q-variate minimal stationary processes

by

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Abstract. A complete description of non full rank in general q-variate minimal stationary processes over discrete Abelian groups are given. This result subsumes the minimality theorems of various authors in special cases.

1. Introduction. In his fundamental paper [1] A. N. Kolmogorov introduced the important concept of minimal processes. Next the concept have been extended to the q-variate case (cf. [3] and [6], Section 10).

The interpolation problem for q-variate stationary processes over groups was studied by H. Salehi and J. K. Scheidt [8] and by A. Weron [9], [10]. Furthermore in those papers characterizations of q-variate minimal processes are also given. In [8] a generalization of Masani's minimality theorem for full rank processes is obtained. Two characterizations of non-full rank processes are given in [10], but unfortunately one of which ([10], Theorem 5.7) contains an error. In this paper a counter example for this (see Example 5.3) and a correct statement of this theorem (see Theorem 4.6(d)) is given. Moreover, we will get a general theorem on characterizations of q-variate minimal (not necessary full rank) processes.

Section 2 is devoted to the preliminary results on the spaces \( L_{p,2} \) — of square integrable matrix-valued functions and \( H_{p,2} \) — of Hellyinger square integrable matrix-valued measures. Section 3 treats on q-variate stationary processes over a discrete Abelian group. Using methods of the earlier work [10] on stationary processes over locally compact Abelian (LCA) groups, we obtain an analytical characterization of a subspace \( N \), which is important in the minimality problem. In Section 4 we discuss the minimality problem and give some characterizations of minimal processes. As a corollary we then deduce Kolmogorov's and Masani's minimality theorems. Finally in Section 5 we give several examples to show that conditions in the presented theorems are essential ones as well as to illustrate them.

2. \( L_{p,2} \) and \( H_{p,2} \) spaces. Let \( \mathcal{B} \) be a \( \sigma \)-algebra of subsets of a space \( \Omega \) and let \( \Phi = \{ \phi_{ij} \}, 1 \leq i, j \leq q \), be a matrix-valued function on \( \Omega \). Throughout this paper all matrices have complex entries and \( C \) denotes
the set of complex numbers. A function $\Phi$ is $\mathcal{B}$-measurable if each function $\varphi_\theta$ is $\mathcal{B}$-measurable. If $m$ is a non-negative real-valued measure on $\mathcal{B}$, then by $L^m_\mathcal{B}$ we denote the class of all $\mathcal{B}$-measurable functions $\Phi$ such that each $\varphi_\theta$ is integrable with respect to (abbreviated to “w.r.t.”) $m$. For $\Phi \in L^m_\mathcal{B}$, we put

$$\int \Phi dm = \int \varphi_\theta dm.$$ 

By a matrix-valued measure on a $\sigma$-algebra $\mathcal{B}$ we shall mean a function $M$ from $\mathcal{B}$ into the set of all $g \times g$-matrices over $G$, the complex numbers, such that for every disjoint sequence of sets $A_1, A_2, \ldots$ in $\mathcal{B}$ with union $A, M(A) = \sum M(A_k)$. Obviously, $M = [M_{\theta}]$ is a matrix-valued measure if and only if each of its entries $M_{\theta}$ is a complex-valued measure on $\mathcal{B}$. If $m$ is a non-negative $\sigma$-finite measure on $\mathcal{B}$, we say that $M$ is absolutely continuous w.r.t. $m$ ($M \ll m$) if each entry $M_{\theta}$ is absolutely continuous w.r.t. $m$.

Let now $\Phi$ and $\Psi$ be $\mathcal{B}$-measurable matrix-valued functions on $\mathcal{B}$ and let $M$ be a matrix-valued measure on $\mathcal{B}$. If $m$ is non-negative $\sigma$-finite measure on $\mathcal{B}$ such that $M \ll m$ and if $\Phi \in L^m_\mathcal{B}$ and $\Psi \in L^m_\mathcal{B}$, then we define

$$\int \Phi M \Psi^* dm = \int \Phi dm \Psi^* dm,$$

where $A^*$ denotes the conjugate transpose of $A$. It is known ([2], 3.1) that $\int \Phi M \Psi^* dm$ is independent of the choice of $m$.


The unique solution of these equations is called the generalized inverse of $A$ and written $X = A^\#$.

