$C_X(T)$ has the DP property from Theorem 3 and Theorem 8 of [15]. If $T$ is locally compact, let $T'$ be the one-point compactification of $T$ with $\infty$ denoting the point at infinity. Then $C_X(T, X)$ is isometrically isomorphic to the closed subspace $\Gamma$ of $C_X(T')$ consisting of those functions which vanish at $\infty$. But $\Gamma$ is complemented in $C_X(T')$ via the projection $P: f \to f(\infty)$ and $C_X(T')$ has the DP property, so $\Gamma$, and hence $C_X(T, X)$, has the DP property ([10], 9.4.3).

Remark 5. Partial solutions to this problem were given in [3], [2], and [16]; for the scalar versions see [9], VI, 7.4.

It also follows from Theorem 4 that if $Z$ is a complemented subspace of a space $O(S)$, then $Z^{\mathcal{S}, Y}$ ([14], 7.1.1) has the DP property when $X$ has the DP property for $Z^{\mathcal{S}, Y}$ is then a complemented subspace of $O(S)^{\mathcal{S}, Y} = C_X(S)$. This suggests the conjecture that if $X$ and $Y$ have the DP property, then $X^{\mathcal{S}, Y}$ also has the DP property.

References


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On the Vitali covering properties of a differentiation basis

by

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Abstract. A functional analysis technique is introduced to relate differentiation and covering properties of a basis.

A. Let $\mathcal{B}$ be a Bairean-Pettis differentiation basis in $\mathbb{R}^n$. That is, for each $x \in \mathbb{R}^n$ we have a collection of bounded open sets $\mathcal{B}(x)$ containing $x$, such that there exists at least one sequence $(R_k) \subset \mathcal{B}(x)$ with diameter $(R_k) \to 0$, and if $x \in \mathcal{B}(x)$, then $x \in \mathcal{B}(x)$.

Given a measurable set $E$ in $\mathbb{R}^n$, we say that $V \subset \mathcal{B}$ is a $\mathcal{B}$-Vitali covering of $E$ if for every $x \in E$ there is a sequence $(R_k) \subset V$ such that $R_k \in \mathcal{B}(x)$ for each $k$ and $R_k \to x$ as $k \to \infty$.

DEFINITION 1. The differentiation basis $\mathcal{B}$ has the covering property $V$ if there exists a constant $C$ such that for every measurable bounded set $E$, $\mathcal{B}$-Vitali covering $V$ of $E$ and any $\varepsilon > 0$, one can select a sequence $(R_k) \subset V$ with the properties:

(i) $|E - \bigcup R_k| = 0$,
(ii) $|\bigcup R_k| \leq C|E|^{1/2}$.

Given a locally integrable function $f$, we define the upper derivative $D(f, x)$ with respect to $\mathcal{B}$ as follows:

$$D(f, x) = \sup_{k \to \infty} \sup_{R_k} \frac{1}{|R_k|} \int_{R_k} f(y) dy,$$

where the "$\sup$" is taken over all the sequences $(R_k) \subset \mathcal{B}(x)$ such that $R_k \to x$ as $k \to \infty$. The lower derivative $D(f, x)$ is defined by setting infinium above.

DEFINITION 2. We say that $\mathcal{B}$ differentiates $f$ if

$$D(f, x) = D(f, x) = f(x)$$

almost every point $x \in \mathbb{R}^n$.

The purpose of this paper is to relate the following two properties of a differentiation basis:
(1) \( \mathcal{A} \) differentiates \( f \) for all \( f \in L^p_0(\mathbb{R}^d) \).
(2) \( \mathcal{A} \) has the covering property \( V_q, 1/p + 1/q = 1 \).

For the particular case \( q = 1, p = \infty \), the equivalence of (1) and (2) is due to de Possel [7]. The implication (2) \(\Rightarrow\) (1) is well known, and Hayes and Pane [4] proved that if \( \mathcal{A} \) differentiates \( f \) for all \( f \in L^p(\mathbb{R}^d) \), then \( \mathcal{A} \) has the covering property \( V_1, 1/p + 1/q = 1 \). In Theorem 1, we prove that for a basis \( \mathcal{A} \) invariant by translations, the properties (1) and (2) are equivalent. For more detailed information about this problem see de Guzman [2] [3].

