Extension of real-valued $\sigma$-additive set functions

by

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Abstract. The extension of real-valued $\sigma$-additive finite finitely additive regular real-valued set functions from an algebra of sets to larger $\sigma$-algebras of sets is given. The extensions are then used to obtain results on $\sigma(A^*, A)$ convergence of $\tau$-additive functionals on an algebra $A$ of real-valued functions on a set $X$.

Introduction. Let $A$ be a uniformly closed algebra of bounded real-valued functions on a set $X$ which separates the points of $X$ and contains the constants. Let $X$ be equipped with the $\tau_0$ topology which is the weakest topology on $X$ which makes each $f \in A$ continuous. In [4] the concept of $\sigma$-additive set functions on a paving $\mathcal{W}$ of subsets of $X$ was introduced to represent the $\sigma$-additive functionals in $A^*$. It was also observed that the $\sigma$-additive set functions could be extended to $\sigma$-additive elements on a larger paving (this includes the fact that $\tau$-additive Baire measures in $\mathcal{C}(X)$ can be extended to Borel measures on $X$). We shall establish this extension process which depends on which definition of outer measure is chosen. We then employ the extension to questions about weak, $\sigma(A^*, A)$, convergence of elements in $A^*$. We anticipate that working with a paving and that working with subalgebras of $\mathcal{C}(X)$ will prove useful in probability theory, and in this direction we obtain a weakened form of Prochorov’s theorem. Also for subalgebras $A_0 = A$, we give sufficient conditions for weak convergence of $\tau$-additive $\varphi$ in $A^*_0$ to be determined by the elements of $A_0$.

The authors wish to thank the referee for pointing out that our results in Section 1 should extend to exhaustive functions with range a suitably endowed topological group. He also noted some of the rich literature on the subject such as done by Drewnowski [2], Sion [8] and Trayner [7]. The referee is of course correct and the authors intend to show this and that the weak additivity condition does yield the usual additivity condition in a different paper.

§ 1. Extension. We refer the reader to [4] for many of the basic definitions and results; however, we shall indicate here some of the essential definitions.

A paving on $X$ is a family $\mathcal{W}$ of subsets which contains $\emptyset$, is closed under finite unions and intersections, and has $X = \bigcup \mathcal{W}$. The paving is full if $X \in \mathcal{W}$ and in this paper all pavings will be assumed to be full.
Let $\mathcal{F}(\mathcal{W})$ be the algebra of subsets of $X$ generated by $\mathcal{W}$, then $M(\mathcal{F})$ will denote the set of all finite, finitely additive real-valued set functions on $\mathcal{F}$ which are regular in the sense that for each $F \in \mathcal{F}$ there is a $W \in \mathcal{W}$ such that $W \subseteq F$ and $|m(G)| \leq \varepsilon$ whenever $G \in \mathcal{F}(\mathcal{W})$ with $G \subseteq F - W$.

For an infinite cardinal $a$, we say that $m \in M(\mathcal{W})$ is $a$-additive if $\sup \{ |m(W_i)| : i \in I \} = 0$ for every collection $\{ W_i : i \in I \}$ which is directed downward to $0$ with card $I \leq a$. The set of $a$-additive elements will be denoted by $M_a(\mathcal{W})$ (or $M_a(\mathcal{W}_a)$) and $\tau$ will denote the least cardinal such that $M_\tau = M_\tau(\mathcal{W})$ when $\tau < \beta$. Finally, $m \in M(\mathcal{W})$ is $a$-singular if there is a family $\{ W_i : i \in I \}$ which is directed downward to $0$ with card $I \leq a$ and such that $|m(V)| = |m(V \cap W_i)|$ for all $V \in \mathcal{W}$ and all $i \in I$.

For the extension process to develop adequately it is essential we choose the proper definition of outer measure; we now give this and remark that if $X = [0, 1]$ and if $m$ is the Lebesgue measure on the Borel sets $\mathcal{W}$, then $m \in M(\mathcal{W})$, and $m$ agrees with the usual outer measure and the extension process yields the Lebesgue measure.

**Definition 1.1.** Let $\mathcal{W}$ be a full paving on $X$ and let $m \in M^*(\mathcal{W})$.

For $A \subseteq X$,

$$m^*_a(A) = \inf \{ \sup m(W) : I \subseteq \mathcal{W}, \text{ I directed downward,} \}$$

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**Lemma 1.2.** Let $\mathcal{W}$ be a paving and let $m \in M^*(\mathcal{W})$. Then $m^*_a$ is an outer measure on $X$.