Let $A(A) = \{y: y = ax\}$ denote the range of $A$, and $A^\perp = \{x: ax = 0\}$ denote the null space of $A$. $P_\perp$ denote the orthogonal projection matrix onto the subspace $A^\perp \subset \mathcal{B}$. If $A^\#$ is the generalized inverse of $A$, then from [4], p. 355, we have

(2.2) $A A^\# = P_{A(A)} A = P_{A(A)}$;
(2.3) $A^\# A = P_{A(A)} = P_{A(A)}$;
(2.4) $(A^\#)^* = A^\#$;
(2.5) $(A^*)^* = (A^\#)^*$.

(2.6) Let $H$ be Hermitian and $G = AHA^*$. Then $A(A) \subset A\{A\}^\perp$ if and only if $A A H A^\# A^* = H$.

We note that (2.3)–(2.5) implies

(2.7) If $A$ is non-negative and Hermitian, then $A^\#$ is also non-negative and Hermitian and $A\{A\} = A\{A\}^\#$.

Let $M$ and $N$ be matrix-valued measures on $\mathcal{B}$. Following [4], p. 361, we say that $N$ is strongly absolutely continuous w.r.t. $M$ ($N \ll M$) if exists a non-negative $\sigma$-finite measure $m$ such that $M \ll m$, $N \ll m$ and $A(dN/dm) = A(dM/dm)$ a.e. We say that the matrix-valued functions $\Phi$ and $\Psi$ are equivalent w.r.t. $m$ if $A(d\Phi/dm) = A(d\Psi/dm)$ for each $\Phi, \Psi \in L^m_\mathcal{B}$.

(2.8) PROPOSITION (cf. [4], 5.4). Let $M$ and $N$ be matrix-valued measures on $\mathcal{B}$. Then $N \ll M$ if and only if there exists a matrix-valued function $\Phi$ such that $\Phi \in L^m_\mathcal{B}$ and $N(E) = \int \Phi dm$ for all $E \in \mathcal{B}$. This $\Phi$ is unique (mod $M$): in fact, $\Phi = (dN/dm)(dM/dm)^\#$, where $m$ is any non-negative measure such that $M \ll m$.

The next Gram–Schmidt decomposition is an easy consequence of [4], 6.14.

(2.9) PROPOSITION. Let $N$ be a non-negative Hermitian matrix-valued measure and let $m$ be a non-negative $\sigma$-finite measure on $\mathcal{B}$. Then $N$ is a unique matrix-valued measures $N^\alpha$ (the absolutely continuous part) and $N^\perp$ (the singular part) such that $N = N^\alpha + N^\perp$, $N^\alpha \ll m$, $N^\perp \perp m$ and $N^\alpha$ are non-negative Hermitian matrix-valued measures.

Let $M, N$ and $F$ be matrix-valued measures on $\mathcal{B}$. If $m$ is a non-negative $\sigma$-finite measure on $\mathcal{B}$ such that $M \ll m$, $N \ll m$, $F \ll m$ and if $A(dM/dm)(dF/dm) = (dN/dm)(dF/dm)$ then we define the Hellinger integral

$$\int A dm = \int A dm = \int dm A dF = \int dm A dF = \int dm A dF = \int dm A dF.$$ 

It is well known ([7]) that $\int dm A dF$ is independent of the choice of $m$.

(2.10) DEFINITION (cf. [5], [7]). Let $F$ be a non-negative Hermitian matrix-valued measure on $\mathcal{B}$.

(a) By $H_{N,F}$ we denote the class of all matrix-valued measures $M$ on $\mathcal{B}$ for which the integral $\{M, M_F = \int dm A dF, dF\}$ exists and $M$ is strongly absolutely continuous w.r.t. $F$.

(b) By $I_{N,F}$ we denote the class of all $\mathcal{B}$-measurable matrix-valued functions $\Phi$ on $\mathcal{B}$ for which the integral $\{\Phi, \Phi_F = \int dm A dF, dF\}$ exists.

We remark that our definition of $H_{N,F}$ space is different from original one, given by I. Sadeh ([7]). Example (5.1) shows that Theorem 2(b) in [7] is not true without the assumption $M \ll \mathcal{F}$ and therefore we add it in the definition.

(2.11) THEOREM (cf. [7], Theorem 2). Let $F$ be a non-negative Hermitian matrix-valued measure on $\mathcal{B}$. Then $M H_{N,F}$ if and only if there exists a function $\Phi$ in $L^m_\mathcal{B}$, such that $M(E) = \int dm A dF$ for each $E \in \mathcal{B}$. 


