta > 0 \) there is an \( \epsilon > 0 \) such that, if \( Y \subseteq Z_\mathcal{E} \) and \( d(\mathcal{E}, l_2) \leq 1 + \epsilon \), then for some integer \( n \), \( Q^n y_i \) differs from \( y_i \) by less than \( \delta \).

If not, then for some fixed \( \delta > 0 \) and any \( \epsilon > 0 \) we can find a normalized block basis \( (y_\mathcal{E}) \) of \( (a_\mathcal{E}) \) with \( \|Q^n y_i\| \geq \delta \) for all \( i \) and such that \( (y_\mathcal{E}) = (1 + \epsilon) \)-equivalent to the unit vector basis of \( l_2 \). By passing to a subsequence of \( (y_\mathcal{E}) \), we obtain disjoint finite subsets \( Z_{\mathcal{E}, n} \subseteq \mathcal{E} \) so that \( \|P_{Z_{\mathcal{E}, n}} y_i\| \geq \delta \) and \( \|P_{Z_{\mathcal{E}, n}} y_i\| = 0 \) for \( i > 1 \). Given \( a > 0 \), since \( \delta < \|P_{Z_{\mathcal{E}, n}} y_i\| \leq 1 \), we may...
assume (by passing to a subsequence) that \(||P_{F_n}y_n|| - \eta| < \alpha\) for all \(i\) and some \(\eta \geq \delta\). Also for some \(\delta < \eta\), \(||P_{F_n}y_n|| < \alpha\). Indeed, the set \(\{k : ||P_{F_n}y_n|| \geq \alpha\text{ for all } i > k\}\) is finite by the disjointness of the \(E_n\) and the fact that \(|y_n|| = 1\) for all \(n\).

For simplicity we thus assume that we have \(y_n, y_n\) and disjoint finite sets \(E, F \subseteq \mathbb{N}\) so that

\[
||P_{E}y_n|| = ||P_{F}y_n|| = \eta \geq \delta
\]

and

\[
||P_{F}y_n|| = ||P_{E}y_n|| = 0.
\]

Then, by Lemma 2.2,

\[
||y_n + y_n||^p \leq ||P_{E}y_n + P_{F}y_n||^p + ||I - P_F||y_n + (I - P_E)y_n||^p
\]

\[
\leq (\eta^p + \eta^p) + ((1 - \eta^p)^{p \beta} + (1 - \eta^p)^{p \alpha})
\]

\[
= 2\eta^p - (2^{\alpha + \beta} - 2)\eta^p < 2\eta^p - (2^{\alpha + \beta} - 2)\eta^p.
\]

But this contradicts the fact that \((y_n)\) is \((1+\epsilon)\)-equivalent to the unit vector basis of \(l_2\) (provided \(\epsilon\) is taken sufficiently small depending upon \(\delta\)) and the claim is proved.

Using the claim and Lemma 3.4 repeatedly, we can find vectors \((y_j) \subseteq X\) and integers \(n_j \to \infty\) so that \((y_j)\) is \(2\)-equivalent to the unit vector basis of \(l_2\) and \(|Q^{c}y_j| < 2^{-j}\) for all \(j\) and \(i\). Assuming without loss of generality that \((y_j)\) is a block basis of \((e_{n_j})\), we see that if \(y = \sum a_j y_j\) then by Lemma 2.2

\[
||Q^{c}y|| = \left\| \sum_j a_j Q^{c}y_j \right\| < \left( \sum |a_j|^2 \right)^{1/2} \leq 2^{-1/2} ||y||.
\]

\textbf{Proof of Theorem 2.1.} We shall construct a sequence of "almost disjoint" Hilbert subspaces of \(X\). First assume \(p > 2\) and let \(\epsilon > 0\). By the hypothesis on \(X\), there are \(K < \infty\) and subspaces \(Y_n \subseteq X\) such that \(d(X, Y_n) \leq K\) for all \(n\) and, if \(y_n \in Y_n\), then

\[
K^{-1} \left( \sum_{j} ||y_j||^2 \right)^{1/p} \leq ||y_n|| \leq K \left( \sum_{j} ||y_j||^2 \right)^{1/p}.
\]

Also we have

\[
\text{(3)} \quad \text{For all integers } N \text{ and } \epsilon > 0 \text{ there is an integer } n_0 \text{ such that, if } n \geq n_0, \text{ then } ||Q^{c}y|| < \epsilon ||y|| \text{ for all } y \in X_n.
\]

Indeed, if (4) is false, there are \(y_n \in X_n\), \(n_0 \to \infty\) with \(|y_n|| = 1\) and \(|Q^{c}y_n|| > \epsilon\) for all \(m\) and some fixed \(N\). By (3), \((y_n)\) is equivalent to the unit vector basis of \(l_2\), but \(|Q^{c}y_n|| > \epsilon\) implies that \((y_n)\) is equivalent to the unit vector basis of \(l_2\), a contradiction.