**Proof.** It is clear that $m^*_a$ is monotone and non-negative. Let $A = \bigcup A_i$, and fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, let $I_n \subseteq \mathcal{W}$ be downward directed with card $I_n \leq a$ such that

$$m^*_a(A_i) + \varepsilon 2^{-n} > \sup \{ m(W) : W \in I_n \}.$$ 

Let $I$ denote the family of all finite intersections of members of $\bigcup I_n : n \in \mathbb{N}$. Then $I$ is directed downward with card $I \leq a$ and $A = \bigcup \{ W : W \in I \}$.

Hence it follows that

$$m^*_a(A) \leq \sup \{ m(W) : W \in I \}$$

$$\leq \sup \{ \sum_{i=1}^{m} |W_i| : W \subseteq \bigcup I_n, i = 1, \ldots, m \}$$

$$\leq \sum_{i=1}^{m} \sup \{ |W_i| : W \subseteq \bigcup I_n, i = 1, \ldots, m \}$$

$$\leq \sum_{i=1}^{m} m^*_a(A_i) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete.
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(4) Let $W_a, W_b, W_c$. Then

$m_a(W_a - W_b) = \sup \{m_a(W): W \subseteq W_a, W_b \subseteq W_c \}.$

Fix $\epsilon > 0$. By (3) there is a $W_{a-b} \subseteq W_a$ and $m_a(W_{a-b}) < m_a(W_a) + \epsilon$. Let $W = W_a \cap W_{a-b}$. Then $W \subseteq W_a, W_{a-b} \subseteq W_c$.

Since $m_a(W_{a-b}) = m_a(W_a) - W = W_{a-b} - W$, it follows that $0 \leq m_a(W_{a-b}) - W < m_a(W) - m_a(W_{a-b}) + \epsilon$. Since $\epsilon > 0$ was arbitrary, (4) follows.

(5) $m_a$ is $\mathcal{S}$-regular on $\mathcal{S}(\mathcal{W}_a)$.

This is immediate from (4) and Proposition [3, 1.2]. The proof of Lemma 1.4 is now complete.

**Lemma 1.5.** Let $\mathcal{G}$ be a full paving and let $m(\{M(\mathcal{W}_a)\})$. Then $m_a = 0$.

Proof. By [4, 4.4] $M(\mathcal{W}_a)$ is a band so there is an increasing net $(m_\alpha)$ of $a$-singular elements of $M(\mathcal{W})$ with $0 \leq m_\alpha$. Since $(m_\alpha)$ is increasing, it is easy to verify that $m_\alpha(F) \uparrow m(F)$ for all $F \in \mathcal{S}(\mathcal{W})$. Fix $\epsilon > 0$ and take $m$ with $m(X) < m(X) + \epsilon$. Let $I \subseteq \mathcal{W}$ be an a-system with $m_\alpha(X) = m_a(W)$ for all $W \subseteq I$. Since $0 \leq m_m - m_\alpha$, it follows that, for each $F \in \mathcal{S}(\mathcal{W})$, $0 \leq m(F) - m_\alpha(F) = m_\alpha(X) - m_\alpha(X) < \epsilon$. Hence

$$0 \leq m_\alpha(X) \leq \sup_{W \subseteq I} m(W) \leq \epsilon + \lim_{W \subseteq I} m_\alpha(W) = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $m_\alpha(X) = 0$, and so $m_a = 0$.

**Proposition 1.6.** Let $\mathcal{G}$ be a full paving, and let $m(\mathcal{M}(\mathcal{S}))$. Assume that $m = m_m + m_a$ where $m_a(\mathcal{M}(\mathcal{S}))$ and $m_a(\mathcal{S}(\mathcal{W}_a))$. Then $m_a$ restricted to $\mathcal{S}(\mathcal{W}_a)$ belongs to $\mathcal{M}(\mathcal{S}(\mathcal{W}_a))$ and $m_\alpha(\mathcal{M}(\mathcal{S}))$. Hence, in particular,

$m_a = m_a$ on $\mathcal{S}(\mathcal{W}_a)$.

Proof. It is easily shown that $m_a = 0(\mathcal{M}(\mathcal{W}_a))$. The result is now immediate from Lemmas 1.4 and 1.5.