Proof. Let $M \in \mathfrak{H}_E$. Since $M \in \mathfrak{H}_E$, then by (2.8) there exists a function $\Phi \in \mathcal{L}_1(E)$ such that $M(E) = \int \Phi dF$ for each $E \in \mathcal{E}$. In fact, $\Phi = (dM/dm)(dF/dm)(mod E)$, where $m$ is any non-negative $\sigma$-finite measure on $\mathcal{E}$ such that $F \leq m$ (for example, $m = \text{tr}F$). Obviously,

$$(2.12) \int \Phi d \mathcal{M}^* dF = \int (dM/dm)(dF/dm)(mod E) \int \Phi dF = \int \Phi dF.$$

Hence $\Phi \in \mathcal{L}_1(E)$.

Conversely, let $M(E) = \int \Phi dF$ for all $E \in \mathcal{E}$ and $\Phi \in \mathcal{L}_1(E)$. Then $\int \Phi dF = \int \Phi dF$, (see [5]) and, by (2.8), $M \in \mathfrak{H}_E$. According to (2.12), the integral $\int \Phi d \mathcal{M}^* dF$ exists and therefore $M \in \mathfrak{H}_E$. ■

(2.13) COROLLARY. Let $F$ be as before. Then $M$ is zero in $H_1$, if and only if $M(E) = 0$ for each $E \in \mathcal{E}$.

Proof. The sufficiency is trivial. For necessity let us consider $M \in H_1$, such that $\int \Phi d \mathcal{M}^* dF = 0$. By (2.11), there exists a function $\Phi \in \mathcal{L}_1(E)$ such that $M(E) = \int \Phi dF$, $E \in \mathcal{E}$. Since $\left(\Phi, \Phi\right) = (M, M)$, from (5) $\Phi = 0$ (mod $F$). Hence $M(E) = \int \Phi dF = 0$ for each $E \in \mathcal{E}$.

3. Stationary processes. Let $G$ be any discrete Abelian group with multiplication. Then $\mathcal{G}$, the dual group of $G$, is a compact Abelian group under compact-open topology. We will denote the elements of $G$ by $g$ and those of $\mathcal{G}$ by $\gamma$. The value of $\gamma g$ at $g \in G$ will be denoted by $(g, \gamma)$. The Borel field of a topological group is the minimal $\sigma$-field generated by the open sets. Throughout this paper the letter $\mathcal{B}$ will denote the Borel field of $\mathcal{G}$. On every locally compact Abelian group there exists a non-negative, finite on compact sets and positive on non-empty open sets, the so-called Haar measure of the group, which is translation-invariant. We denote by $d\gamma$ and $d\gamma$ the Haar measures on $G$ and $\mathcal{G}$. Without loss of generality, we will assume $d\gamma = 1$.

(3.1) DEFINITION. A $(\gamma)$-variant stationary process over $G$ is a function $(X_{g,\gamma})$ such that:

(a) $X_{g,\gamma}$ is a Hilbert space (if $g$ is a fixed complex Hilbert space)

(b) the Grammian matrix $(X_{g,\gamma}, X_{\gamma}) = \langle X_{g,\gamma}, X_{\gamma} \rangle = K\langle gh^{-1}\rangle$ depends only on $gh^{-1}$ for all $g, h \in G$.

$K(g)$ is positive-definite, and, in view of Bochner's theorem, can be written in the form

$$K(g) = (X_{g,\gamma}, X_{\gamma}) = \int (g, \gamma) d\gamma,$$

where $\mathcal{P}$ is the non-negative Hermitian matrix-valued measure on $\mathcal{B}$, the so-called spectral measure of $(X_{g,\gamma})$. This $\mathcal{P}$ is unique.

Let $\mathfrak{F}$ denote the time domain of the stationary process $(X_{f,\gamma})$, i.e., the closed subspace of $\mathcal{P}^*$ spanned over the elements $X_{\gamma}, \gamma \in G$, with $X \times \gamma$-matrix coefficients.

(3.3) THEOREM (cf. [10], Theorem 3.7). If $(X_{g,\gamma})$ is a $\gamma$-variant stationary process over $G$, with the spectral measure $\mathcal{P}$, then the spaces $\mathfrak{F}$, $\mathfrak{F}$, and $\mathfrak{F}$, and $\mathfrak{F}$ are isomorphic, where

(a) the mapping $V_1: X_{g,\gamma} \rightarrow \gamma G$, by denoting the unit matrix, induces an isomorphism between $\mathfrak{F}$ and $\mathfrak{F}$,

(b) the mapping $V_2: \mathcal{P} \rightarrow \mathcal{S}_\mathcal{P}$, for any matrix-valued function $\mathcal{P}$ with values on the set of measures $\mathcal{M}$ on $\mathcal{B}$, is $\mathcal{P}$-isomorphism between $\mathfrak{F}$ and $\mathfrak{F}$.

