Complementably universal Banach spaces

by

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Abstract. There is no separable Banach space which is complementably universal for the class of separable Banach spaces.

I. Introduction. A Banach space $X$ is said to be complementably universal for a class $\mathcal{A}$ of Banach spaces provided every space in $\mathcal{A}$ is isomorphic to a complemented subspace of $X$. A. Pelczyński proved in [11] that there exists a separable Banach space which is complementably universal for the class of all Banach spaces with Schauder basis. This result was extended by Kadec [8], who constructed a separable space which is complementably universal for the class of all separable Banach spaces which possess the bounded approximation property (b.a.p.). Actually, the spaces constructed by Kadec and Pelczyński are isomorphic (cf. [7], [12]).

In Section II we prove

A. Basic Result. There is no separable Banach space which is complementably universal for the class of all separable Banach spaces.

This result is a simple consequence of Enflo’s counterexample to the approximation problem [3].

Section III and Section IV contain extensions of our basic result. In Section III it is shown that, in contrast to Kadec’s theorem,

B. There is no separable Banach space which is complementably universal for the class of separable Banach spaces which possess the approximation property.

In Section IV it is shown that Davis’ construction in [2] yields

C. For each $2 < p < \infty$, there is no separable Banach space which is complementably universal for the class of all subspaces of $\ell_p$.

Of course, results B and C both contain the basic result A. We have included a separate proof of A because it is short and the proof is accessible to anyone who is willing to accept the “axiom” that there are subspaces of some special spaces which fail a certain approximation condition mentioned at the beginning of Section II. The proof in Section III is also not
very difficult, but uses many more "axioms", while the proof in Section IV requires the reader's detailed knowledge of Davie's construction [2].

We use standard Banach space theory notation as may be found, for example, in [30].

II. The Basic Result. Let us say that a Banach space $X$ has the bounded compact approximation property (b.c.a.p.) provided there is a uniformly bounded net of compact operators on $X$ which converges strongly to the identity. It was pointed out to us by T. Figiel that the criterion used by Enflo [3] (and then Davie [2] and Figiel [4]) to guarantee that a space fails the approximation property actually guarantees that it fails the b.c.a.p. (We include in an appendix at the end of the paper Figiel's proof of this assertion.) Thus by Enflo's result [3] (see [2] or [4] for more readable expositions) there is for each $p, 1 < p < \infty$, a subspace $E_p$ of $(l_1^p \oplus l_1^p \oplus l_1^p \oplus \ldots)_p$ which fails the b.c.a.p.

Suppose now that $X$ is a separable Banach space and for each $p$ there is a complemented subspace $Y_p$ of $X$ which is isomorphic to $E_p$. Letting $Q_p$ be a projection of $X$ onto $Y_p$, we have that there is $\lambda < \infty$ and an uncountable set $A \subset (1, \infty)$ so that $\|Q_p\| \leq \lambda$ for each $p \in A$.

Since each $Y_p$ fails the b.c.a.p., there are finite sets $\{y_i^j, y_j^k\}$ of unit vectors in $Y_p$ and $\epsilon_j > 0$ so that if $T$ is a compact operator on $Y_p$ for which $\|y_i^j - Ty_i^j\| < \epsilon_j$ for $j = 1, \ldots, m(p)$, then $\|T\| > 1\alpha$. Choose an uncountable subset $B$ of $A$ so that $\lim_n (n \cdot m(p) \cdot \epsilon) = 0$ as $p \rightarrow \infty$.

Since $B$ is uncountable and $X$ is separable, there exist $p < r$ in $B$ so that $\|y_i^j - y_j^k\| < (\lambda + 1)^{-1} \epsilon$ for $i = 1, \ldots, n$. Also, note that every vector from $Y_p$ to $Y_r$ is compact. Indeed, it is essentially contained in Banach's book [1] (cf. also the appendix to [13]) that every operator from $l_p$ to $l_r$ is compact when $p < r$; the proof goes over to show that every operator from a subspace of $(l_1^p \oplus l_1^p \oplus l_1^p \oplus \ldots)_p$ into $(l_1^r \oplus l_1^r \oplus l_1^r \oplus \ldots)_p$ is compact.