**Proposition 1.7.** Let $\mathcal{G}$ be a full paving, and let $m(\mathcal{M}(\mathcal{S}))$. Then there is a unique element $\mu \in \mathcal{M}(\mathcal{S}(\mathcal{W}_a))$ whose restriction to $\mathcal{S}(\mathcal{W})$ is $m$. In fact, if $\lambda \in \mathcal{M}(\mathcal{W}_a)$ is any element whose restriction to $\mathcal{S}(\mathcal{W})$ is $m$, then

$$\mu = \lambda.$$

Proof. Let $\mu$ denote the restriction of $m_a$ to $\mathcal{S}(\mathcal{W}_a)$. By Proposition 1.6, $\mu(\mathcal{M}(\mathcal{W}_a))$ and $\mu = m$ on $\mathcal{S}(\mathcal{W})$. Now let $\lambda(\mathcal{M}(\mathcal{W}_a))$ and assume that $\lambda = m$ on $\mathcal{S}(\mathcal{W})$. Fix $W \subseteq \mathcal{W}_a$. Let $I \subseteq \mathcal{W}$ be downward directed with $\{\mathcal{M}(\mathcal{S}(\mathcal{W}_a))\}$. Then $0 < \lambda(W) < \inf_{U \subseteq I} \lambda(U) = \inf_{U \subseteq I} \mu(U)$. Then $W \subseteq \mathcal{M}(\mathcal{W}_a)$ was arbitrary, and $\mathcal{M}(\mathcal{W}_a)$ is a band, it follows that $\lambda(\mathcal{M}(\mathcal{W}_a))$. Hence $\lambda(W) = \inf_{U \subseteq I} \lambda(U) = \inf_{U \subseteq I} \mu(U) = \mu(W)$. Thus $\lambda(W) = \mu(W)$ for all $W \subseteq \mathcal{W}_a$. Since $\mathcal{W}_a$ is a band, it follows that $\lambda(\mathcal{W}_a) = \mu(\mathcal{W}_a)$. Since $\lambda(\mathcal{W}_a)$ is an ideal, it follows that $\lambda(\mathcal{M}(\mathcal{W}_a))$. Hence $\lambda(W) = \inf_{U \subseteq I} \lambda(U) = \inf_{U \subseteq I} \mu(U) = \mu(W)$. Thus $\lambda(W) = \mu(W)$ for all $W \subseteq \mathcal{W}_a$. Since $\mathcal{W}_a$ is a band, it follows that $\lambda(\mathcal{W}_a) = \mu(\mathcal{W}_a)$. Since $\lambda(\mathcal{W}_a)$ is an ideal, it follows that $\lambda(\mathcal{W}_a) = \mu(\mathcal{W}_a)$. Hence $\lambda(W) = \inf_{U \subseteq I} \lambda(U) = \inf_{U \subseteq I} \mu(U) = \mu(W)$. Thus $\lambda(W) = \mu(W)$ for all $W \subseteq \mathcal{W}_a$. Since $\mathcal{W}_a$ is a band, it follows that $\lambda(\mathcal{W}_a) = \mu(\mathcal{W}_a)$. Since $\lambda(\mathcal{W}_a)$ is an ideal, it follows that $\lambda(\mathcal{W}_a) = \mu(\mathcal{W}_a)$. Hence $\lambda(W) = \inf_{U \subseteq I} \lambda(U) = \inf_{U \subseteq I} \mu(U) = \mu(W)$. Thus $\lambda(W) = \mu(W)$ for all $W \subseteq \mathcal{W}_a$. Since $\mathcal{W}_a$ is a band, it follows that $\lambda(\mathcal{W}_a) = \mu(\mathcal{W}_a)$.
Define a map \( T_\mu \) from \( M_\mu(\mathcal{W}) \) into \( M_\mu(\mathcal{W}) \) as follows: (We continue to assume that \( \mathcal{W} \) is a full paving.) For \( m \in M_\mu(\mathcal{W}) \), let \( T_\mu(m) \) denote the restriction of \( m \) to \( R(\mathcal{W}) \). Then \( T_\mu(m) = M_\mu(\mathcal{W}) \). By Proposition 1.7. For arbitrary \( m \in M_\mu(\mathcal{W}) \), define \( T_\mu(m) = T_\mu(m-\mu) \).