Let \(\epsilon > 0\) be arbitrary. Using (4) and Lemma 2.5, we can inductively construct integers \(m_1 \to \infty\), subspaces \(X_1 \subseteq X_{m_1}\) and disjoint finite subsets \(E_1 \subseteq \mathbb{N}\) with \((-E_1)\) the complement of \(E_1\) such that

\[
||P_{E_1}y|| < \epsilon ||y|| \text{ for all } y \in X_1.
\]

For each \(i\) choose unit vectors \((x_{n_i})_{n_i=1}^{\infty} \subseteq X_1\) so that

\[
||P_{E_1}x_{n_i} - y|| < \epsilon_1 2^{-i}.
\]

where \((y_0)_{0=1}^{\infty} \subseteq ((e_{n_i})_{n_i=1}^{\infty} : i < K)\) is a block basis of \((e_{n_i})\) satisfying

\[
||P_{E_1}y_0|| = \lambda_0 \quad \text{for } k \in E_1
\]

\((\lambda_0)\) is independent of \(j\).

By Lemma 2.3,

\[
||\sum_{j} a_j y_j|| = \left( \sum_{j} |a_j|^2 ||y_j||^2 \right)^{1/2}
\]

and \((||y_j||_{E_1}^{2^{-i}})\) is norm-1 complemented in \(((e_{n_i})_{n_i=1}^{\infty} : i < K)\). Thus \((||y_0||_{E_1}^{2^{-i}})\) is isometric to \(Z_0\) and norm-1 complemented in \(Z_0\).

By standard perturbation arguments, the proof will be completed if we show that the operator \(T:\((a_{n_l})_{n_l=1}^{\infty} := (||y_0||_{E_1}^{2^{-i}})\) given by \(T x_0 = y_0\) satisfies \(||T|| = ||T^{-1}|| \leq 1 + \epsilon\) (provided the \(\epsilon_i\)'s are taken sufficiently small).

By (6), it suffices to show that the operator \(S_0:\((a_{n_l})_{n_l=1}^{\infty} := ((P_{E_1}x_{n_l})_{n_l=1}^{\infty})\)

defined by \(S_0 x_0 = P_{E_1}x_0\) satisfies

\[
||S_0|| \leq \left( 1 + \delta \right)^{1/2}\text{ if the } \epsilon_i\text{'s are chosen sufficiently small.}
\]

Let \((a_{n_l})_{n_l=1}^{\infty}\) be a finitely non-zero sequence of scalars. Then, if \(x = \sum_{j} a_j x_j\),

\[
||S_0 x|| = \left( \sum_{j} P_{E_1} \left( \sum a_j x_j \right) \right)
\]

\[
\leq \left( \sum_{j} \sum a_j x_j \right) + \left( \sum P_{-E_1} \left( \sum a_j x_j \right) \right)
\]

\[
\leq ||x|| + \sum_{j} \epsilon_j \left( \sum a_j x_j \right)
\]

\[
\leq ||x|| + \sum_{j} \epsilon_j K ||x|| = \left( 1 + K \sum \epsilon_j \right) ||x||.
\]

Here we have used (3) and (5).

Similarly,

\[
||x|| = \left( \sum_{j} \sum a_j x_j \right)
\]

\[
\leq \left( \sum P_{E_1} \left( \sum a_j x_j \right) \right) + \left( \sum P_{-E_1} \left( \sum a_j x_j \right) \right)
\]

\[
\leq ||x|| + \sum_{j} \epsilon_j K ||x||.
\]
or

\[(1 - K \sum \epsilon_i) ||w|| \leq ||w||.\]

(7) follows by taking \((\epsilon_i)\) small enough to ensure that

\[1 - K \sum \epsilon_i \leq \frac{1}{1 + K \sum \epsilon_i} = 1 - \delta + \delta^2,\]

and this completes the case \(p > 2\).

