Well embedded Hilbert subspaces in $C^*$-algebras

by

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Abstract. A Hilbert subspace $E$ of a normed linear space $X$ is a well embedded Hilbert subspace of $X$ if there exists a linear subspace $Z \subset X$ such that $X = E \oplus Z$ and that the linear operator $U \oplus I_Z$ (where $I_Z$ is the identity operator on $Z$) is an isometry for every unitary operator $U$ acting on $X$. We characterize such subspaces in $C^*$-algebras.

1. Introduction. Recent advances in study of the structure of finite dimensional Banach spaces have brought the definition of a well embedded Hilbert subspace ([6], Definition 1 below). Let e.g. $X$ be a finite dimensional complex Banach space with such a basis that the norm of any vector in $X$ depends only on the absolute values of its components in this basis. Then $X$ is a direct sum of well embedded Hilbert subspaces ([8] and [6]). We characterize completely well embedded Hilbert subspaces in $C^*$-algebras with identity, showing that they are rather “uncommon” in a sense explained below.

2. Definitions and preliminary results. Let $X$ be a normed linear space (real or complex) and $Y$ a linear subspace in $X$. We say that $Y$ is a Hilbert subspace of $X$ if it is a Hilbert space in the norm it inherits from the space $X$. A vector $x \in X$ is orthogonal to a vector $y \in X$ if $\|x + ay\| = \|x\|$ for all scalars $a$.

**Definition 1.** A Hilbert subspace $Y$ of a normed linear space $X$ is a well embedded Hilbert subspace of $X$ if there exists a linear subspace $Z \subset X$ such that $Y \oplus Z = X$ and that the linear operator $U \oplus I_Z$ (where $I_Z$ is the identity operator on $Z$) is an isometry for every unitary operator $U$ acting on $X$.

**Remark.** If $X$ is a Hilbert space, every closed subspace $Y$ in $X$ is a well embedded Hilbert subspace. In this case the space $Z$ is the orthocomplement of the space $Y$.

**Lemma 1.** Let $Y$ and $Z$ be as in Definition 1, $y \in Y$, and $z \in Z$. Then $y$ and $z$ are mutually orthogonal.

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Proof. Set $U = -J_Y$ in Definition 1. Thus $y$ and $z$ are eigenvectors corresponding to distinct eigenvalues of a linear isometry. Apply [2], Corollary 3.

Our approach to the problem is based upon Kadison's characterization of linear isometries between $C^*$-algebras with identities. We will use the following (weaker) result:

Let $A$ be a $C^*$-algebra with identity $1$ and $T$ a linear isometry of $A$ onto itself. Then $u = T1$ is a unitary element in $A$ and $f = u^*T$ preserves the self-adjoints.

A nice elementary proof of Kadison's result is given in [3].

A non-zero idempotent $g$ in a Banach algebra $A$ is called minimal if the algebra $gA$ is one-dimensional.

3. Well embedded Hilbert subspaces in $C^*$-algebras.

Theorem 1. Every well embedded Hilbert subspace in a $C^*$-algebra with identity is necessarily one-dimensional and is spanned by a central minimal projection.

Conversely, every central minimal projection in a $C^*$-algebra $A$ spans a one-dimensional well embedded Hilbert subspace of $A$.

Proof. We prove the second statement first. Let $y \in A$ be a central minimal projection. Set $X = \{x | x = xy \}$ and $Z = \{x | xy = 0 \}$. Clearly, $X = X \cap Z$. If $x \in A$, then $xy = yx = yx = ty$ for some scalar $t$. Hence $y$ is one-dimensional. In dimension 1 a unitary operator means multiplication by a complex number of modulus 1. Let $x \in Z$. Then

$$|\|y| + z|^2 = |\|x^*y + xz\||^2 = |\|y + z\||^2,$$

for all real $t$ and the proof is complete.

We return to the first statement. Let $Y'$ be a well embedded Hilbert subspace in a $C^*$-algebra $A$ with identity 1. Clearly, every closed linear subspace of $Y'$ is also a well embedded Hilbert subspace of $A$. We may assume that $\dim A > 1$.