**Theorem 1.8.** Let \( \mathcal{W} \) be a full paving. Then the map \( T_\mu \) is a Banach lattice isomorphism of \( M_\mu(\mathcal{W}) \) onto \( M_\mu(\mathcal{W}) \). Furthermore, for \( \mu_2 \in M_\mu(\mathcal{W}) \), \( T_\mu(\mu_2) \) is the restriction of \( \mu_2 \) to \( \mathcal{W} \).

**Proof.** It is easy to verify that \( T_\mu \) is linear on \( M_\mu(\mathcal{W}) \). From this it is immediate that \( T_\mu \) is a positive linear transformation on \( M_\mu(\mathcal{W}) \). Let \( \mu \in M_\mu(\mathcal{W}) \) with \( T_\mu(\mu) = 0 \), then \( T_\mu(m) = 0 \) for all \( m \in R(\mathcal{W}) \).

In order to verify that \( T_\mu \) is norm-preserving, it is sufficient to show that \( \|T_\mu(m)\| = \|m\| \) for all \( m \in M_\mu(\mathcal{W}) \). Since \( T_\mu \) is a positive transformation, \( \|T_\mu(m)\| \leq \|m\| \) for all \( m \in M_\mu(\mathcal{W}) \). Let \( m \) denote the restriction of \( T_\mu(m) \) to \( \mathcal{W} \). Then \( \|m\| \leq \|T_\mu(m)\| \).

If \( \mu \) is an ideal, it follows that \( \|m\| \leq \|T_\mu(m)\| \). Thus \( \|m\| \leq \|T_\mu(m)\| \) for all \( m \in M_\mu(\mathcal{W}) \).

We will now show that the image of \( M_\mu(\mathcal{W}) \) under \( T_\mu \) is an ideal in \( M_\mu(\mathcal{W}) \). Since \( T_\mu \) is a lattice isomorphism, \( M_\mu(\mathcal{W}) \) is a Riesz subspace of \( M_\mu(\mathcal{W}) \).

Finally, in order to demonstrate that \( T_\mu(M_\mu(\mathcal{W})) \) is a band, let \( \{T_\mu(m)\} \) be an upward directed net in \( T_\mu(M_\mu(\mathcal{W})) \) with \( T_\mu(m) \uparrow \mu \in M_\mu(\mathcal{W}) \). Let \( m \) denote the restriction of \( \mu \) to \( \mathcal{W} \). Since \( m \in M_\mu(\mathcal{W}) \), \( m \uparrow \mu \). Thus \( m(\mathcal{W}) \leq m(\mathcal{W}) \).

Hence \( \mu \) is \( \mathcal{W} \)-regular. Since \( m \in M_\mu(\mathcal{W}) \), \( m \) is \( \mathcal{W} \)-regular. Since \( m \) is \( \mathcal{W} \)-regular, \( m \) is \( \mathcal{W} \)-regular.

The map \( T_\mu \) is not onto \( M_\mu(\mathcal{W}) \) in general as the following example shows.

**Example.** Let \( \mathcal{W} = [0, 1] \). Define \( W_n = [0, 1/n] \) and \( W_n = [1 - 1/n, 1] \) for \( n \in N \). Let \( \mathcal{W} \) be the smallest paving on \( X \) containing \( \{W_n : n \in N\} \). (Thus \( \mathcal{W} = X \times W_0 \cup \{W_1 : n \in N\} \cup \{W_0 : n \in N\} \).) Note that \( \mathcal{W} \) is a full paving.

**Proof.** For \( \mu \in \mathcal{W} \), define \( \mu \) as is easily seen. If \( m \) denotes the restriction of \( \mu \) to \( \mathcal{W} \), then \( m \not\in M_\mu(\mathcal{W}) \). (Indeed, \( m(W_0) = 1 \), but \( \sup(m(\mathcal{W})) = 0 \).)

The following gives a simple condition on \( \mathcal{W} \) which guarantees that \( T_\mu \) is onto \( M_\mu(\mathcal{W}) \). (Note that the family \( \mathcal{W} \) of all zero sets on a topological space satisfies the condition, and this accounts for the fact that the usual Baire measure on a topological space is \( \mathcal{W} \)-regular.)