Let $g$ be a fixed element of $G$. By $\mathfrak{F}$, we will denote the closed subspace of $\mathcal{P}^*$ spanned by $X_{g,\gamma}, \gamma \neq g$, and $\mathfrak{F}_g = \mathfrak{F} \oplus \mathfrak{F}_g$.

The following theorem in the general case of locally compact Abelian groups is proved in [9], 2.8.

(3.4) THEOREM. Let $(X_{g,\gamma})$ be a $(\gamma)$-variant stationary process over $G$, and $\mathcal{P}$ its spectral measure. Then $V_1 V_2 \mathfrak{F}$ consists of all matrix-valued measures $\mathcal{S}_\mathcal{P}$ from the space $\mathfrak{F}$, where $\mathcal{S}_\mathcal{P}(\mathcal{B}) = \mathbb{A} \mathcal{P}(\mathcal{B})$ for each $E \in \mathcal{E}$, and $\mathcal{A}$ is any $\gamma$-matrix.

Remark. Let $X \in \mathcal{F}$. From the diagram which is presented in [10], p. 175, it follows that, for each $E \in \mathcal{E}$,

(3.5) $\mathcal{P}(\mathcal{B}) = \mathcal{P}(\mathcal{B}) \mathcal{P}(\mathcal{B}) = \mathcal{P}(\mathcal{B}) \mathcal{P}(\mathcal{B})$, where $\mathcal{P}(\mathcal{B}) = V_1 V_2 \mathcal{P}$.

Let $\mathcal{P}$ and $\mathcal{P}$ denote the absolutely continuous and singular parts of $\mathcal{P}$ in the Crème's decomposition (2.9). w.r.t. the Haar measure $d\gamma$ and let $\mathcal{P}$ denote the derivative of $\mathcal{P}$. If we put $\mathcal{P} = V_2 \mathcal{P}$, then, by (3.3) and (3.5), we have

$$\int (X_{g,\gamma}) d\gamma = \mathcal{P}(\mathcal{B}) = \int \mathcal{P}(\mathcal{B}) + \int \mathcal{P}(\mathcal{B}) = \int \mathcal{P}(\mathcal{B}) + \int \mathcal{P}(\mathcal{B}) = \int \mathcal{P}(\mathcal{B}).$$

Since $\mathcal{P}(\mathcal{B}) = \mathcal{P}(\mathcal{B})$, we conclude that

(3.6) $\mathcal{P}(\mathcal{B}) = \mathcal{P}(\mathcal{B})$, $\mathcal{P}(\mathcal{B}) = \mathcal{P}(\mathcal{B})$.

(3.7) LEMMA. Let $F$ be the spectral measure of a $(\gamma)$-variant stationary process $(X_{g,\gamma})$ over $G$ and let $\mathcal{N}_\mathcal{A} = \mathcal{P}(\mathcal{B})$, where $\mathcal{A}$ is any $\gamma$-matrix. Then $\mathcal{N}_\mathcal{A}$ is $\mathfrak{F}$ and if and only if $\mathcal{A} \mathcal{A}(\mathcal{B}) = \mathcal{P}(\mathcal{B})$, o.d. a.e. and the integral $\int \mathcal{A}(\mathcal{B}) d\mathcal{A}(\mathcal{B})$ exists.

Proof. Necessity. Let $\mathcal{N}_\mathcal{A}$ be $\mathfrak{F}$, and by (2.11) there exists a function $\mathcal{A} \mathcal{A}(\mathcal{B})$ such that for each $E \in \mathcal{E}$

$$\mathcal{N}_\mathcal{A}(\mathcal{B}) = \int \mathcal{A}(\mathcal{B}) d\mathcal{A}(\mathcal{B}).$$
Since, for all \( E \in \mathcal{B} \),
\[
\mathcal{A} \mathcal{B}(E) = N_A(E) = \int_E \Phi_A dE = \int \Phi_A dF^* d\gamma + \int \Phi_A dF^d d\gamma,
\]
we conclude that \( \Phi_A F^* = A \) \( \gamma \)-a.e. and \( \Phi_A = 0 \) on \( S \), where \( S = \text{supp} \mathcal{F}^d \). Thus \( \mathcal{A}(A) = \mathcal{A}(\Phi_A F^*) = \mathcal{A}(F^d) \) \( \gamma \)-a.e. Furthermore
\[
\int_E dN_A dN_A^* dE = \int \Phi_A dF^* F_A^* d\gamma = \int \Phi_A dF^d F_A^* d\gamma = \int A (F^d)^* A^* d\gamma
\]
and consequently the integral \( \int A (F^d)^* A^* d\gamma \) exists. The sufficiency is trivial. □

