Weak star and bounded weak star continuity of Banach algebra products

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1. Introduction. A variety of common Banach algebras are dual Banach spaces. For example, a well-known result of Sakai (see [20]) states that von Neumann algebras are characterized, among all $C^*$-algebras, by this property. In Sections 6 through 11 of this paper we consider several instances of this phenomenon, notably, convolution measure algebras, $L^p$-spaces, $H^\infty$ spaces of plane domains, operator algebras in Hilbert space and second duals of Banach algebras under Arens products.

The question we consider in such a Banach algebra is whether the product is continuous for the weak-$^*$ convergence or for the bounded weak-$^*$ convergence, in each variable, or jointly.

The natural set up for this problem is considered in the first part of the paper, where we study the following situation. Let $A$ be a (real or complex) Banach space with dual $A' = A^*$, so that each $b \times B$ acts on $A$ as a linear functional. Suppose that a second action is defined making

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each \( b \in B \) act on \( A \) as an operator \( A \rightarrow A \). Under these hypotheses, a product \( bb' \) can be defined on \( B \) by just taking the composition of the maps \( b \) and \( b' \), where \( b \) is considered as an operator and \( b' \) as a functional: \( A \rightarrow A \rightarrow B \) (or \( C \) as the case may be). Of course such a product need not be associative in general. When it is, it makes a Banach algebra of \( B \). In all the examples mentioned above, the Banach algebra products are obtained in this way.

In Sections 3, 4 we determine some relations between continuity properties of the product and operator properties of the action of \( B \) on \( A \).

We use the notations of [7] and [4]. In particular, if \( a \in A, b \in A^* \), \((a, b)\) is the duality map. For \( X, Y \) Banach spaces, \( L(X, Y) \) is the Banach space of all bounded linear operators \( T : X \rightarrow Y \), and \( T^* : Y^* \rightarrow X^* \) is the adjoint of \( T \). The sequence spaces are denoted, as usual, by \( c_0 \), \( c \), and their non-separable analogues are denoted by \( c_0(S) \), \( c(S) \), \( S \) an arbitrary set. Inner products in Hilbert spaces are denoted by \((x|y)\).

Finally, the weak* topology on a dual space is denoted by \( w^* \) and the bounded weak* topology by \( w_0^* \). We will say that \( b_{m} \rightarrow 0 \) for the bounded weak* convergence (or for \( w_0^* \)) when \( b_{m} \rightarrow 0 \) weak* and moreover \( [b_{m}] \) is bounded. (We use the symbol \( w_0^* \) rather than the more traditional \( b \)-convergence to conform with the general set up of [12].)

Some of these results were announced in [10]. We received encouragement or comments from Felix Ruzstalio, Ana Roth, and Leo A. Rubel, and want to express our gratitude.

2. Bilinear maps. The notations and abbreviations introduced here will be used throughout.

We shall denote by \( A \) an arbitrary Banach space and by \( B \) its dual \( B = A^* \); \( a, a', ... \) will be typical elements of \( A \) and \( b, b', ... \) typical elements of \( B \). We shall consider bilinear bounded transformations \( F : B \times A \rightarrow A \). We think of them as "mixed products" and in fact the notation \( F(b, a) = b \cdot a \), emphasizing this attitude, will be used. We can dualize \( b \rightarrow a \) to obtain a bilinear map \( B \times B \rightarrow B \). More precisely,

2.1. Definition. The product associated to \( (b, a) \rightarrow b \cdot a \) is the bilinear transformation \( (b, b') \rightarrow b \cdot b' \) from \( B \times B \) into \( B \) defined by \((a, b') = (b \cdot a, b')\).

It is clear that \( b \cdot b' \) is also bounded and bilinear. A bounded bilinear map \( B \times B \rightarrow B \) arising in this fashion will be called a \( \cdot \cdot \cdot \) product.

Consider the family of all operators \( T : B \rightarrow A \) of the form \( b \rightarrow b \cdot a \), where \( a \) varies in \( A \). We define the properties \( (w^*, w), (r), (w^*, n), (w_0^*, w), (w_0^*, n) \) for \( \cdot \cdot \) by requiring that all \( T \) have the corresponding operator property described below:

- \( (w^*, w) \) weak* weak continuous;
- \( (r) \) the range of the transpose \( T^* : B \rightarrow A \) is contained in \( A \);
- \( (w^*, n) \) \( w_0^* \) norm continuous.

For \( \cdot \cdot \cdot \cdot \) we need a new definition.

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\( (w^*, w) \) weakly compact;

\( (k) \) compact;

\( (f) \) finite rank.

In order to clarify the mutual relations between various types of continuity of bilinear maps, we shall introduce a list of abbreviations. Let \( Z \) be any (real or complex) linear topological space and let \( F : B \times B \rightarrow Z \) be a bilinear transformation. Consider the following continuity properties of \( F \):

- \( \{ F \} \) \( [a_0, y_0] \rightarrow F(a_0, y_0) \) when \( x \rightarrow a_0 \) and \( y \rightarrow y_0 \);
- \( [F] \) has \( [F] \{ a_0, y_0 \} \) for all \( (a_0, y_0) \in X \times Y \);
- \( [F] \) for each \( a_0, y_0 \), \( F(x, y) \rightarrow F(a_0, y_0) \) when \( x \rightarrow a_0 \);
- \( [F] \) for each \( a_0, y_0 \), \( F(a_0, y) \rightarrow F(a_0, y_0) \) when \( y \rightarrow y_0 \).

where, depending on the case, \( x \rightarrow a_0 \) and \( y \rightarrow y_0 \) will stand for weak* or bounded weak* convergence.

2.2. Proposition. For the weak* convergence, \( [F] \{ 0, 0 \} \) and \( [F] \) are equivalent. For the bounded weak* convergence, \( [F] \{ 0, 0 \} \rightarrow [L] \rightarrow [F] \) and none of the properties on the left is redundant in general.

It is perhaps somewhat disappointing that \( [F] \{ 0, 0 \} \) is not equivalent to \( [F] \). The proof of the equivalence in Proposition 2.3 is elementary. The redundant statement will be easily established after some examples are introduced (see Section 5 below).

In the sequel, properties in square brackets refer to continuity properties of a \( \cdot \cdot \cdot \cdot \) product in the convergence alluded to.

3. The \( w_0^* \)-convergence.

(3.1) Theorem. Let \( b \cdot a \) be the product associated to \( b \cdot a \). Then for the weak* convergence:

- \( (3.1.1) \) \( \{ F \} \) always holds;
- \( (3.1.2) \) \( (r) \) and \( (w^*, w) \) are equivalent;
- \( (3.1.3) \) \( [F] \{ 0, 0 \} \), \( [F] \), \( (f) \rightarrow [L] \), and \( (f) \rightarrow (w^*, w) \) are equivalent;
- \( (3.1.4) \) \( [L] \) holds if \( A \) is reflexive; moreover \( [L] \) holds for all \( \cdot \cdot \cdot \cdot \) products if and only if \( A \) is reflexive.

Proof. (3.1.1) follows trivially from the formula \( \langle a_0, b_0 \cdot c \rangle = \langle b_0 \cdot a_0, c \rangle \).

The equivalence between \( [L] \) and \( (w^*, w) \) is proved as follows: assume that \( b_0 = 0 \) then \( b_0 \cdot c = 0 \) if \( a_0, b_0, c \cdot 0 \rightarrow 0 \) if \( b_0 \cdot a_0 = 0 \).