Let $T : Y_r \rightarrow Y_p$ be the restriction of $Q_pQ_Q$. In the above discussion, $T$ is compact, and obviously $\|T\| < \lambda^2$. But note that for $i = 1, \ldots, n$,

$$|y_i^j - Ty_i^j| = |y_i^j - Q_pQ_Q y_i^j|$$
$$\leq \|y_i^j - Q_p y_i^j\| + \|Q_pQ_Q y_i^j - Q_pQ_Q y_i^j\|$$
$$= \|Q_p y_i^j - y_i^j\| + \|Q_p y_i^j - y_i^j\|$$
$$\leq (\lambda + 1)^{-1} \|y_i^j - y_j^k\| < \epsilon.$$

This contradicts the choice of $\|y_i^j\|_{l^p_2}$ and completes the proof of the basic result.

III. Spaces with the approximation property. A basis $(a_n)$ is said to be block Hilbertian (resp., block Besselian) provided that there is a constant $K$ so that for each normalized block basic sequence $(y_k)$ of $(a_n)$ and scalars $(\alpha_k), \|a_n y_k\| < K \|\sum \alpha_k a_k\|^{1/2}$ (resp., $\|a_n y_k\| > K \|\sum \alpha_k a_k\|^{1/2}$). It is clear that a basis is block Hilbertian if and only if the biorthogonal functionals to $(a_n)$ are block Besselian.

In [9], Lindenstrauss extended the technique of James [6] to show that for each separable Banach space $X$ there is a space $Z$ so that $Z^* = Z \oplus X^*$ isometrically with the projection of $Z^*$ onto $X^*$ perpendicular to $Z$ having norm at most 2. (Here we have $Z$ embedded in $X^*$ in the canonical way.) Further, $Z$ was constructed to have a shrinking basis $(a_n)$ so that the biorthogonal functionals to $(a_n)$ are block Besselian, and hence $(a_n)$ is block Hilbertian. We will need the fact that if $X$ has a block Hilbertian basis then every operator from $X$ into $(l_1^p \oplus l_1^p \oplus l_1^p \oplus \ldots)_p$ is compact for $1 < p < \infty$. (The proof of this assertion is identical to the proof in the special case $X = l_1^1$.)

For $1 < p < 2$, let $E_p$ be a subspace of $(l_1^p \oplus l_1^p \oplus l_1^p \oplus \ldots)_p$ which fails the b.c.a.p. and let $Z_p$ be a James-Lindenstrauss space which has a shrinking block Hilbertian basis and which satisfies $Z^*_p = Z_p \oplus E_p$. Note that if $S$ is an operator on $E_p$ and $S$ factors through $Z^*_p$ for some $\nu \neq p$, then $S$ is compact. Using this and a lemma from [5], we will show:

Proposition 1. There does not exist a separable Banach space $X$ and a $\lambda < \infty$ so that for each $1 < p < 2$ and each equivalent norm $|\cdot|$ on $Z_p$, $(Z_p^*, |\cdot|)$ is $\lambda$-isomorphic to a $\lambda$-complemented subspace of $X$.

Note that since $Z_p$ has a shrinking basis, $Z_p^*$ has a basis and hence the approximation property.

The result B is an immediate consequence of Proposition 1, Proposition 2 below, and the obvious fact that $(X \oplus Y \oplus \ldots)_p (1 < p < \infty)$ has the approximation property if each $Y_k$ does.

Proposition 2. Suppose that $X$ is complementably universal for a class of and there is $1 < p < \infty$ so that for every sequence $(Y_k)$ in $\mathcal{A}$, $(X_k \oplus Y_k \oplus \ldots)_p (1 < p < \infty)$ is in $\mathcal{A}$. Then there is $\lambda < \infty$ so that every $Y$ in $\mathcal{A}$ is $\lambda$-isomorphic to a $\lambda$-complemented subspace of $X$.