4. The minimality theorem. Let \((X_n)_{n\in\mathbb{N}}\) be a \( \mathcal{G} \)-valued minimal process over discrete abelian group \( G \) and \( F \) its spectral measure. The derivative of the absolutely continuous part of \( F \) in the Cramer's decomposition w.r.t. the Haar measure \( \gamma \) will be called the spectral density of \((X_n)_{n\in\mathbb{N}}\) and will be denoted by \( F^d \).

(4.1) DEFINITION. We say that the stationary process \((X_n)_{n\in\mathbb{N}}\) is minimal if \( X_n \mathcal{E} \mathcal{M} \).

For the proof of the minimality theorem, we need the following lemmas.

(4.2) LEMMA (4.1, 3.2). Let \( E \) be a measurable set and let \( \Phi \) be a non-negative Hermitian matrix-valued function on \( \Omega \) such that \( \Phi \mathcal{A} \mathcal{B}(E) \) exists. Then

(a) for all \( \omega \in \Omega \), \( \mathcal{A}(\int \Phi \mathcal{A} \mathcal{B}(E) \) exists,
(b) for all \( \omega \in \Omega \), \( \mathcal{A}(\Phi(\omega)) = \mathcal{A}(\int \Phi \mathcal{A} \mathcal{B}(E) \)
(c) if \( \forall \omega \in \Omega \), \( \mathcal{A}(\Phi(\omega)) = \mathcal{A}(\int \Phi \mathcal{A} \mathcal{B}(E) \), then \( \mathcal{A}(\Phi(\omega)) = \mathcal{A}(\int \Phi \mathcal{A} \mathcal{B}(E) \).

The following lemma is easy to prove.

(4.3) LEMMA. Let \( \Phi \) be a matrix-valued function on \( \Omega \) and let \( A, B \) be any matrices. Then

(a) the integral \( \int A \Phi \mathcal{A} \mathcal{B}(E) \) exists if and only if the integral \( \int A \int \Phi \mathcal{A} \mathcal{B}(E) \) exists; if these integrals exist, then
\[
\int A \Phi \mathcal{A} \mathcal{B}(E) = A \int \Phi \mathcal{A} \mathcal{B}(E),
\]
(b) the integral \( \int \Phi \mathcal{A} \mathcal{B}(E) \) exists if and only if the integral \( \int \Phi \mathcal{A} \mathcal{B}(E) \) exists; if these integrals exist, then
\[
\int \Phi \mathcal{A} \mathcal{B}(E) = \int \Phi \mathcal{A} \mathcal{B}(E).
\]

(4.4) LEMMA (cf. [10], p. 151). If \( \Phi \) is a non-negative Hermitian matrix-valued function, then the integral \( \int \Phi \mathcal{A} \mathcal{B}(E) \) exists if and only if the integral \( \int \Phi \mathcal{A} \mathcal{B}(E) \) exists.

Let \( Y_n \) be the orthogonal projection of \( X_n \) onto \( \mathcal{M}_n \). The letter \( J \) will denote the orthogonal projection matrix onto the range of \((Y_n, X_n)\). Now, we prove the following theorem which gives a characterization of the space \( \mathcal{A}(\{Y_n, X_n\}) \).

(4.5) THEOREM. Let \((X_n)_{n\in\mathbb{N}}\) be a \( \mathcal{G} \)-valued stationary process over discrete abelian group \( G \) with the spectral measure \( F \). The range \( \mathcal{A}(\{Y_n, X_n\}) \) is a maximal closed linear subspace \( \mathcal{A} \) of \( \mathcal{G} \) satisfying the following conditions:

(a) the integral \( \int A \mathcal{A} \mathcal{B}(E) \) exists,
(b) \( \mathcal{A} \subset \mathcal{A}(F^d) \) \( \gamma \)-a.e.,
(c) \( \mathcal{A}(\int \Phi \mathcal{A} \mathcal{B}(E) \) exists.