The equivalence between \( [L] \) and \( (r) \) is proved as follows. First, let \( T : B \rightarrow A \) be the operator \( T(b) = b \cdot a_0 \) then \( T(b) \rightarrow (w^*, w) \rightarrow (w^*, n) \rightarrow (a_0, b) \rightarrow a_0, b \cdot a_0 \rightarrow 0 \) so that \( [L] \) holds iff \( T^* \) is a weak* continuous functional on \( B \), i.e., an element of \( A \). This concludes the proof of (3.1.2). We want to prove now that \( [F] \{ 0, 0 \} \) implies \( (f) \). Assume that \( [F] \{ 0, 0 \} \) holds and that \( \dim (b \rightarrow a_0) \sim \infty \) for a suitable \( a_0 \in A \). Consider the \( w^*\)-

neighborhood of \(0_B\) defined by \(V = \{b \in B; \left< a, b \right> < 1\}\). Let now \(U, W\) be arbitrary weak*-neighborhoods of \(0_B\). We are going to show that there are elements \(b_k \in U, c_k \in W\) such that \(b_k \rightarrow c_k \rightarrow V\). First, there is a finite family \(a_1, a_2, \ldots, a_n\) in \(A\) such that \(\left< a_j, b \right> = 0, j = 1, 2, \ldots, n\), implies \(b \rightarrow U\) and a finite family, which we can call \(a_1, a_2, \ldots, a_n, a_{n+1}\) in \(A\) such that \(\left< a_j, b \right> = 0, j = m + 1, \ldots, n\), implies \(b \rightarrow W\). Consider now the subspace \(S = \{b \in B; \left< a_j, b \right> = 0, 1 \leq j \leq n\}\). Clearly, \(S \subset U \cap W\) and \(S\) has finite codimension in \(B\), and so dim(\(S^\perp = \left< a_j \right>\)) = \(\infty\). Hence, there is a \(b_k \in S\) such that \(\left< b_k, a_j \right> \neq 0\) does not belong to the linear span of \(a_1, a_2, \ldots, a_n\). Therefore, by Hahn–Banach, there is a \(c_k \in B\) such that \(\left< a_j, c_k \right> = 0, 1 \leq j \leq n\) and \(\left< b_k, a_j \right> = \left< c_k, a_j \right> = 1\). Thus \(b_k, c_k \in S\) and a fortiori, \(b_k, c_k \in W\) while \(b_k \rightarrow c_k \rightarrow V\). This proves that \([J, (0, 0)]\) implies \([f]\). In order to prove that \([J, (0, 0)]\) implies \([L]\) we shall need:

(3.2) For an arbitrary Banach space \(X\), an element \(x^* \in X^*\) belongs to \(X^{\text{iff}}\) for each net \(x^*_\alpha \rightarrow x^*\), there is \(\beta\) with \(\left< x^*_\alpha, \beta^* \right> \neq 0, \beta \geq \beta_0\) bounded.

The "only if" part of (3.2) is trivial. Assume now that \((x^*_\alpha)\) is a standard argument implies to show that for each finite set \(F = \{x_1, \ldots, x_n\}\) there is \(x^* \in X^*\) such that \(\left< x^*, x_\alpha \right> = 0, 1 \leq j \leq k\) and \(\left< x^*, x^*_\alpha \right> = k\). Then \(x^*_\alpha \rightarrow x^*\) as \(F\) increases while \(\left< x^*_\alpha, \beta^* \right>\) is unbounded for each tail \(F \supset F'\).

Assume \([J, (0, 0)]\) holds for \(b \rightarrow c\). Then \(f\) holds and so for each \(a \in A\) there are linearly independent \(a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in B\) such that \(b_1 \rightarrow a_1, \ldots, b_m \rightarrow a_m\). We will use now (3.2) to show that each \(b_j \rightarrow A\). Fix \(j\) with \(1 \leq j \leq m\). Assume that \(b_j \rightarrow A\). If \(\left< b_j, b_j^* \right> = 0\), then \(\left< b_j, a_j \right> \neq 0\). But then \(\left< a_1, b_j \rightarrow a_1 \right> = \left< b_j, a_j \right> = \left< \sum (b_j, b_j^*), a_j \right> = \left< \sum (b_j, b_j^*), a_j \right> = \left< b_j, a_j \right> = 0\), which contradicts \([J, (0, 0)]\). Thus \(b_j \rightarrow A\) for each \(j = 1, \ldots, m\), and so relabelling \(b_j = a_j\), we have \(b \rightarrow a = \sum (b_j, b_j^*) a_j \rightarrow (b_j, b_j^*) a_j \rightarrow (b_j, b_j^*) (a_j, a_j) = (0, 0)\), clearly \([L]\) follows.

Now the fact that \([J, (0, 0)]\) implies \([J]\) follows from this and (3.1.i) using Proposition (2.1). If \(A\) is reflexive, \((r)\) holds trivially and according to (3.1.i), \([L]\) follows. The second half of (3.1.iv) will follow from Example 5.1. This completes the proof of 3.1.

Remark. The equivalent conditions in (3.1.vii) obviously imply the equivalent conditions in (3.1.ii). For the fact that the converse is false in general, see Section 5.

4. The weak* convergence.

4.1. Theorem. Let \(b \rightarrow c\) be the product associated to \(b \rightarrow a\). Then for the bounded weak* convergence:

(4.1.i) \([J, (0, 0)]\) is equivalent to \((k)\);

(4.1.ii) \([J, (0, 0)]\) (for all \(c\), \([J, (0, 0)] + [L]\)) (for all \(c\), \([J, (0, 0)] + [L]\)) and \([L] + [L] \rightarrow [L]\) are equivalent;

(4.1.iii) \([L] \implies (bc)\).

Proof. We begin with \((k) \implies (J, (0, 0))\), so let us assume that \((k)\) holds and let \(b_k \rightarrow a_k, c_k \rightarrow c, \|b\| \leq 1, \|c\| \leq 1\). Consider the compact subset \(K \subset A\) defined by \(K = \text{closure}(b \rightarrow a_k, b \rightarrow B, \|b\| \leq 1)\). From the fact that \(c_k \rightarrow 0\) pointwise on \(K\) and that \(c_k\) are norm bounded, and hence equicontinuous, we conclude that \(c_k \rightarrow 0\) uniformly on \(K\). Then, \(\left< b_k, a_k \rightarrow c_k \rightarrow 0\right.\) uniformly on \(\beta\) as \(\gamma\) increases; hence \(b_k \rightarrow a_k \rightarrow c\).

Conversely, assume that \([J, (0, 0)]\) holds. Suppose there is an element \(a \in A\) with \(b \rightarrow b \rightarrow a\) not compact, and let \(M = \{b \rightarrow a, b \rightarrow B, \|b\| \leq 1\}\). \(M\) is bounded but not precompact. Hence there is an \(r > 0\) such that for each finite dimensional subspace \(Y \subset A\), there is an \(m \in M\) with dist \((m, Y) > r\). For each finite subset \(F = \{a_1, \ldots, a_n\} \subset A\), let \(X_F\) be the linear span of \(F\) and let \(m_F = b_F \rightarrow a_F\) with \(\|b_F\| \leq 1\), satisfy dist \((m_F, Y_F) > r\). From a corollary of Hahn–Banach (see [7], Lemma II, 3.12) there is a \(c_F \in B\), satisfying \(c_F = 0\) on \(Y_F, \|m_F, c_F\| = 1\) and \(\|c_F\| = 1\). Clearly, \(c_F \rightarrow 0\). Moreover, \(\left< b_F, a_F \rightarrow c_F \right.\) \(= \left< m_F, c_F \right.\) \(= 1\). By \(w^*-\text{compactness}\), there is a \(w^*-\text{convergent}\) subsequence \(b_{F_k} \rightarrow b_k\) and so \(\left< b_{F_k} \left< b_{F_k}, c_F \right.\) \(= \left< b_F \rightarrow a_F \rightarrow c_F \right. \rightarrow b_F \rightarrow a_F, c_F \rightarrow a, c_F \rightarrow 0\right.\) \(= 1\). Assume that \(\left< J, (0, 0) \right.\) holds for all \(c \in A\) and \(b_k \rightarrow 0\). Pick \(a \rightarrow A\) and let \(b_j \rightarrow 0\) be an arbitrary subnet of \(b_j\). Clearly for each \(j\) there is a \(c_j \in B\) with \(\left< b_j \rightarrow a_j \rightarrow c_j \right.\) \(= \left< a_j, b_j \rightarrow a_j \rightarrow c_j \right.\). By \(w^*-\text{compactness}\), there is a \(c_j \rightarrow c\) \(= \left< a_j, b_j \rightarrow c_j \right.\) \(= \left< a_j, b_j \rightarrow a_j \rightarrow c_j \right.\) \(= \left< c_j \rightarrow a_j \rightarrow c_j \right.\) \(= \left< a_j, b_j \rightarrow a_j \rightarrow c_j \right.\). \(\rightarrow c_j \rightarrow c\) \(= \left< b_j \rightarrow a_j \rightarrow c_j \right.\) \(= \left< a_j, b_j \rightarrow a_j \rightarrow c_j \right.\) \(= \left< a_j, b_j \rightarrow a_j \rightarrow c_j \right.\) \(= \left< a_j, b_j \rightarrow a_j \rightarrow c_j \right.\).\(\rightarrow c\).}