Let $Y$ be an arbitrary one-dimensional linear subspace of $X'$ and $Z$ a complementary subspace to $X$ such that $X$ and $Z$ satisfy the requirements of Definition 1. For every real $t$, let $T_t: A \to A$ be the linear operator:

$$T_t(y + x) = e^{ty} + x \quad (y \in Y, x \in Z).$$

The set $\{T_t\}$ is a one-parameter group of isometries. Write, as before, $u_t = T1$ and $f_t = u_t^*T_t$. We prove that $1 \in Y'$ and $1 \in Z$. If $1 \in Y'$, then $u_t = T1 = e^{t1}$. Let $x \in Z$ be an arbitrary non-zero element. We choose the scalar $s$ such that $s^* - ay \in Z$ and compute

$$f_t(s^*) = u_t^*T_t(s^* - ay) = e^{-t}(e^{ty} + s^* - ay) = ay + e^{-t}(s^* - ay).$$

On the other hand,

$$f_t(\bar{s}^*) = f_t(s^*) = (u_t^*T_t\bar{s})^* = (u_t^*\bar{s})^* = e^{t\bar{s}}$$

for all real $t$. This implies readily that $s^* = 0$, a contradiction. If $1 \in Z$, then $u_t = T1 = 1$ and a similar argument leads to $Y = \{0\}$, a contradiction. Thus $1 \in Y'$ and $1 - y \in Z$ for some non-zero $y \in X$. Since $u_t = T1 = e^{ty} + 1 - y$ and

$$1 = u_t u_t^* = 1 + y - y^* + e^{ty}(y - y^*) + e^{ty}(y - y^*),$$

for all real $t$, we see immediately that $y = y^* = y^*$. Once more, let $z \in Z$ be arbitrary. Choose $a \in C$ so that $z^* - ay \in Z$. Then

$$f_t(s^*) = u_t^*T_t(ay + s^* - ay) = (e^{-ty} + 1 - y)(a + z - s) - ay = z^* + (ay - y^*) + e^{ty}(y - y^*).$$

Also

$$f_t(\bar{s}^*) = f_t(s^*) = (u_t^*\bar{s})^* = z^* - z^*y + e^{t\bar{s}}y.$$

Comparing the two results we see that $z^*y = 0 = ay - y^*$. But $ay = (ay)y = (y^*)y = y(sty) = 0$. Thus $y^* = 0 = y - ay$. This proves that $y$ is central and minimal.

Suppose now that $\dim Y' \geq 2$ (see the beginning of the proof). This follows from our results that there exist at least two linearly independent central minimal projections, say $Y, y', y''$. Since $y'y = y'yy = y'y''$ and both $y, y'$ are minimal, it follows that $yy' = 0$. Thus $\|y + y'\|^2 = \|y + y''\|^2 = \|y + y'\|^2$ and so $\|y + y'\| = \|y - y'\| = 1$. The parallelogram law fails.

Remark. Using the generalization in [4] of Kadison's theorem to arbitrary $C^*$-algebras, it is possible to prove Theorem 1 in the absence of an identity in a $C^*$-algebra, too. However, the only proof the author knows of is rather complicated (although elementary) and consequently it will not be published here.

References

Abstract. A complete description of non full rank in general \( q \)-variate minimal stationary processes over discrete Abelian groups are given. This result subsumes the minimality theorems of various authors in special cases.

1. Introduction. In his fundamental paper [1] A. N. Kolmogorov introduced the important concept of minimal processes. Next the concept have been extended to the \( q \)-variate case (cf. [3] and [6], Section 10). The interpolation problem for \( q \)-variate stationary processes over groups was studied by H. Salehi and J. K. Scheidt [8] and by A. Weron [9], [10]. Furthermore in those papers characterizations of \( q \)-variate minimal processes are also given. In [8] a generalization of Masani's minimality theorem for full rank processes is obtained. Two characterizations of non-full rank processes are given in [10], but unfortunately one of which ((10), Theorem 5.7) contains an error. In this paper a counter example for this (see Example 5.3) and a correct statement of this theorem (see Theorem 4.6(d)) is given. Moreover, we will get a general theorem on characterizations of \( q \)-variate minimal (not necessary full rank) processes.

Section 2 is devoted to the preliminary results on the spaces \( L_{\alpha,F} \) — of square integrable matrix-valued functions and \( H_{\alpha,F} \) — of Helfinger square integrable matrix-valued measures. Section 3 treats on \( q \)-variate stationary processes over a discrete Abelian group. Using methods of the earlier work [10] on stationary processes over locally compact Abelian (LCA) groups, we obtain an analytical characterization of a subspace \( Y_q \) which is important in the minimality problem. In Section 4 we discuss the minimality problem and give some characterizations of minimal processes. As a corollary we then deduce Kolmogorov's and Masani's minimality theorems. Finally in Section 5 we give several examples to show that conditions in the presented theorems are essential ones as well as to illustrate them.

2. \( L_{\alpha,F} \) and \( H_{\alpha,F} \) spaces. Let \( \mathcal{B} \) be a \( \sigma \)-algebra of subsets of a space \( \Omega \) and let \( \Phi = \{ \phi_{ij}, 1 \leq i, j \leq q \} \) be a matrix-valued function on \( \Omega \). Throughout this paper all matrices have complex entries and \( \mathbb{C} \) denotes...