Proof. Let \( M = V, V, Y_n \). By (3.5), \( M(F) = \{X, Y\}d\gamma \mathcal{A}(E) = \{X, Y\}d\gamma \mathcal{A}(E) \) for all \( E \in \mathcal{B} \). Furthermore, by (3.7), \( \mathcal{A}(\{Y_n, X_n\}) = \mathcal{A}(F^d) \) \( \gamma \)-a.e. and the integral \( \int J(F^d)^* Jd\gamma \) exists, \( J = \mathcal{A}(F^d) \). These facts in combination with (2.1)–(2.3) imply
\[
\{Y_n, X_n\}d\gamma = \{X, Y\}d\gamma \mathcal{A}(E) = \{X, Y\}d\gamma \mathcal{A}(E) \mathcal{A}(F^d)^* Jd\gamma = \{X, Y\}d\gamma Jd\gamma \mathcal{A}(F^d)^* Jd\gamma = J(F^d)^* Jd\gamma.
\]

Consequently, by (2.7), \( \mathcal{A}(\{Y_n, X_n\}) = \mathcal{A}(J(F^d)^* Jd\gamma) \). Thus, conditions (a), (b), (c) hold for \( \mathcal{A} = \mathcal{A}(\{Y_n, X_n\}) \).

Let \( \mathcal{A} \) be a closed linear subspace of \( \mathcal{G} \) satisfying conditions (a), (b), (c). We shall show that \( \mathcal{A} = \mathcal{A}(\{Y_n, X_n\}) \). Put \( B = \mathcal{A}(F^d)^* Jd\gamma \). By (3.4) and (3.7), \( X_n = V, V, Y_n \) where \( X_n(F) = Bd\gamma(E) \) for all \( E \in \mathcal{B} \). Let \( Y \) be an element of \( \mathcal{M}_n \) such that \( Y_n = V, V, Y_n \); then, by (3.5), \( B = \{X, Y\} \). Since \( X_n = Y_n = Y, D \) for any matrix \( D \). Hence
\[
\mathcal{A}(\{Y_n, X_n\}) = \mathcal{A}(\{X_n, D\}) = \mathcal{A}(B) = \mathcal{A}.
\]

Now we state the main result of this paper.

(4.6) THEOREM. The following properties of \( \mathcal{G} \)-valued stationary process \((X_n)_{n\in\mathbb{N}}\) over a discrete abelian group \( G \) are equivalent:

(a) the process \((X_n)_{n\in\mathbb{N}}\) is minimal,
(b) the Hellinger integral \( \int dX_n dX_n = \mathcal{A}(F^d) \) exists and is non-zero, where for each \( E \in \mathcal{B}, N(E) = Jd\gamma(E) \).
(c) there exists a linear closed subspace \( M \subset \mathcal{H}(F) \) dy-a.e. such that the integral \( \int F(x)\,d\mu_x \) exists and is non-zero,

(d) there exists a matrix \( A \) with \( \mathcal{A}(A) \subset \mathcal{H}(F) \) dy-a.e. such that the integral \( \int \text{tr}(A(F)^{\tau}A^*) \,dy \) exists and is non-zero.

**Proof.** First of all we remark that in view of (4.4) we may use also the trace in the integrals in conditions (b) and (c).

(a) = (b). Let \( (X_0)_{0<\gamma} \) be minimal. If we denote \( M = V_1^*Y_1X_1 \), then by assumption and by (3.3), \( (M, M)_F = (Y_1, X_1) \neq 0 \). Using the same argument as that in the proof of (4.5), we obtain

\[
(Y_1, X_1)_{F} = \int J(F)^{\tau}d\gamma = \int dN_1'dN_1/dF.
\]

Hence, the integral \( \int dN_1'dN_1/dF \) exists and is non-zero.

(b) = (c). Suppose that \( \int dN_1'dN_1/dF \) exists and is non-zero. Put \( \mathcal{A} = \mathcal{A}(Y, X) \). From (4.5), \( \mathcal{A} = \mathcal{A}(Y, X) \subset \mathcal{H}(F) \) dy-a.e. Since \( \int F(x)\,d\mu_x \) is the first integral exists and is non-zero.

(c) = (d). Let us suppose that (c) holds. If we put \( A = P_{S_1} \), then it is obvious that the integral \( \int \text{tr}(A(F)^{\tau}A^*) \,dy \) exists and \( \mathcal{A}(A) = \mathcal{A} \subset \mathcal{H}(F) \) dy-a.e. Since \( F \) is the non-negative Hermitian function, then, by (2.7), \( A(F)^{\tau}A^* \) is also non-negative and Hermitian. By the inequality \( 0 \leq \mathcal{A}(F) = (\text{tr}(F))I \), it is satisfied for any non-negative and Hermitian matrix, we deduce that the integral \( \int \text{tr}(A(F)^{\tau}A^*) \,dy \) exists and is non-zero.