5. Examples.

5.1. Let \(A\) be a non-reflexive Banach space, \(B = A^*\), and pick \(a \rightarrow A^*\) with \(a \rightarrow A\). Define \(b \rightarrow a\in A\). Clearly, \(b \rightarrow c \rightarrow a\). Since \([L]\) does not hold for \(w^*\), this completes the proof of (3.1.i).
5.2. Let $A$ be any real Hilbert space, $B = A$ and pick $0 \neq a \in A$.
Define $b \leftarrow a = (a, a) b$, so that $b \leftarrow b = (b, a) a$.

Using $\rightarrow$ and $\leftarrow$ to mean "it holds" and "it does not hold", respectively, these examples satisfy, for the bounded weak$^*$ convergence:

\[
\begin{array}{cccccc}
(\ell) & (\ell) & (\ell) & [a] & [f] & [f] \\
5.1 & + & + & + & - & - \\
5.2 & - & - & + & - & - \\
\end{array}
\]

6. Convolution algebras. Let $G$ be a locally compact group. We denote by $C_b(G)$ the Banach space of all complex continuous functions vanishing at infinity and by $M(G)$ the space of all bounded regular measures on $G$.

$C_b(G)$ and $M(G)$ are Banach spaces under the sup norm $\| \cdot \|_\infty$ and the total variation norm $\| \cdot \|_w = \| \cdot \|_G$, respectively, and the pairing $\langle f, \mu \rangle = \int f(x) \mu(dx)$ identifies $M(G)$ with the dual of $C_b(G)$. In the sequel, the "weak" topology on $M(G)$ refers to the weak* topology of $M(G)$ as the dual of $C_b(G)$ (for the relationships with other standard measure topologies — e.g., the vague topology — see [3], Chapter VIII, Section 3, Example 11).

If $f \in C_b(G)$, $\mu \in M(G)$ and $\ast \in G$, we define $(\mu \ast f)(s) = \int f(tx) \mu(dx)$, i.e.,

\[(6.1) \quad (\mu \ast f)(s) = \gamma(s)f(s), \quad \gamma(s) : C_b(G) \rightarrow C_b(G) \text{ is the regular (right) representation } \gamma(s)f(t) = f(ts).
\]

We have

\[(6.2.1) \quad \mu \leftarrow f \ast \gamma(s) \mu,
\]

\[(6.2.2) \quad \mu \leftarrow f \ast \gamma(s) \mu,
\]

\[(6.2.3) \quad \mu, \gamma(s) \mu \leftarrow f = \text{a bilinear map } M(G) \times C_b(G) \rightarrow C_b(G).
\]

For (6.2.1) see [3], dépliant II; (6.2.2) follows from $\| \mu \leftarrow f \|_w = \| \gamma(s)f \|_w = \| f \|_G \| \mu \|_G$ and (6.2.3) is obvious.

This shows that we are in the situation described in Section 2 with $A = C_b(G)$, $B = M(G)$ and $\leftarrow$ defined by (6.1) above.

Next assume $\mu, \nu \in M(G)$. Then for $f \in C_b(G)$, $\langle f, \mu \ast \nu \rangle = \int f(x) \mu(dx) \nu(dx) = \int f(tx) \mu(dx) \nu(dx)$, which shows that the product associated to $\ast$ is the ordinary convolution of measures $\mu \ast \nu = \nu \ast \mu$.

The following theorem extends Example 1.4 in [6] and Corollary 1 on page 284 of [21] (cf. also Theorem 3 of [21]):

\[(6.3) \text{THEOREM Let } G \text{ be a locally compact group, } M(G) \text{ the Banach space of regular bounded measures on } G, \text{ and } \mu \ast \nu \text{ the convolution of the measures } \mu, \nu \in M(G). \text{ Then:}
\]

\[(6.3.1) \quad \nu \ast \mu \ast \nu \text{ is } \ast \text{-continuous for each } \nu;
\]

\[(6.3.2) \quad \mu \ast \nu \ast \mu \text{ is } \ast \text{-continuous for each } \mu;
\]

\[(6.3.3) \quad \nu \ast \mu \ast \nu \text{ is jointly } \ast \text{-continuous if and only if } G \text{ is finite;}
\]

\[(6.3.4) \quad \mu \ast \nu \ast \mu \text{ is jointly } \ast \text{-continuous if and only if } G \text{ is compact.}
\]

Proof. (6.3.1) follows from (3.1.1). Let $f^\gamma(s) = f(s^{-1})$ and define $\mu^\gamma(s) = \gamma(s)f^\gamma(s)$. Then $\langle f, \mu^\gamma(s) \rangle = |f|^{-1} |\int f(x) \mu(dx)|$ and therefore $\langle f, \mu^\gamma \ast \nu \rangle = \langle f, \mu^\gamma \ast \nu \rangle = \langle f, \mu^\gamma \ast \nu \rangle$, which proves (6.3.3).

From 1.4.3 in [6] follows that if $G$ is compact, then $\nu \ast \mu$ is jointly $\ast$-continuous (the proof of 1.4.3 in [6] actually shows that $\mu \rightarrow \mu \ast f$ is a compact operator for each $f \in C(G)$). Conversely, assume that $\nu$ is not compact and for each compact set $K \subset G$ pick $\mu_K \in \nu \ast \gamma(s) \nu$ to be a unit point mass measure with support $\{s \in K \}$ outside $K$. Clearly $\nu \ast \gamma(s) \nu \rightarrow 0$. For the same reasons, if $\nu \ast \gamma(s) \nu$ is the unit point mass measure with support $\{s \in K \}$, then also $\nu \ast \gamma(s) \nu \rightarrow 0$. Hence, $\mu \ast \nu$ is not jointly $\ast$-continuous at $(0, 0)$, and (6.3.4) follows.

Assume now that $\nu \ast \mu$ is jointly $\ast$-continuous. Then it is also jointly $\ast$-continuous and, from (6.3.1), $G$ is compact. Then 1.4.3 in [6] applies and $\nu \ast \mu$ is finite. The converse is obvious, so (6.3.3) follows and the theorem is proved.

Observe that (6.3.4) together with (4.1.1) and the fact that $C(G)$ has the approximation property, imply the Peter–Weyl Theorem (as formulated, for instance, in [7], p. 940).