Proof. If not, then there are $Y_k$ in $\mathcal{A}$ so that $P_{X_k} d(X_k, Y_k) > n$ for any projection $P_k$ from $X_k$ onto a subspace $X_k$. (Here $d(Y, Z)$ is the Banach-Mazur distance coefficient $\inf \{|\cdot| : |\cdot| : Z \rightarrow Y$ is an isomorphism from $Y$ onto $Z$.) It is clear that $(X_k \oplus Y_k \oplus \ldots)_p$ is not isomorphic to a complemented subspace of $X$.

Before proving Proposition 1, we restate Proposition 1 of [5] in a form suitable for our needs.

Lemma 1. Let $(Y, |\cdot|)$ be a Banach space, $\beta < \infty$, $\epsilon > 0$, and $U$ a finite subset of $Y^*$. There is an equivalent norm $|\cdot|$ on $X', \beta > 0$, and a finite set $F$ of $|\cdot|$-unital vectors in $X'$ so that if $T$ is an operator on $X'$ which satisfies $|Tz - u| < \delta$ for $z \in F$ and $|T| < \beta$, then $|Tz| < (1 + 2\epsilon^{-1}\beta)$ for each $u \in U$. 
We turn now to the proof of Proposition 1. Assume, for contradiction, that there is such an $X$ and $\lambda$. Fix $0 < \varepsilon < 2^{-1}$ and $1 < p < 2$. Since $(E_p, \| \cdot \|_p)$ fails the b.c.a.p., the proof of Proposition 2 in [5] shows that there is a finite set $U_p$ of unit vectors in $E_p$ so that if $S$ is a compact operator on $E_p$ and $\| S u - u_p \| < 2 \varepsilon$ for $u \in U_p$, then $\| S \| > 2[1 + 2^{-1/2}]^2$. By Lemma 1, there is an equivalent norm $\| \cdot \|_p'$ on $Z_p'$, a $\delta_p > 0$, and a finite set $\{ u'_p \}$ of unit vectors in $Z_p'$ so that if $T$ is an operator on $Z_p'$ which satisfies $\| Tu' - u'_p \| < \delta_p$ and $\| T_u' \| < \lambda$, then $\| T \| > [1 + 2^{-1/2}]^2/\delta_p$ and $\| T u' - u_p \| < \varepsilon$ for all $u \in U_p$. (Here we regard $Z_p'$ as being isometric to $Z_p \hat{\otimes} E_p$, where $E_p$ has norm $\| \cdot \|_p'$, and denote the norm on $Z_p'$ and $Z_p^\ast$ also by $\| \cdot \|_p'$.) By the hypothesis on $X$, we can assume that $Z_p^\ast$ is contained in $(X, \| \cdot \|_p)$, that $\lambda \cdot \| \cdot \|_p' \leq \| \cdot \|_p$ for $\varepsilon < \lambda$, and that there is a projection $Q_p$ from $X$ onto $Z_p^\ast$ with $\| Q_p \| \leq \lambda$.

As in Section 1, we have that there exists $1 < r < 2 < \alpha$ so that $n(\alpha) = n(\alpha) = n$ and for $1 \leq i \leq n$, $\| e_i - e_i' \| < (\alpha - 1)^{-1} \lambda$. Letting $T$ be the restriction of $Q_p Q_p^\ast$ to $Z_p^\ast$, we have just as in Section 1, that $\| T \| \leq \lambda$ and $\| T e_i - e_i' \| < \delta_p$ so that also $\| T u' - u' \| \leq \lambda$. Therefore, we have that $\| T \| \leq (1 + 2\varepsilon)^2$ and $\| T u' - u_p \| < \varepsilon$ for all $u \in U_p$. Finally, let $P_t$ be the projection of $Z_p^\ast$ onto $E_t$, perpendicular to $Z_t$, and let $S$ be the restriction of $P_t^\ast T^\ast$ to $E_t$. Since $\| T \| \leq 2$, it follows that $\| S \| \leq 2(1 + 2\varepsilon)^2 \lambda$ and $\| S u - u_p \| < 2 \varepsilon$ for all $u \in U_p$. However, $S$ factors through the space $Z_p^\ast = Z_p \hat{\otimes} E_t$ and is therefore compact. This contradiction completes the proof.