The case \(p < 2\) may be proved in a similar fashion once we have established the following

**Lemma 2.6.** Let \(X\) be a subspace of \(Z_p\) (1 \(< p \leq 2\)) which is isomorphic to \(Z_p\). Then for every \(n \epsilon \mathbb{N}\) and \(\epsilon > 0\) there is a subspace \(W \cong Y, W \sim_l l_2\) such that

\[||Q_n w|| \leq \epsilon ||w|| \quad \text{for all } w \in W.\]

Proof. Let \(Y = \left\{ (y_i) \mid i > 1\right\}\), where \((y_i) \epsilon l_2\), is \(K\)-equivalent to the unit vector basis of \(l_2\) for each \(i\), and, if \(w_i = \left[ (y_i) \epsilon l_2\right]\), then

\[K^{-1} \left( \sum ||y_i||^2 \right)^{1/2} \leq \left( \sum ||y_i||^2 \right)^{1/2} \leq \left( \sum ||y_i||^2 \right)^{1/2}.\]

By passing to subsequences (using a diagonal process), we may assume that \((y_i) \epsilon l_2\), is a block basis of \((\epsilon_i)\).

To prove the lemma we need only show that for all integers \(m\) and \(\delta > 0\) there is a normalized block basis \((v_i)\) of \((y_i)\) which is equivalent to the unit vector basis of \(l_2\) and such that

\[||Q_n v_i|| \leq \delta \quad \text{for all } i.\]

Indeed, if this is true, then by passing to a subsequence we may assume \((Q_m v_i ||Q_n v_i||^{-1})\) is \(K\)-equivalent to the unit vector basis of \(l_2\) (here we are using that \(p < 2\) (cf. [14])). Then by Lemma 2.2,

\[||Q_n \left( \sum \epsilon_i v_i \right)|| = \left( \sum ||Q_n v_i||^2 \right)^{1/2} \leq 2 \left( \sum ||Q_n v_i||^2 \right)^{1/2} \leq 2 \left( \sum ||v_i||^2 \right)^{1/2} \leq 2 \left( \sum ||v_i||^2 \right)^{1/2} = 2 \left( \sum ||Q_n v_i||^2 \right)^{1/2},\]

which proves (8).

Thus let \(n\) and \(\epsilon > 0\) be arbitrary and assume that \(||Q_n y_i|| \geq \delta\) for all \(i\) and \(j > N_\epsilon\). We next observe that there is an \(\eta > 0\) such that for all \(m\) there is an \(i\) with

\[||Q_m^* y_i|| \geq \eta \quad \text{for an infinite number of } j.\]

If not, then for all \(\eta > 0\) there is an \(m\) such that for all \(j\)

\[||Q_m^* y_i|| \leq \eta \quad \text{for all but a finite number of } j.\]

Thus there are \((y_{i,j})\) and \(m_{i,j} \uparrow \infty\) so that for all \(i,\)

\[||Q_m^* y_{i,j}|| < 2^{-i} \quad \text{for all } j.\]

But then, by a result of Araszy and Lindenstrauss (proof of Theorem 1 of [1]), a subsequence of \((y_{i,j})\) is equivalent to the unit vector basis of \(l_2\), contradicting (9). Thus by relabeling \(y_i\)’s necessary, we may assume that we disjoint sets \(E_0 \subseteq X\) and an \(\eta > 0\) such that

\[|E_i| Y_i > \eta \quad \text{for all } i \text{ and } j.\]

We now employ an averaging argument to produce the desired sequence \((w_i)\). Let \(n\) and \(\delta > 0\) be given. For a fixed integer \(k\) (to be chosen below) and arbitrary \(j\), let \(x_j = \sum_{i=1}^{k} y_{i,j} = ||y_{i,j}||^{-1} y_{i,j}\). Since \(y_{i,j} \epsilon \{y_{i,j} \epsilon l_2\}^m\), \((w_i)\) is equivalent to the unit vector basis of \(l_2\). We shall show that, if \(k\) is taken sufficiently large, then \(||Q_n w_i|| < \delta\). For any \(j\),

\[||Q_n w_i|| = \left( \sum_{i=1}^{k} ||Q_n y_{i,j}||^2 \right)^{1/2} < \epsilon \left( \sum_{i=1}^{k} ||Q_n y_{i,j}||^2 \right)^{1/2} \leq \epsilon ||y||^{1/2}.\]