We close this section with two remarks. First, point out that the results in Theorem 6.3 actually hold for an arbitrary invariant closed subspace $A$ of $C_b(G)$ and its dual.

Second, minor modifications will also apply to the following situation.

Let $S \subset G$ be an open semigroup of $G$, $A = C_b(S)$, $B = M(S)$. Here again the product associated to $\mu \leftarrow f(s) = \int f(s) \mu(dx)$ is the convolution $\mu \ast f$ in $M(S)$. The conclusion is: if $S$ is not finite then the convolution product is not jointly $\ast$-continuous and if $S$ is not compact, then it is not jointly $\ast$-continuous. This requires a different version of 1.4.1 in [6]; namely: if $\theta$ is locally compact and not compact, for each $f \in C_b(G)$ not identically zero there are infinitely many linearly independent translates of $\psi$ ("the anti-Peter–Weyl lemma"). This fact can in turn be proved as follows: assume that all translates $\gamma(s)f$ of $f$ are linear combinations of $\delta s$ linearly independent translates $f_j = \gamma(s)\delta_j$. For each $s \in G$, there are $\delta_j, \delta_j > 0$ and $c_{j, m} \in \mathbb{C}$, $j = 1, \ldots, n$ with $\sum_{j=1}^n c_{j, m} = 1$ and $\gamma_j(s)f_j = \sum_{j=1}^n c_{j, m} \delta_j$. Let $\frac{a}{s} \rightarrow 0$ so that $|\gamma(s)f(t)| \rightarrow 0$ for each $t$ and pick a subset $i \rightarrow s$ so that $\gamma_j(s) \delta_j + \sum_{j=1}^n c_{j, m} \delta_j = 1, 2, \ldots, n$. Then

\[
\sum_{j=1}^n c_{j, m} \delta_j \left[ \frac{a}{s} \right] = \lim \left[ -\delta_j \gamma(s)\delta_j \right] = 0.
\]
so that $a^0_j = 0$ for each $j$. Then $1 = \beta_\omega + \sum |a^0_j|$ implies $\beta_\omega = 1$. But now, taking $i = (s')^{-1}a$, we get

$$\beta_{s'} f(a) = \beta_{s'} \sum |a^0_j| f_j((s')^{-1}a) = - \sum |a^0_j| f_j((s')^{-1}a)$$

and therefore

$$f(a) = \lim \beta_{s'} f(a) - \lim \sum |a^0_j| f_j((s')^{-1}a) = \lim \sum |a^0_j| f_j = 0,$$

as claimed.

A typical application of the second generalization is the case $A = e_0$, $B = 1$, with $G = Z = S = N$.

7. The space $L^0(\mu)$. In [21], J. Shapiro proved that $L^0(T)$, the unit circle, is not a topological algebra under pointwise multiplication and the le$^*$ topology (which has an intrinsic meaning since $L^0(T)$, as any other $W^*$-algebra, is the dual of only one Banach space — namely, $L^1(T)$). We aim to explain this result in terms of measure theoretic properties (see (7.3.1) below).

All measures considered below are positive measures. We recall that a measure space $(X, \Sigma, \mu)$ is localizable if for each family $(U_i)$ of measurable sets, there is a measurable set $U$ such that (1) $\mu(U_i \cap U) = 0$ for all $i$ and (2) if $\mu(U_i \cap U) = 0$ for all $i$, then $\mu(U \cap U_i) = 0$ also; a measure space $(X, \Sigma, \mu)$ has the finite subset property if any set of positive measure has a subset of finite positive measure (see [23] for both definitions). The following important fact can be found in Theorem 4, Section 50, Chapter 12 of [23].

(7.1) Let $(X, \Sigma, \mu)$ be a measure space with $\mu \geq 0$. Necessary and sufficient for $L^0(\mu)$ to be the dual of $L^1(\mu)$ is that the measure space be localizable and have the finite subset property.

Under these conditions, we can take $A = L^1(\mu)$, $B = L^0(\mu)$ and define $g \mapsto f \cdot L^0(\mu)$ for $g \in L^0(\mu)$, $f \in L^1(\mu)$ by $(f \cdot g)(s) = f(s)g(s)$. The $-$ product is the ordinary pointwise product in $L^0(\mu)$.

We shall adopt the following definition: an atom in a measure space $(X, \Sigma, \mu)$ is a measurable set $A \subseteq E$ with $0 < \mu(E) < +\infty$ such that for no $B \subseteq E$ we have $0 < \mu(E) < \mu(B)$. The measure space is purely atomic if for each measurable set $A \subseteq X$, the set $A' = A - \{ E \mid E \subseteq A \text{ and } \mu(A') > 0 \}$ is measurable and $\mu(A') = 0$. With these definitions we have (see [13]):

(7.2) If $(X, \Sigma, \mu)$ is purely atomic, $L^1(\mu)$ is isometric to $l^1(S)$ for an appropriate set $S$, and an isometry $T$ can be picked to also map pointwise products $f \cdot g$ when $f$ and $g$ are in $L^1(\mu)$ into $(Tf)_i(Tg)_iW_{\omega}$ for an appropriate weight $(W_{\omega})_{\omega \in \omega}$, $0 < W_{\omega} < +\infty$.

The set $S$ is the set of atoms and $T: L^1(\mu) \to l^1(S)$ is the map defined by: for each $a \in S$, $(Tf)_i = f(a)\mu(a)$, where $f(a)$ is the common value $f(0)$, $Te = a$; also $W_{\omega} = \mu(a)$.

We may now state our result:

(7.3) THEOREM. Let $(X, \Sigma, \mu)$ be a localizable measure space with the finite subset property (so that $L^0(\mu)$ is the dual of $L^1(\mu)$). Then for the pointwise product in $L^0(\mu)$,

(7.3.1) $T (0, 0)$ holds for le$^*$ if and only if $(X, \Sigma, \mu)$ is purely atomic;

(7.3.2) $T (0, 0)$ holds for le$^*$ if and only if $\dim L^0(\mu) < +\infty$ (i.e., $(X, \Sigma, \mu)$ is purely atomic and it has only finitely many atoms).

Proof. The "if" part in (7.3.1) is obvious and the "if" part in (7.3.2) reduces, in view of (7.2), to the statement in (8.4) below.

Assume now that $(X, \Sigma, \mu)$ is not purely atomic. Since $(X, \Sigma, \mu)$ is localizable and has the finite subset property, we can pick a measurable set $T$ with $0 < \mu(T) < +\infty$ and containing no atoms. Let $\{T_{00}, ..., T_{0n}\}$ be a dyadic partition of $T$, i.e., for each $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq \epsilon \leq \theta$,

$(T_{00} \cup \cdots \cup T_{0n}) \cap T = 0$. Then $T_{00} \cup \cdots \cup T_{0n}$ is a measurable subset of $T$ such that (1) $T_{00} \cup \cdots \cup T_{0n} = T_{00} \cup \cdots \cup T_{0n}$, $T_{0k} = T_{00} \cup \cdots \cup T_{0n}$, $T_{0k} = T_{00} \cup \cdots \cup T_{0n}$, $T_{0k} = T_{00} \cup \cdots \cup T_{0n}$, $T_{0k} = T_{00} \cup \cdots \cup T_{0n}$, etc.; (2) $T_{00} \cup \cdots \cup T_{0n} \subseteq T$ and $T_{0k} \cap T = \emptyset$ if $\epsilon \neq \delta$ (for some $j = 1, 2, \cdots, k$, and $3 \mu(T_{00} \cup \cdots \cup T_{0n}) = \mu(T_{00} \cup \cdots \cup T_{0n})$ for each $k = 1, 2, \cdots, k$. Define the function $\phi_k$ on $X$ by:

$\phi_k = 0$ if $k = 1$ and $x \in T_{0k-1}$;

Let $g = \sum_{k=1}^{\infty} g_k$ and set $\phi_n = \mu(T_{00} \cup \cdots \cup T_{0n})$. Clearly, $\mu(T_{00} \cup \cdots \cup T_{0n})$ is an orthonormal system in $L^0(\mu)$ and therefore $\mu(T_{00} \cup \cdots \cup T_{0n})$ is le$^*$-continuous at $0$, so that $\phi_n$ does not converge to le$^*$ and this proves that the product is not jointly le$^*$-continuous at $(0, 0)$.