(4.8) **COROLLARY** (Kolmogorov's minimality theorem). Let \( (X_n)_{0<\gamma} \) be a univariate stationary process over the group \( Z \) of integers, and \( F \) its spectral density. Then \( (X_n)_{0<\gamma} \) is minimal if and only if \( \int \text{tr}(F)^{\tau}F \,dy < \infty \).

**Proof.** This assertion is an easy consequence of (4.6) and the fact that \( \mathcal{A} \) is a closed linear subspace of \( G \) if and only if \( \mathcal{A} = (0) \) or \( \mathcal{A} = G \). 

(4.9) **COROLLARY** (Masani's minimality theorem). Let \( (X_0)_{0<\gamma} \) be a g-variate stationary process over discrete Abelian group \( G \) and \( F \) its spectral density. Then \( (X_0)_{0<\gamma} \) is minimal if and only if \( \int \text{tr}(F)^{\tau}F \,dy < \infty \).

**Proof.** Let \( (X_0)_{0<\gamma} \) be minimal and rank \( (Y_1, X_1) \). Then by the proof of (4.5), \( \mathcal{A}(Y, X) \subset \mathcal{H}(F) \) dy-a.e. and the integral \( \int \text{tr}(F)^{\tau}F \,dy \) exists. Since \( \mathcal{A}(Y, X) = G \), it follows that \( (F)^{-1} \) exists dy-a.e. and \( (F)^{-1} \) is \( L_1 \).

Conversely, let \( (F)^{-1} \) exist dy-a.e. and \( (F)^{-1} \) is \( L_1 \). By (4.5) and (4.2), \( \mathcal{A}(Y, X) = G \). Hence the process \( (X_0)_{0<\gamma} \) is minimal and rank \( (Y_1, X_1) = q \).

Finally we give the formulas for the linear interpolation. Let \( (X_0)_{0<\gamma} \) be a g-variate stationary process over discrete Abelian group \( G \). Suppose that all \( X_n \) for \( n \in g \) are known, \( g \) is a fixed element of \( G \). We say that \( X_n^* \) is a prediction of \( X_n \) based on observations from the complement of the element \( g \) if \( X_n^* \cap R \) and

\[
\|X_n - X_n^*\| = \min_{x \in R} \|X_n - x\|.
\]

It follows that \( X_n^* \) is the projection of \( X_n \) onto \( R \). We note that the closed subspace of \( \mathcal{H}(F) \) spanned by \( X_n - X_n^* \) is exactly the space \( R \) which was defined in Section 3. Thus, \( X_n^* = X_n - Y_n \), where, as before, \( Y_n \) denotes the orthogonal projection of \( X_n \) onto \( R \). If we put \( M_n^* = V_n^*X_n^*, M_n = V_n^*X_n \), then, from (4.7) and (4.8),

\[
(Y_n, X_n) = \left\{ \int J(F)^{\tau}F \,dy \right\}^*.
\]

Consequently, by (3.3), (3.5), (3.6), we have the following formulas for the prediction:

\[
M_n^*(F) = F(E) - \left\{ \int J(F)^{\tau}F \,dy \right\}^*d\mu(E), \quad E \in \mathcal{B},
\]

\[
\Phi_n^*(y) = I - \left\{ \int J(F)^{\tau}F \,dy \right\} \Phi(F)'(y)^* (\text{mod} F),
\]

where \( J \) denotes the orthogonal projection matrix onto the range of \( (Y_n, X_n) \).

It is well known that if \( \{U_{n} \}_{n>0} \) is a shift groups of unitary operators on \( \mathcal{H}(F) \) defined by equality \( U_{n} X_n = X_{n+1}, g \in G \), then for each \( g \in G \) and each \( x \in \mathcal{H}(F) \),

\[ x = \sum_{n=-\infty}^{\infty} \langle x, U_{n} X_n \rangle U_{n} X_n. \]
\[ Y \in \mathcal{B}, \quad V_i U_{i, Y} = \langle \xi, Y \rangle V_i Y. \] Consequently, from the obviously equality \( X_0 = U_{i, Y}^k \), we obtain
\begin{align*}
\Phi^i_C(\gamma) &= \langle \xi, \gamma \rangle \cdot \Phi^i_C(\gamma) \quad \text{(mod } \mathcal{F}), \\
M^i_C(\varphi) &= \int \langle \xi, \gamma \rangle \cdot dM^i_C.
\end{align*}

Let us note that the space \( \mathcal{M}(\{X_i, Y_i\}) \) used in these formulas has the characterization given in (4.5).