(Here \(\epsilon\) is a constant depending only on \(d(Q_n Z_p, l_2)\).) Since the \(E_i\)’s are disjoint, Lemma 2 of [8] yields

\[||y_i|| \geq \left( \sum_{i=1}^{k} ||F_{E_i} y_i||^2 \right)^{1/2} > \eta ||y||^{1/2}.\]

Thus \(||Q_n y_i|| \geq \eta^{-1} k^{-1/2} \epsilon \eta^{1/2}\), which is turn smaller than \(\delta\) if \(k\) is sufficiently large. This completes the proof of Lemma 2.6 and Theorem 2.1. \(\blacksquare\)

**Remarks and Questions.** 1. The third named author has recently shown that, if \(X\) is a complemented subspace of \(Z_2\), then \(X\) is isomorphic to one of the four spaces: \(l_1, l_1 \oplus l_1, \text{ or } Z_2\). This result was obtained in [16] under the assumption that \(X\) has an unconditional basis.

2. G. Schechtman has proved that there are no infinite types of distinct isomorphic types of \(L_p\) subspaces of \(Z_p\) (\(p > 2\)) [17].

3. If \(X\) is a subspace of \(Z_2\) (\(p > 2\)) which does not contain an isomorphic to \(Z_2\), is \(X\) isomorphic to a subspace of \(l_1 \oplus l_1\)? If the answer is yes, does the same result hold for subspaces of \(L_p\)?

References


A general result on the equivalence between derivation of integrals and weak inequalities for the Hardy–Littlewood maximal operator

by

IRENEO PERAL ALONSO (Madrid)

Abstract. In this paper we consider integrals of functions belonging to \( p \)-classes, and their differentiation properties with respect to a translation invariant \((B^\infty)\) differentiation basis. We prove that the differentiation of certain integrals is equivalent to a certain property of weak type for the maximal function of Hardy–Littlewood, which is associated to the basis. In a sense, this is a sharp result (see Peral [21]).

Introduction. We consider for each \( x \in \mathbb{R}^n \), a family of open bounded sets \( \mathcal{B}(x) \) such that each \( B \in \mathcal{B}(x) \) verifies:

(i) \( x \in B \); 

(ii) there is a sequence \( \{B_k\}_{k \in \mathbb{N}} \subseteq \mathcal{B}(x) \) such that \( \delta(B_k) \to 0 \) as \( k \to \infty \) (\( \delta(B_k) \) stands for the diameter of \( B_k \)).

If these conditions are satisfied, we say that \( \mathcal{B}(x) \) contracts to \( x \), and that \( \mathcal{B} = \bigcup_{x \in \mathbb{R}^n} \mathcal{B}(x) \) is a differentiation basis in \( \mathbb{R}^n \).

\( \mathcal{B} \) is a Besovmann–Feller \((B^\infty)\) basis, if for each \( B \in \mathcal{B} \) with \( y \in B \), we have \( B \in \mathcal{B}(y) \).

A differentiation basis \( \mathcal{B} \) is translation invariant, if each translation of a set \( B \in \mathcal{B} \) belongs also to \( \mathcal{B} \).

We denote by \( \mathcal{B}_r \) and \( \mathcal{B}_r(x) \) the elements in \( \mathcal{B} \) and \( \mathcal{B}(x) \) with a diameter less than \( r \).

If \( B \) is a measurable set, then \( |B| \) will be its measure.

Let \( f \) be a locally integrable function on \( \mathbb{R}^n \), i.e., \( f \in L^1_{loc}(\mathbb{R}^n) \); we define the upper and lower derivatives of the integral of \( f \) with respect to \( \mathcal{B} \) by:

\[
D^+ \left( \int f; x \right) = \sup_{k \to \infty} \left( \limsup_{k} \frac{1}{|B_k|} \int_{B_k} f(y) \, dy : B_k \subseteq \mathcal{B}(x) \right),
\]

\[
D^- \left( \int f; x \right) = \inf_{k \to \infty} \left( \liminf_{k} \frac{1}{|B_k|} \int_{B_k} f(y) \, dy : B_k \subseteq \mathcal{B}(x) \right).
\]