Assume finally that the product in $L^0(\mu)$ is jointly le$^*$-continuous at $(0, 0)$. In particular it is jointly le$^*$-continuous at $(0, 0)$ and so $L^0(\mu) = l^0(S)$. Then (8.5) applies to show that $S$ is finite and $\dim L^0(\mu) < +\infty$.

Remark. It is clear that the failure to joint le$^*$-continuity in $L^0(\mu)$ follows from our proof from the existence of an orthonormal system $(\{\phi_n\})$ with $\phi_n$ some fixed constant on a set of positive measure. Such a system can be defined easily on any cube $[0, 1)^2$ ($Q$ an arbitrary set) by just taking $\phi_k = \exp(ik\xi_1)$, where $\xi_1$ is a distinguished index. But then the existence of similar systems for arbitrary $(X, \Sigma, \mu)$ follows from Maharam's theorem ([12]). This provides an alternative proof of (7.4).
the dual of $A = L^p(\mu)$ in $B = L^q(\mu)$, where $(1/p) + (1/q) = 1$ (and $q = 1$ if $p = \infty$). When $1 < p < +\infty$ (as always for reflexive spaces) all products (i.e., bounded bilinear maps $B \times B \to B$) on $B = L^q(\mu)$ are $\to$-products. Hence, from Theorem 3.1 we get:

(8.1) For $1 < p < +\infty$, all norm continuous products on $L^p(\mu)$ ($\mu$ purely atomic) satisfy $[L]$ and $[B]$ for the $w^*$ and the $lu^*$ topologies.

On the other hand, Rosenthal's generalization of the classical result of Pisz (see [16], Theorem A2.5 and Remark 2 on p. 211) implies that all operators $T: L^p(\mu) \to L^q(\mu)$ are compact when $\mu$ is purely atomic and $1 \leq s < r < +\infty$ or $1 \leq s < 2$, $r = +\infty$. Then, using (2.2), (8.1) and (4.1.i), we conclude that

(8.2) If $2 < p < +\infty$, all norm continuous products on $L^p(\mu)$, $\mu$ purely atomic, are jointly lu*-continuous everywhere.

It is easy to see that the conclusion is not generally true for $1 \leq p \leq 2$ or for $p = \infty$. In the case $p = \infty$, the space $L^\infty(\mu)$ being no longer reflexive, the products are not automatically $\to$-products. We can only conclude that

(8.3) All $\to$-products on $L^\infty(\mu)$, $\mu$ purely atomic, satisfy $[J, (0, 0)]$ (and of course $[E]$) for the bounded weak* convergence.

In order to interpret the pointwise product in $L^p(\mu)$ as a particular case of pointwise products in $L^p(\mu)$ for each $p$, we first observe that (7.2) can be used to identify $L^p(\mu)$, $\mu$ purely atomic, with the space $P(S, W)$ of all $(\alpha_n, W_n)$ such that $\sum_n |\alpha_n|^p W_n < +\infty$, where $W = (W_n)_{n \in \mathbb{N}}$ is a "weight" satisfying $W_n > 0$. The norm in $P(S, W)$ is $\sum_n |\alpha_n|^p W_n^{1/p}$, $S = \{\alpha\}$ is the set of atoms of $\mu$ and $W = \mu(\alpha)$. We can now define the pointwise product on $L^p(\mu) = P(S, W)$ by $(\alpha_n, y_n) = (\alpha_n, y_n)$, where $\alpha_n = \alpha_n, y_n W_n^{1/p}$. Clearly this is a $\to$-product with $\to$-defined by the same formula. It is now easy to verify that (cf. 1.3.1 and 1.3.5 of [6]):

(8.4) The pointwise product in $L^p(\mu)$, $\mu$ purely atomic, $1 \leq p \leq +\infty$, is jointly lu*-continuous everywhere.

Routine arguments (and 3.1.i) also yield:

(8.5) The pointwise product in $L^p(\mu)$, $\mu$ purely atomic, $1 \leq p \leq +\infty$, satisfies $[J, (0, 0)]$ for the $w^*$ topology if and only if dim $L^p(\mu) < +\infty$ (i.e., $\mu$ has finitely many atoms).

In the last two statements, $L^p(\mu) = P(S, W)$ is to be interpreted as the dual of $\mathcal{E}_S(S, W)$. We leave the details to the reader.

9. The Arens product. Let $X, Y, Z$ be Banach spaces and let $m: X \times Y \to Z$ be a bounded bilinear map. Define $m^*: Y \times X \to Z$ and $m^{**}: Z^* \times X \to Y^*$ by:

$m^*(x, y) = y (x, 1),$  
$m^{**}(z^*, x) = m(z^*(x), 1).$

Iteration of this procedure yields the maps

$m: X \times Y \to Z,$  
m^*: Z^* \times X \to Y^*,$  
m^{**}: Y^{**} \times Z^* \to X^*,$  
m^{***}: X^{***} \times Y^{**} \to Z^{***}.$

These definitions were introduced by Arens who studied several properties of them in [1], [2].

In the sequel only the case $X = Y = Z$ will be considered and we shall use the symbols $A = X^*, B = X^{**}$. Thus we have

$m: X \times X \to X,$  
m^*: A \times X \to A,$  
m^{**}: B \times A \to A,$  
m^{***}: B \times B \to B.$

Clearly, $m^{**}$ can be considered to be a $\to$-product. From the definition follows that:

(9.1) The $\to$-product associated to $m^{**}$ is $m^{***}$.

Thus, by (3.1.i):

(9.2) $m^{***}$ always has property $[L]$ for $w^*$ (and then for lu*).

We borrow from [2] the following

(9.3) Definition. $m$ is regular when $m^{**} = m^{***}$. Arens proved that ([2], Theorem 3.3):

(9.4) $m$ is regular if and only if $m^{***}$ has property $[E]$ for $lu^*$.

In view of (9.2) and (9.4) it is quite natural to take up the question of joint lu* continuity. This has been considered by Fyn (115); cf. also Theorem 2.1 of ([11]) and by McMillan and White ([11], Theorem 2.2). Fyn proved that $m$ is regular iff all $a \cdot A$ are weakly almost periodic and McMillan and White proved that $m^{***}$ has property $[J]$ for lu* iff all $a \cdot A$ are almost periodic. Also for the "scalar case" (i.e., $X = Y = Z = \mathbb{R}$), this is considered in [8].

Our goal here is to give some other conditions closely related to $[J]$ or $[J, (0, 0)]$ for the $\to$-product $m^{***}$, but in the spirit of our general setup.

We begin by defining an auxiliary space $X^+$ as follows: $X^+$ is the vector space $L(S, B)$ of all linear bounded maps $w: X \to B$ made into a locally convex space by means of the seminorms:

$$p(w) = \sup \{|(x, w(x))|; \|x\| \leq 1\}.$$
where $a$ varies in $A$. It is clear that this topology is weaker than the norm topology of $L(X, B)$ and therefore the natural inclusion $L(X) \to X^\#$ is continuous.

Let us use the notation

$$m_k : X \to X^\#,$$  

$$m_d : X \to X^\#$$

for the maps defined by $m_k : x \mapsto u$, $m_d : x \mapsto v$ with $u(y) = m(x, y)$, $v(y) = m(y, x)$.