5. Examples. In this section we will give several examples illustrating the above results.

(5.1) Example. Let \( M, N \) be the real-valued measures on the Borel field of \([0, 2\pi] \), absolutely continuous w.r.t. the Haar measure \( dt \), and let their derivatives be given by
\[ M'(t) = I_{[0,\pi]}(t), \quad N'(t) = I_{[0,\pi]}(t), \quad t \in [0, 2\pi], \]
where \( I_A(t) \) denotes the indicator of \( A \). Obviously, the integral \( \int dM M' \cdot dN \) exists. But it does not exist a function \( \Phi \) such that for each measurable \( E, M(E) = \int \Phi(t) dN'(t). \). Therefore the assumption \( \mathcal{M} \) is strongly absolutely continuous w.r.t. \( N \) in (2.11) is essential one.

(5.2) Example. Let \( X_{n, m, Z} \) and \( Y_{n, m, Z} \) be mutually orthogonal stationary processes over the group \( Z \), of the integers, with the absolutely continuous spectral measures \( F_1 \) and \( F_2 \). Let \( F_1(t) = I_{[0,\pi]}(t), F_2(t) = I_{[\pi,2\pi]}(t), t \in [0, 2\pi]. \) Then \( W_n = [X_n, Y_n], n \in Z \), is a bivariate stationary process over \( Z \) with the spectral density given by
\[ F'(t) = \begin{bmatrix} I_{[0,\pi]}(t) & 0 \\
0 & I_{[\pi,2\pi]}(t) \end{bmatrix}, \quad t \in [0, 2\pi]. \]

From the univariate Kolmogorov's minimality theorem we have that the processes \( X_{n, m, Z} \) and \( Y_{n, m, Z} \) are not minimal, and therefore the process \( W_{n, m, Z} \) is not minimal. We note that
\[ (F')^{-1}(t) = \begin{bmatrix} I_{[0,\pi]}(t) & 0 \\
0 & I_{[\pi,2\pi]}(t) \end{bmatrix}, \quad t \in [0, 2\pi]. \]

According to formulas (4.10) and (4.11) we obtain:
\begin{align*}
(dM_1/dt)(t) &= \begin{bmatrix} 1 & 0 \\
0 & 0 \end{bmatrix}, \quad t \in [0, 2\pi], \\
\varphi_0^i(t) &= \begin{bmatrix} 1 & 0 \\
0 & 0 \end{bmatrix}, \quad t \in [0, 2\pi].
\end{align*}

References

The range of vector measures into Orlicz spaces

by

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Abstract. It is shown that the range of a σ-additive vector measure having values in an Orlicz space \( L_\phi(X, A, \mu) \), where \( \phi \) is unbounded and satisfies condition \( A_2 \), is bounded. This implies that every scalar-valued, bounded measurable function can be integrated with respect to any vector measure taking values in such a space \( L_\phi(X, A, \mu) \). In the special case of the sequence spaces \( \mathcal{F} \), \( 0 < p < 1 \), the range is relatively compact, and the closure is even convex and compact if the measure is nonatomic.

1. It is known that the range of every σ-additive vector measure with values in a locally pseudoconvex vector space is bounded (cf. [1]). On the other hand P. Turpin has shown in [11] that there exists a non-locally pseudoconvex E-space and a vector measure having unbounded range in that space. With regard to integration theory it would be important to know whether a vector measure has always bounded range in an Orlicz space \( L_\phi(X, A, \mu) \) (cf. [5]). P. Turpin states this question in [9] and [11].

In this note we answer the question positively for the class of Orlicz spaces \( L_\phi(X, A, \mu) \), where \( \phi \) is unbounded and satisfies condition \( A_2 \). It is done by showing that every normbounded, convex set in \( L_\phi(X, A, \mu) \) is bounded and then using the fact that the convex hull of the range of such a vector measure is normbounded. The latter follows from an inequality for Orlicz spaces, which is essential for the proof that in these spaces unconditonal convergence is equivalent to bounded multiplier convergence ([4], [10]).

As a consequence every scalar-valued, bounded measurable function can be integrated with respect to any vector measure taking values in such a space \( L_\phi(X, A, \mu) \).

In the special case of the sequence spaces \( \mathcal{F} \), \( 0 < p < 1 \), the range is even relatively compact. When such a vector measure is also nonatomic, the closure of its range is compact and convex.

2. Throughout the paper, \( \mathcal{O} \) will denote a set and \( \Sigma \) a σ-algebra of subsets. Let \( Y \) be an E-space (i.e. a complete metric topological linear