**Theorem 1.9.** Let \( \mathcal{W} \) be a full paving. Assume that if \( W_n \in \mathcal{W} \), then there is a sequence \( \{W_n \} \) in \( \mathcal{W} \) with \( W_n \to \mathcal{W} \). Then for every infinite cardinal \( \alpha \), \( T_\mu \) maps \( M_\mu(\mathcal{W}) \) onto \( M_\mu(\mathcal{W}) \).

**Proof.** Let \( m \in M_\mu(\mathcal{W}) \) and let \( m \) denote the restriction of \( m \) to \( \mathcal{W} \). Then \( m \) is a non-negative, finite, finitely-additive function on \( \mathcal{W} \). All that need be verified is that \( m \) is \( \mathcal{W} \)-regular. Hence let \( W_n \in \mathcal{W} \) and choose an increasing sequence \( \{W_n \} \) in \( \mathcal{W} \) with \( W_n \subset \mathcal{W} \). Since \( m \) is \( \mathcal{W} \)-additive, \( m(W_n) \leq \lim m(W_n) \leq \sup(m(\mathcal{W})) \). By Proposition 4.1.4.5, it follows that \( m \) is \( \mathcal{W} \)-regular.

**§ 2. Applications of the extension.** In this section, we wish to apply the extension theorems to obtain certain results on weak convergence in \( A^* \) where \( A \) is a uniformly closed algebra of bounded real-valued functions on \( X \) which satisfies the point of \( X \) and contains the constants.

We shall denote the paving of zero sets of \( A \) by \( \mathcal{W} \). If \( \mathcal{W} \) is a full paving on \( X \), then a standard representation of \( A \) is an isometric isomorphism \( I \) of \( A^* \) onto \( M_\mu(\mathcal{W}) \) such that \( I(\mathcal{W}) = m(\mathcal{W}) = \{x : \mathcal{W} \text{ is finite and } \mu \in M_\mu(\mathcal{W}) \}

**Lemma 2.1.** Let \( A_1, A_2 \) be algebras on \( X \) with \( A_1 \leq A_2 \), and let \( \tau_1, \tau_2 \) be \( \tau_1 \), \( \tau_2 \). Let \( \mathcal{W}_1, \mathcal{W}_2 \) be two full pavings of closed sets in \( X \) which are bases for the \( \tau_1, \tau_2 \). Let \( \phi(x) \) be a \( \phi \) on \( X \) with \( A_1 \leq A_2 \), and let \( \tau_1, \tau_2 \) be \( \tau_1 \), \( \tau_2 \). Let \( \mathcal{W}_1, \mathcal{W}_2 \) be two full pavings of closed sets in \( X \) which are bases for the \( \tau_1, \tau_2 \). Let \( \phi(x) = \phi_1(x) \) on \( A_1 \), \( \phi(x) = \phi_2(x) \) on \( A_2 \), \( \phi(x) = \lim_{\tau_1, \tau_2} \phi(x) \) on \( \mathcal{W}_1, \mathcal{W}_2 \).

**Proof.** Since \( \mathcal{W}_1 \), \( \mathcal{W}_2 \) are bases for the closed sets for \( \tau_1, \tau_2 \), \( \mathcal{W}_1 \), \( \mathcal{W}_2 \) is the family of all closed sets for \( \tau_1, \tau_2 \). By Theorem 1.8, it is sufficient to prove that \( m(\mathcal{W}_1) \leq m(\mathcal{W}_2) \) for all \( \tau_1, \tau_2 \). Hence \( \mathcal{W}_1, \mathcal{W}_2 \) is a basis for \( \tau_1, \tau_2 \).

Since \( \mathcal{W}_1 \), \( \mathcal{W}_2 \) are bases for the closed sets, \( \mathcal{W}_1 \), \( \mathcal{W}_2 \) is a \( \tau_1, \tau_2 \). Hence, since \( \mu \) is a \( \tau \)-additive, there is \( W_n \in \mathcal{W}_1 \), \( W_n \in \mathcal{W}_2 \) for all \( \tau_1, \tau_2 \). Furthermore, there is \( \phi_1(x) \) on \( \mathcal{W}_1 \), \( \phi_2(x) \) on \( \mathcal{W}_2 \). Hence, \( \phi_1(x) = \phi_2(x) \). The proof is complete.

We note that \( \mathcal{W} \) is a normal base if \( \mathcal{W} \) is paving of closed sets on \( X \) which is a base for the closed sets of \( X \) and satisfies:
(i) If $G$ is a closed set in $X$ and if $x \in G$, then there are $W_1, W_2 \in \mathcal{W}^*$ with $W_1 \cap W_2 = \emptyset$ and $x \notin W_1, G \subseteq W_2$.