(9.6) DEFINITION. We shall say that $m$ is $s$-compact (resp., $d$-compact) when $m : X \to X^\#$ (resp., $m_d : X \to X^\#$) is a compact operator.

Then we have:

(9.7) THEOREM. Let $X$ be a Banach space and $m : X \times X \to X$ a bounded bilinear map. Consider the following conditions:

1. $m$ is $s$-compact;
2. $m$ is $d$-compact;
3. $m^{***}$ has property $[J, (0, 0)]$ for $lu^*$;
4. $m^{***}$ has property $[J]$ for $lu^*$.

Then:

1. (3) and (4) are equivalent;
2. (3) implies (3) and (4);
3. (3) implies (3) and (4).

Moreover, if $X$ is separable, then (1), (2), (3) and (4) are equivalent.

Proof. We still use the notation $A = X^\#, B = X^{**}$.

Clearly, (4) implies (3). In order to prove the converse we first observe that for fixed $a \in A$, if the operator $T : b \mapsto \langle a, b \rangle$ is compact from $B$ to $A$, then it is also $lu^*$-norm continuous. In fact, let $\langle b, a \rangle \in B$ with $\|b\| < 1$ and $b \mapsto \langle a, b \rangle$ for $u$. Let $b_j$ be an arbitrary subnet of $b_j$. By compactness, there is a subnet $\langle b_j \rangle$ of $b_j$ with $(T_{b_j})$ convergent, so that $T_{b_j} \to a$ in norm for some $a \in A$. However, $\langle a, m^{**}(b_j, a) \rangle = \langle m^{*}(a, b_j), a \rangle$ implies that $T_{b_j} = m^{**}(b_j, a) \to 0$ in the norm, as claimed. Now, using (4.1) we conclude that (3) implies (4).

Next we shall establish (1) $\Rightarrow$ (3). It will suffice to show that if $m$ is $s$-compact and $a \in A$ is fixed, then $T : b \mapsto m^{**}(b, a)$ is compact. Since $T$ is the adjoint of $S : X \to A, Sx = m^{*}(a, x)$, this amounts to proving that $S$ is compact. Let then $\{a_k\}$ be a bounded sequence in $X$.

From (1) follows that there is a subnet $\{y_k\}$ of $\{a_k\}$ such that $m_k(y_k)$ is convergent in $X^\#$. This means that there is a bounded linear operator $u : X \to B$ such that for each seminorm $p, p(m_k(y_k) - u) \to 0$. Then, if $a$ is the element of $X^\#$ that defines $p$ as in (9.5):

$$\sup \{\langle a, m_k(y_k) \rangle - \langle a, u \rangle \delta \mid ||\delta|| \leq 1\} \to 0$$

and so, by definition

$$\sup \{||\langle a, m_k(a, y_k) \rangle - \langle a, u \rangle || \delta \mid ||\delta|| \leq 1\} \to 0,$$

where $u^* : B^* \to A$ is the adjoint of $u$. This means that for each $a \in A, ||m_k(a, y_k) - u(a)|| \to 0$, and so, in particular, $S(y_k) \to u(a)$. Hence $S$ is compact.

In order to show that (2) $\Rightarrow$ (4), we use implication (1) $\Rightarrow$ (4) applied to $m^*$ to conclude that $m^{***}$ has $[J]$ for $lu^*$. But then (9.4) implies that $m^*$ is regular. Now it follows from Theorem 3.2 in [2] (implication $u \Rightarrow (3.3.3')$) applied to $m^{***}$ that $m^{***} = m^{***}$.

Since $m^{***}$ (and therefore $m^{***}$) also has $[J]$ for $lu^*$ as observed above.

We assume now that $A$ is separable and pick a sequence $\delta = \{a_k\}_{k=1}^\infty$ dense in $A$. Assume further that (4) holds. By the argument used in the proof of (3) $\Rightarrow$ (4), we conclude that $S_k : x \to m^{**}(a_k, x)$ is a compact operator $S_k : X \to A$ for each $a \in A$. We shall abbreviate $S_k = S_k$ when $a = a_k$.

Let now $\{a_k\}$ be a bounded sequence in $X$. Since $S_k$ is compact, there is a subsequence $\{a_{k_l}\}$ with $(S_k a_{k_l})$ convergent. Similarly, there is a subsequence $\{a_{k_l}\}$ with $(S_k a_{k_l})$ convergent, and so on. Denote by $y_{k_l}$ the diagonal sequence produced by the iteration. Then $(S_k y_{k_l})$ is convergent for each $k = 1, 2, \ldots$; let $d_k a_{k_l}$ be defined as $d_k = \lim S_k y_{k_l}$.

We define $T : D \to A$ by $T y_{k_l} = d_k$. Since $||S_k y_{k_l} - S_k y_{k_l}|| = ||m^*(a_k - a_{k_l})|| \leq K ||a_k - a_{k_l}||$, where $K = ||m^*|| ||\sup||$, then $||d_k a_{k_l} - d_k a_{k_l}|| \leq ||a_k - a_{k_l}||$ and therefore $T$ can be extended to a continuous map (with the same name) $T : A \to A$. We claim that $T$ is linear. In fact, let $k, j, i$ be positive integers. Then

$$||T(a_k + a_j) - T a_k - T a_j|| \leq ||T(a_k + a_j) - T a_k|| + ||T a_k - T a_j||$$

$$= ||T(a_k + a_j) - T a_k|| + ||S_k y_{k_l} - S_k y_{k_l}||$$

$$\leq ||T(a_k + a_j) - T a_k|| + \lim ||m^*(a_k - a_k, a_{k_l})||$$

$$\leq ||T(a_k + a_j) - T a_k|| + ||a_k - a_{k_l}||$$

Picking the indices $i$ so that $a_{k_l} \to a_k + a_j$, we get $||T(a_k + a_j) - T a_k - T a_j|| = 0$ and then, by continuity, $||T(a_k + a_j) - T a_k|| = 0$ for all $a_k, a_j \in A$.

Therefore, $T : A \to A$ is a continuous linear operator. It is now easy to see that $m^{**}(a_k, y_{k_l}) - Ta_k$ for each $a$. Let $u : X \to B$ be the restriction $u = m^{**} : X$ to the adjoint of $T$. Our final claim is that $m_u(y_{k_l}) \to u$ in $X^\#$. In fact, for $a \in X$,

$$\langle a, m_u(y_{k_l}) \rangle - \langle a, u \rangle = \langle m_u(y_{k_l}, a) \rangle - \langle a, T a \rangle$$

$$= \langle a, m^*(a, y_{k_l}) - Ta \rangle$$

so that, with $p$ the seminorm defined by $a^* = a$ in (9.5),

$$p(m_u(y_{k_l}) - u) = ||m^*(a, y_{k_l}) - Ta|| \to 0$$

and we are done.
We have therefore proven that \((2) \Rightarrow (4)\) and (when \(A\) is separable), \((4) \Rightarrow (1)\). Thus \((2) \Rightarrow (1)\) and so, by symmetry, when \(A\) is separable, (1) and (2) are equivalent, and equivalent to (3) and (4), which completes the proof of (8.7).

It is clear from (9.4) and (9.7) that (9.8) \(If\) (1) or (2) is satisfied, then \(m\) is regular.

One can see that the implications in (9.8) cannot be reversed in general by examining Example 9.2.