IV. Subspaces of $l_1$, $2 < p < \infty$. Throughout this section $p$ is a fixed number with $2 < p < \infty$ and $\mathcal{A}$ denotes the class of all subspaces of $l_1$. The result C is an immediate consequence of Lemma 2 and the construction of the $X_t$'s given below.

**Lemma 2.** There is no separable Banach space which is complementably universal for $\mathcal{A}$, provided

(*) for every $m = 1, 2, \ldots$ there exist an uncountable family $\{ X_t \}_{t \in \mathbb{R}_+}$ of spaces $X_t$ and points $e_1, \ldots, e_m \in X_t$, $t \neq \varepsilon \in \mathbb{R}_+$ such that whenever $T : X_t \rightarrow X_{t'}$, $t \neq \varepsilon \in \mathbb{R}_+$ satisfies

$$\sum_{i=1}^{m} \| T e_i - e_i' \| < 1,$$

then $\| T \| > \alpha_m$ with $\alpha_m \rightarrow \infty$.

Proof. Suppose $X$ is complementably universal for $\mathcal{A}$. The class $\mathcal{A}$ obviously satisfies the assumptions of Proposition 2 so let $\lambda < \infty$ be such that every $Y$ in $\mathcal{A}$ is $\lambda$-isomorphic to a $\lambda$-complemented subspace of $X$.

Let $m$ be so big that $\alpha_m > \lambda$. For each $t \neq \varepsilon \in \mathbb{R}_+$ fix an embedding $T_t : X_t \rightarrow X$ and a projection $P_t : X \rightarrow X_t$ so that

$$\| P_t \| \leq \lambda \quad \text{and} \quad \| T_t \| \leq \lambda \| T_t \| \quad \text{for every} \quad t \neq \varepsilon \in \mathbb{R}_+.$$

Fix an $\varepsilon > 0$. Since $X$ is separable and $P_m$ is uncountable, we can find a pair $t \neq \varepsilon \in \mathbb{R}_+$ such that

$$\| T_t e_i - T_t e_i' \| < \varepsilon \quad \text{for} \quad i = 1, \ldots, m.$$

Define now $T : X_t \rightarrow X_t$ by $T = T_t^\ast P_t T_t$. We have $T_t^\ast P_t (T_t e_i - T_t e_i') = T_t e_i - e_i'$ and therefore

$$\| T_t e_i - e_i' \| \leq \| T_t \| \| P_t \| \| e_i \| \leq \lambda \cdot \varepsilon,$$

on the other hand, $\| T \| \leq 1 \cdot \lambda$. By taking $\varepsilon = \lambda^{-1} m^{-1}$ we get a contradiction.

Construction of the $X_t$'s. We exploit Davie's construction to such an extent that it seems more reasonable to emphasize the changes rather than to rewrite the major part of [2].

The main point is that we strengthen condition (1) in [2] by requiring the following four conditions to hold simultaneously for $e = 0, 1$ and $\eta = 0, 1$

$$(1+j)^\eta \sum_{j = 0}^{1 \cdot \varepsilon} e_j(t e_i) = \left| \sum_{j = 0}^{1 \cdot \varepsilon} e_j(t e_i) \right| < A \varepsilon^j 2\varepsilon_n \quad \text{for all} \quad g \in \mathcal{G}_t, \quad e = 1, 2, \ldots$$

(this $\mathcal{A}$ is an absolute constant).

For $e = 0, 1$, we put $J(t, k, \varepsilon) = e \cdot 2^{k-1} + 1, e \cdot 2^{k-1} + 2, \ldots, e \cdot 2^{k-1}$.

For $t = \{ t(n) \}_{n=0}^{1 \cdot \varepsilon} \in (0, 1)^{\varepsilon}$.