(ii) If $W_1, W_2 \in \mathcal{W}^*$ with $W_1 \cap W_2 = \emptyset$, then there are $V_1, V_2 \in \mathcal{W}$ with $V_i \cup V_j = X$ and $V_i \subseteq V_j^*$ for $i = 1, 2$.

We remark that a normal base $\mathcal{W}^*$ gives rise to a compactification $X_{\mathcal{W}^*}$ and if this compactification is $X_{\mathcal{W}}$ (the compactification such that every element of $\mathcal{W}$ is a continuous map on $X_{\mathcal{W}}$), then $\mathcal{M}(\mathcal{W})$ represents $A^*$ by Proposition 2.6.

**Theorem 2.11.** Let $A_e, A_f$ be algebraic algebras on $X$ with $A_e \subseteq A_f$, and $A_e(A) = A_f(A)$; and let $\mathcal{W}^*$, $\mathcal{W}^*$ be normal bases with $X_{\mathcal{W}^*}$ the $\mathcal{W}^*$-compactification for $j = 1, 2$. Let $\varphi_1$ be a net in $(A_e^*)^*$, let $\varphi_2(A_e)^*$, and assume that $\varphi_i(f) \to \varphi_i(f)$ for all $f \in A_i$. Then $\varphi_1(f) \to \varphi_2(f)$ for all $f \in A_1$, if either of the following two conditions hold.

1. $\mathcal{W}^* \subseteq \mathcal{W}^*$.
2. $\varphi_i(A_e)^*$ for all $i$.

**Proof:** First assume that $\mathcal{W}^* \subseteq \mathcal{W}^*$. (That is, condition (1) holds). Let $m_1, m_2 \in M^*(\mathcal{W})$ represent $\varphi_1$ and $\varphi_2$, respectively, and let $m_{1, e}, m_{2, e} \in M^*(\mathcal{W}^*)$ represent the restrictions of $\varphi_1$ and $\varphi_2$ to $A_e$. By Proposition 2.6, $m_{1, e} \in M^*(\mathcal{W}^*)$ and $m_{2, e} \in M^*(\mathcal{W}^*)$. Fix $W_1, W_2 \in \mathcal{W}^*$, and $0 < \epsilon$. Since $\mathcal{W}^*$ is a normal base, there is a $W_1 \in \mathcal{W}^*$ with $x \subseteq W_1$ and $\mu(W_1) < T_1(\mu(W_1)) + \epsilon$. For all $i$, $m_i(W_1) = \inf \{ \mu(f) : f \in A_i, x \subseteq W_1 \} \leq \inf \{ \mu(f) : f \in A_i, x \subseteq W_1 \} = \mu(W_1)$. Using this together with Theorem 2.1 and Lemma 2.1 we obtain that

$$\limsup m_1(W_1) \leq \limsup m_1(W_2) \leq \limsup \mu(W_1) + \epsilon = m(W_1) + \epsilon.$$

The result is now an immediate consequence of Theorem 2.6.

Now assume that condition (2) holds. Again let $m_{1, e}, m_{2, e} \in M^*(\mathcal{W}^*)$ represent $\varphi_1$ and $\varphi_2$, respectively, and let $m_{1, e}, m_{2, e} \in M^*(\mathcal{W}^*)$ represent the restrictions of $\varphi_1$ and $\varphi_2$ to $A_e$. By Lemma 2.1, $T_{1, e} = T_{2, e}$ and $T_{1, e} = T_{2, e}$. Fix $W_1, W_2 \in \mathcal{W}^*$ and $0 < \epsilon$. Choose $W_1 \in \mathcal{W}^*$ such that $W_1 \subseteq W_2$ and $0 < \mu(W_1) < \mu(W_2) = \mu(W_2) - \mu(W_1)$. Then by Theorem 2.6 and Lemma 2.1 we obtain

$$\limsup m_1(W_1) = \limsup m_1(W_2) \leq \limsup m_2(W_1) \leq \mu(W_1) - \epsilon = m(W_1) + \epsilon.$$
Theorem 2.6. Let $A$ be an algebra on $X$, and let $B \subseteq (A_+^*)^\ast$. If $B$ is relatively weakly countably compact in $(A_+^*)^\ast$, then $B$ is relatively weakly compact in $(A_+^*)^\ast$.