We close this section with the observation that \(s\)-compactness or \(\tilde{\delta}\)-compactness of \(m\) are properties that generalize weak compactness in the scalar case, as in [5], p. 5. In fact, the Grothendieck canonical extensions and the Arens products are closely related, as follows. For \(X, Y, Z\) Banach spaces and \(m: X \times Y \to Z\) bounded and bilinear, define the right canonical extension \(m^r: X \times Y \to Z\) by first interpreting \(m\) as a map \(X \to L(Y, Z)\) and then following it by the double-adjoint map \(L(Y, Z) \to L(X, Z^{**})\). The resulting map \(X \to L(Y, Z^{**})\) determines \(m^r\) in the obvious way. In a similar way we can define \(m_l: X^{**} \times Y \to Z^{**}\) as the composition \(Y \to L(X, Z) \to L(X^{**}, Z^{**})\). The two iterated maps \(I(m)\) and \(I(h)\) have the same domain \((X^{**} \times Y^{**})\) and the same range \((Z^{***})\). It is (tedious but) routine matter to determine the following identities involving Arens products and Grothendieck extensions:

\[ m^{***} = P \circ (m^l), \quad m^{**} = P \circ (h), \]

where \(P: Z^{***} \to Z^{**}\) is the canonical projection.

10. The Hardy space \(H^\infty\). Let \(\Omega\) be a plane domain. Denote by \(B_H(\Omega)\) the vector space of all bounded holomorphic functions on \(\Omega\). We assume that \(A\) supports non-constant bounded holomorphic functions.

For \(b \in B_H(\Omega)\) and \(\mu\) a regular complex measure on \(\Omega\), write (with \(\varepsilon = \omega_{\gamma+i}z\))

\[ \langle \mu, b \rangle = \int_{\Omega} b(z) \mu(\omega_{\gamma+i} z) \, dz. \]  
(10.1)

This defines a duality between the space \(M^1(\Omega)\) of all regular complex measures on \(\Omega\) and \(B_H(\Omega)\), with degeneracy. Thus, we can take \(N = \{\mu \in M^1(\Omega) : \langle \mu, b \rangle = 0, \forall b \in B_H(\Omega)\}\) and define \(A = M^1(\Omega)/N\) as the quotient Banach space. It follows from [19], (4.3), (4.4), and (4.5) that the dual of \(A\) can be isometrically identified via the duality (10.1) with \(B_H(\Omega)\) under the sup norm (that is, the Hardy space traditionally denoted by \(H^\infty(\Omega)\), cf. [17], where the notations used here are introduced).

Also we can define, for \(b \in B_H(\Omega)\), \(a = \mu + N \in M^1(\Omega)/N\), the element \(b \mapsto a = \mu_N + N\), where \(\mu_N \in M^1(\Omega)\) is the ordinary product of a bounded function and a measure. The product in \(B_H(\Omega)\) associated to \(\langle \cdot, \cdot \rangle\) is again

the ordinary product \(b(\varepsilon) - b(z)\, c(\varepsilon)\) of holomorphic functions. Thus we find again the situation in Section 2 with \(A = M^1(\Omega)/N, B = B_H(\Omega)\) and the ordinary product in \(B_H(\Omega)\) as \(\cdot \cdot\cdot\) product.

The \(l^\infty\)-continuity properties of this product are known:

(10.2) Theorem (Rudin and Shields). The product of \(B_H(\Omega)\) is jointly \(l^\infty\)-continuous.

This result, however, depends on the fact, due to Rudin and Byrd, that \(l^\infty\) is the strict topology of \(B_H(\Omega)\), i.e., \(l^\infty\) is the topology defined by the seminorms \(\sup_{z \in \Omega} |b(z)|\), where \(x \cdot O(\Omega)\) with \(x \cdot O(\Omega)\) and \(\partial \Omega\). This result is hard to get (see [18]). We can, however, get Theorem (10.2) from Theorem (4.1) and the following easy consequence of the normal families theorem.

(10.3) Proposition. Let \(b_\varepsilon(\varepsilon)\) be holomorphic functions on \(\Omega\) with \(|b_\varepsilon(\varepsilon) - M|\leq M\) and assume that \(b_\varepsilon(\varepsilon) \to 0\) for each \(\varepsilon \in \Omega\), as \(\varepsilon \to \infty\).

\[ \int_{\Omega} |b_\varepsilon(\varepsilon)|^2 \, d\mu(\varepsilon) \to 0 \]

for each \(\mu \in M^1(\Omega)\).

Proof. Write

\[ \left\| \int_{\Omega} b_\varepsilon(\varepsilon)|^2 \, d\mu(\varepsilon) \right\| \leq \int_{\Omega} \left\| |b_\varepsilon(\varepsilon)|\right\|^2 |d\mu(\varepsilon) + M \int_{\Omega} |d\mu(\varepsilon),\]  

where \(K \subset \Omega\) is a compact subset. Given \(\varepsilon > 0\), we can pick \(K\) such that the last integral does not exceed \(\varepsilon/2M\). Now, from the normal families theorem follows that the integral in the middle is also small for a large, and we are done.

This proposition just means that \(b\mapsto b(\varepsilon)\) is \(l^\infty\)-norm continuous from \(B_H(\Omega)\) into \(M^1(\Omega)/N\), and therefore from (4.1.ii) follows that the product is jointly \(l^\infty\)-continuous, i.e., the statement in (10.2).

Similarly, we can use Theorem (3.1) to prove that:

(10.4) Theorem. The product in \(B_H(\Omega)\) is not jointly \(w^\ast\)-continuous.

Proof. We first give an elementary proof for \(\Omega\) bounded. Pick \(z_0 \in \Omega\) and choose \(a > 0\) with \(z_0 + t \in \Omega\) for all \(0 \leq t < a\). Define the measure \(\mu_t, \mu_t = 0, 1, \ldots, \) by

\[ \int_{\Omega} f(z) \mu_t(\omega_{\gamma+i} z) \, dz = \int_0^a f(z_0 + t) \, dt. \]

Let \(\lambda_1, \lambda_2, \ldots, \lambda_a\) be complex numbers and suppose that \(\sum \lambda_a \mu_t \in N\), the annihilator of \(B_H(\Omega)\) in \(M^1(\Omega)\). Then, for \(b \in B_H(\Omega)\),

\[ 0 = \int_{\Omega} b(z) \left( \sum_{t=0}^a \lambda_t \mu_t \right) \, dz = \int_{\Omega} b(z_0 + t) P(t) \, dt, \]

where \(P(s) = \sum \lambda_s \mu_s\). Thus, if \(Q(z_0) = \sum \lambda_a (z_0 - z_a)^k\), then

\[ 0 = \int_{\Omega} Q(z_0 + t) P(t) \, dt = \int_{\Omega} |P(t)|^2 \, dt, \]
and therefore the polynomial $P(z)$ is identically zero, whence $a_i = a_i = a_0 = 0$. But then $\mu_i + N, \mu_i + N, \ldots, \mu_i + N$ are linearly independent in $\mathbb{M}^1(\Omega)$. Since $z^a - (\mu_i + N) = \mu_i + N$, it follows that the range of $b^a - a_0$ is infinite dimensional when $a_0 = \mu_i + N$. Then (10.4) follows from (3.1.ii).

The following proof that (3.1.iii) implies (10.4) for general $\Omega$ is due to L. A. Rubel, to whom we wish to express our appreciation.