We define

$$X_t = \text{span} \{ e_j(t e_i) : j \neq k \}, \quad \varepsilon = 1, 2, \ldots$$

(here the $e_j$'s are defined as in (2) of [2]), and we set $X = \text{span} \{ X_t : t \in (0, 1)^{\varepsilon} \}$.

For any operator $T : X_t \rightarrow X$ we define

$$\beta_T(T) = 2^{-k+1} \sum_{j \neq k} e_j(T e_j)$$

(the $e_j$'s are defined as in (3) or (4) of [2]).

The further argument of [2] yields also in our case

$$| \beta_T(T) - \beta_T(T) | \leq \sup \{ \| T e_j \| : e \in \mathcal{G}_t \}$$

for some $\mathcal{G}_t$, where $\| \mathcal{G}_t \| = 0 (k^{-1})$, uniformly on $t$ and $\varepsilon$.

This gives us that for every $\delta > 0$ there exists a $k = k(\delta)$ such that for every $m > k$,

$$\| \beta_T(T) - \beta_T(T) \| \leq \delta \| T \|$$

for every $t \in (0, 1)^{\varepsilon}$ and for every operator $T : X_t \rightarrow X$. 

All the above properties are uniformly valid for $X$.
Suppose now that \( t, s \in (0, 1)^m \) coincide on the first \( k \) places but are different, i.e., \( t(n) \neq s(n) \) for some \( n > k \). Suppose also that \( T : X_k \to X_{k} \) is such that

\[
\sum_{j \in (j(x, y))} \| T_0^j - T_y^j \| \leq 1.
\]

This gives

\[
\beta^n(T) \geq 1 - 2^{-k+1}.
\]

On the other hand, since \( t(n) \neq s(n) \) and \( (e_{n_1}^n, e_{n_2}^n) \) is a biorthogonal system we have

\[
\beta^n(T) = 0
\]

and therefore \( \| T \| \geq 1/2^{k-1} \).

Now, for \( m \geq 2^{k-1} \) we take \( I_m = \{ t \in (0, 1)^m : t_i = 0 \text{ for } i = 1, \ldots, k \} \) and \( e_{j}^n = e_{n}^j \) for \( j = 1, \ldots, 2^{k-1} \) and all \( t \in I_m \); \( e_j^n \) are arbitrary for \( j = 2^{k-1} + 1, \ldots, m \).

Appendix. We wish to thank T. Figiel for permission to include the following lemma, which was mentioned already in Section II.

Lemma. Let \( E \) be a Banach space, \( (e_n^m) \) a bounded sequence in \( a \), and \( (e_n^m) \) a bounded sequence in \( X^* \). Suppose that \( (t_n) \) is a sequence of positive reals and \( (t_n) \) is a pairwise disjoint sequence of sets of positive integers such that

\[
(\ast) \quad \sup_n \sum_{i \in a_n} t_i < \infty.
\]

For \( T \in L(E) \), set \( \varphi_n(T) = \sum_{i \in a_n} t_i e_i^m(T e_i) \). If either \( e_{n_1}^m \to 0 \) or \( e_{n_2}^m \to 0 \), then \( \varphi_n(T) \to 0 \) for each compact operator \( T \).

Proof. Assume, e.g., that \( e_{n_1}^m \to 0 \). Since the set \( K = \text{closure}(T e_i) \) is compact, it follows that \( \limsup_n \| e_i^m(a) \| = 0 \). This and (\ast) yield the conclusion.

The space \( E \) constructed in [3], [2], or [4] is endowed with a sequence \( (\varphi_n) \) as above and a compact set \( K_a \) such that for any \( A \in L(E) \),

\[
\lim_n A(\varphi_n) = \sup_n A(\varphi_n) = 1, \quad \text{where } A = \text{identity on } X.
\]

Consequently, for any compact operator \( T \) on \( E \), one has

\[
\sup_{x \in K_a} \| (I - T)x \| \geq \lim_n \varphi_n(I - T)x
\]

which implies that \( E \) fails the h.e.a.p.

References