Proof. It is enough to prove the theorem in the special case, $A = C^0(X)$. The general result then follows from Lemma 2.3 as in the proof of Theorem 2.3 above. But if $A = C^0$, then $M_+(X)$ is complete for the Mackey topology $\mathcal{m}(M_+(X), C^0)$. (Indeed, it is shown in [3] that $M_+$ is complete for a topology $\mathcal{d}$ for which the dual of $M_+$ is $C^0$. Hence it is complete for the Mackey topology.) It then follows from Eberlein's theorem that if $B \subseteq M_+(X)$ is relatively weakly countably compact, then it is relatively weakly compact. The proof is complete.

Remark. We have shown above that any relatively weakly sequentially compact subset of $M_+(X)$ is necessarily relatively weakly compact.

We note from [4] that for a paving $\mathcal{F}$, a set $S \subseteq X$ is $\mathcal{F}$-compact if for every filter $\mathcal{F} \subseteq \mathcal{F}$ with $S \cap U \neq \emptyset$ for all $U \in \mathcal{F}$, then $S \cap \bigcap (S \cap U : U \in \mathcal{F}) \neq \emptyset$.

Definition 2.7. Let $\mathcal{F}$ be a paving on $X$. A set $S \subseteq M(\mathcal{F})$ is tight if $\sup \{|m| : m \in B\} < \infty$ and if for every $\varepsilon > 0$, there is a $\mathcal{F}$-compact set $W \subseteq \mathcal{F}$, such that $|m|(W) < \varepsilon$ for all $W \subseteq \mathcal{F}$ with $W \cap W_0 = \emptyset$ and all $m \in B$.

It is clear that the mapping $S$ of Lemma 2.4 preserves tight sets. Hence using Lemma 2.4 and Theorem 31 of [8, p. 66], we obtain the following weakened version of Prohorov's theorem.

Theorem 2.8. Let $A$ be an algebra and assume that $(X, \tau_A)$ is locally compact and that $(X, \tau_A)$ is metrizable with a complete metric. Then $B \subseteq M_+(X)$ is relatively weakly compact if and only if it is tight.

We conclude this paper with a last application to obtain a generalization of a known result (see for example [5; 5.1(d)]). We denote the set of tight elements of $M(\mathcal{F})$ by $M_\tau(\mathcal{F})$.

Theorem 2.9. Let $\mathcal{F}$ be a normal base with compactification $X_\mathcal{F}$ or let $\mathcal{F} = X(A)$ for $A$ an algebra on $X$. If $m \in M_\tau(\mathcal{F})$, then for each compact regular Borel measure $\mu$ on $X$ such that $\mu(\mathcal{F}) = m^+$ and $\mu^-(\mathcal{F}) = m^-$.

Proof. We note that the hypothesis implies the $\tau_A$-compact sets are $\mathcal{F}$-compact and conversely. By [4; 5.6] $m \in M_\tau(\mathcal{F})$ is in $M_\tau(\mathcal{F})$. Therefore there exist unique $\mu^+, \mu^- \in M_\tau(\mathcal{F})$ such that $\mu^+(\mathcal{F}) = m^+$ and $\mu^-(\mathcal{F}) = m^-$. Since $\mathcal{F}$ is a base for the $\tau_A$ closed sets of $X_\mathcal{F}$, the paving of all closed sets so that $\mu^+$ and $\mu^-$ are Borel measures on $X$.

Finally, since $m$ is tight, for any $\varepsilon > 0$ there is a $\mathcal{F}$-compact set $W_\varepsilon$ such that $|m|(W) < \varepsilon$ if $W \cap W_\varepsilon = \emptyset$. Since $W_\varepsilon$ is $\tau_\mathcal{F}$-compact and $\mathcal{F}$ is a basis for the $\tau_\mathcal{F}$ closed sets, it follows that for any closed set $P$ with $P \cap W_\varepsilon = \emptyset$, there is a $W \subseteq \mathcal{F}$ such that $P \subseteq W$ and $W \cap W_\varepsilon = \emptyset$. Consequently, $|\mu|(P) = |\mu|(W) = |m|(W) < \varepsilon$ so that $|\mu|$ is compact regular.

References


Received October 23, 1974
Revised version May 30, 1975