First, take $f \in \mathcal{B}_H(\Omega)$ non-constant and with $||f||_m = 1$. Let $a_n \in \Omega$ be all different for $n = 1, 2, \ldots$ with $|f(a_n)| = 1$ and denote $z_n = f(a_n)$. It follows from [9], Corollary, p. 204, that a suitable subsequence (also denoted $(z_n)$) is an interpolating sequence for $\mathcal{B}_H(\Omega)$. Hence, for each bounded sequence $(w_n)$ there is a $g \in \mathcal{B}_H(\Omega)$ with $g(a_n) = w_n$. The same is true, of course, for $(z_n)$: there is $b \in \mathcal{B}_H(\Omega)$ with $b(z_n) = w_n$ (take $b = g(f)$). Let now $\mu$ be the measure $\mu = \sum_n 2^{-n} \delta_{z_n}$, where $\delta_z$ is the unit point mass at $z$. Clearly, if $b \in \mathcal{B}_H(\Omega)$ with $b(a_0) = w_0$, then for $a_0 = \mu + N$ we have

$$b \in a_0 = \left\{ \sum_{n=0}^\infty 2^{-n} \delta_{z_n} \right\} + N$$

and this shows that the range of $b^a - a_0$ contains all the (finite) linear combinations of the measures $\delta_z$ and therefore it is not of finite rank.

Theorems (10.2) and (10.4) obviously imply that $w^* \neq \lambda w^*$ (see Theorem 3.14 of [18]); in fact this can be obtained directly from the first two propositions and on p. 190 of [18] (cf. remark of last two lines of [18], loc. cit.).

We remark that the statements above extend with no changes to domains in $\mathbb{C}^n$. In fact, in this case it is also true that the $\lambda w^*$-topology coincides with the strict topology (222), p. 476) and that $\lambda w^* \neq w^*$.

11. Operator algebras. The supporting reference for this section is [5].

Let $H$ be a Hilbert space, $L(H)$ the Banach algebra of all bounded linear operators in $H$ under the operator norm and $L_n(H)$ the trace class, that is to say, the space consisting of all $T \in L(H)$ with $\sum \lambda_j(T) < +\infty$ where $(\lambda_j(T))_j$ is an arbitrary complete orthonormal system of $T = \sum \lambda_j\delta_j$. For $T \in L_n(H)$ the trace is the number $\text{tr}(T) = \sum \lambda_j(T)$. It is known that $L_n(H)$ and $\text{tr}(T)$ do not depend on the complete orthonormal system $(\delta_j)$; also, $L_n(H)$ is a two-sided ideal in $L(H)$. $T \in L_n(H)$ if and only if $\text{tr}(T) = 0$.

$T \neq 0$ if and only if $\text{tr}(T) = 0$. $T \in L_n(H)$ if and only if $\text{tr}(T) = 0$.

We define a pairing of $L_n(H)$ and $L(H)$ that identifies $L(H)$ with the dual of $L_n(H)$.

With this background, we can consider $A = L_n(H)$ and $B = L(H)$ from the point of view of Section 2, and a natural $\lambda$-map suggests itself:

$S \subset T = TS$, the composition of operators. It is easy to see that $\lambda S \subset T$ is the composition of operators:

$$(T, S) \to (S, T, S) = (S, T, S) = T \lambda S = \text{tr}(TS) = \text{tr}(TS) = (T, S, S) = (T, S, S).$$

The $w^*$ and $\lambda w^*$-continuity properties of the product in $L(H)$ are not hard to get (see [5], [3], 3.5, Example 2, [6], 2-nd paragraph on p. 290) and can be summed up as follows:

11.1. The product in $L(H)$ always properties $[L]$ and $[E]$ for $w^*$ and $\lambda w^*$; it has property $[J, (0, 0)]$ for $w^*$ or $\lambda w^*$ if only if $\dim(H) < +\infty$.

We aim to show here that our previous theorems are quite related to 11.1. A further property of the trace will be used (see [5], [6], Property 1), namely: if $T \in L_n(H)$, $S \in L(H)$ then $\text{tr}(TS) = \text{tr}(ST)$. It follows that if $U \in L_n(H)$, then also $(S, T) = (S, T, U) = (S \subset T, U)$. But then it is clear from this identity that if $S_n \notin 0$, then $S_n \subset T \to 0$ in the weak topology of $L_n(H)$. Consequently, (3.1.iii) implies that the product in $L_n(H)$ has property $[L]$ for $w^*$ (and hence also for $\lambda w^*$); since it always has $[R]$ by (3.1.iii), the first half of 11.1 follows.

The $w^*$ and $\lambda w^*$-discontinuity when $\dim(H) = +\infty$ is not hard to prove directly (see again [6], 2-nd paragraph on p. 290). It can also be obtained from (4.1) as follows: pick a countable orthonormal system $(\delta_j)$ and define $T_j \in L_n(H)$, $S_j \in L_n(H)$ by $T_j = (S_j, (S_j, z_1)\delta_1, (S_j, z_2)\delta_2, \ldots, (S_j, z_n)\delta_n, \ldots)$, $S_j = (S_j, (S_j, z_1)\delta_1, (S_j, z_2)\delta_2, \ldots, (S_j, z_n)\delta_n, \ldots)$ for $j \neq 1, j \neq n$, and $S_n = 0$ on the orthogonal complement to the span of $(\delta_n)$. One has $\|S_j\| = 1$ and $\|S_n\| = T_j \to 0$, hence $S \to S$ is not compact. This shows that $\lambda w^* \neq w^*$.

References


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The structure of $L$-ideals of measure algebras

by

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Abstract. This paper shows that there are $L$-ideals $I_1$ and $I_2$ of the measure algebra on a L.C.A. group such that $Z(I_1) = Z(I_2)$ and there are no $L$-ideals $I$ such that $I_1 \subseteq I \subseteq I_2$.

1. Introduction. Let $G$ be a non-discrete L.C.A. group and let $\hat{G}$ be the dual group of $G$. Let $L^1(G)$ and $M(G)$ be the group algebra on $G$ and the measure algebra on $G$, respectively. We denote by $\text{Rad}L^1(G)$ the radical of $L^1(G)$, that is, $\text{Rad}L^1(G)$ is the intersection of all maximal ideals of $M(G)$ which contain $L^1(G)$. For $\mu \in M(G)$, $\nu \ll \mu$ means that $\nu$ is absolutely continuous with respect to $\mu$, and $\nu \perp \mu$ means that $\nu$ and $\mu$ are mutually singular. For $\mu \in M(G)$, we put $L^1(\mu) = \{\lambda \in M(G); \lambda \ll \mu\}$. A closed subspace (ideal, subalgebra) $N$ is called an $L$-subspace (L-ideal, L-subalgebra) if $L^1(\mu) \subseteq N$ for every $\mu \in N$.

Taylor [3] showed a characterization of the maximal ideal space of $M(G)$ as follows: There exist compact topological abelian semigroup $\bar{S}$ and an isometry isomorphism $\theta$ of $M(G)$ into $M(\bar{S})$ such that the maximal ideal space of $M(G)$ is identified with $\bar{S}$, the set of all continuous semicharacters of $S$, and the Gelfand transform of $\mu \in M(G)$ is given by $\hat{\mu}(f) = \int f \overline{\theta(\mu)}$.

For a closed ideal $I$ of $M(G)$, we put

$$Z(I) = \{f \in L^1(\mu); \hat{\mu}(f) = 0 \text{ for every } \mu \in I\}.$$ 

H. Helson ([2]) showed that: If $I_1$ and $I_2$ are closed ideals of $L^1(G)$ with $I_1 \subseteq I_2$ and $Z(I_1) = Z(I_2)$, then there is a closed ideal $I$ such that $I_1 \subseteq I \subseteq I_2$.

In this paper, we show that Helson’s theorem is not true in the category of $L$-ideals of $M(G)$ or in the category of closed ideals of $M(G)$. Our results are the following.

Theorem 1. There are two $L$-ideals $I_1, I_2$ of $M(G)$ such that $I_1 \subseteq I_2$ and $Z(I_1) = Z(I_2)$, but there are no $L$-ideals $I$ so that $I_1 \subseteq I \subseteq I_2$. 

The $L$-ideals $I_1$ and $I_2$ mentioned above are constructed by means of a suitable closed ideal $K$ of $M(G)$.