B. Suppose that \( \mathcal{A} \) is a differentiation basis invariant by translations.\(^{(1)} \)

That is, there exists a family \( \mathcal{A}(0) \) of bounded open sets containing the origin such that the fiber of \( \mathcal{A} \) at the point \( x \) is given by \( \mathcal{A}(x) = \{ x \in E, R \in \mathcal{A}(0) \} \). Then we have:

**Theorem 1.** \( \mathcal{A} \) differentiates \( f \) for all \( f \in L^p_0(\mathbb{R}^d) \) if and only if it has the covering property \( V_q, 1/p + 1/q = 1, 1 < q < \infty \).

**Proof.** (1) \(\Rightarrow\) (2): Assume \( \mathcal{A} \) is translation invariant and differentiates \( f \) for all \( f \in L^p(\mathbb{R}^d) \), allowing us to apply the theorems of Stein [9] and Sawyer [1], to conclude that there exists \( r > 0 \) such that the maximal function \( M_r \) is of weak type \((p, p)\). Further generalizations of this argument have been obtained by B. Rubio [9] and I. Peral [6].

Given a measurable bounded set \( E \) and \( E \) of positive \( \varepsilon \), we pick an open set \( \Omega \) s.t. \( \Omega > \varepsilon \) and \( |\Omega - E| \leq \varepsilon \). From now on, we shall consider only the elements of the Vitali covering of \( E \) which are contained in \( \Omega \) and have diameter less than \( r \). Obviously they constitute another Vitali covering of \( E \); we shall denote by \( V \) that covering.

Since the measures of the elements of \( V \) are bounded, we can choose an element \( E_k \) such that \( |E_k| \geq \frac{1}{2}|E| \).

Suppose that we have chosen \( E_1, ..., E_n \). Then we divide the family \( V \) in two classes:

1) Elements \( E \) s.t. \( |E \cap \bigcup_{i \in A} E_i| \leq \frac{1}{2}|E| \);
2) Elements \( E \) s.t. \( |E \cap \bigcup_{i \in A} E_i| > \frac{1}{4}|E| \).

We eliminate the second class and observe that the first class constitutes a Vitali covering of \( E - \bigcup_{i \in A} E_i \).

Now we choose \( R_{q;1} \) to be an element of the first class such that \( |R_{q;1}| \geq \frac{1}{2} \sup \{|R_i| : E_i \text{ is in the first class} \} \).

By induction we get a sequence \( \{ R_{q,j} \} \) such that

\[
|R_{q,j}| \geq \frac{1}{2} |R_{q,j-1}| \quad \text{where} \quad R_{q,j} = R_{q,j-1} - \bigcup_{j < k} R_{q,k}
\]

and furthermore \( |R_{q,j}| \) is of the order of the biggest possible from this. From this, and using the fact that \( \mathcal{A} \) differentiates integrals of functions in \( L^q \), it is easy to see that \( |B - \bigcup R_{q,j}| = 0 \). The relation \( |B - \bigcup R_{q,j}| \leq \varepsilon \) is an immediate consequence of the fact that \( R_{q,j} \subset \Omega \) for every \( k \).

Next we consider the linear operator

\[
Tf(x) = \sum_{|R_{q,j}|} \int_{R_{q,j}} f(y) dy \cdot \chi_{\Omega}(x)
\]

and its formal adjoint

\[
Sf(x) = \sum_{|R_{q,j}|} \int_{R_{q,j}} f(y) dy \cdot \chi_{\Omega}(x).
\]

Observe that \( |Tf(x)| \leq M_{r}(f) \) and \( S(\chi_{\Omega}) \geq \frac{1}{2} \sum \chi_{\Omega} \).

Since \( M_r \) is of weak type \((p, p)\), we have that the family of operators like \( T \) (corresponding to different sequences \( \{ R_{q,j} \} \)) is a uniformly bounded family of linear operators from \( L^p(\mathbb{R}^d) \) to the Lorentz space \( L(p, \infty) \). Therefore their duals \( T^* \) are uniformly bounded operators from \( \langle L(p, \infty) \rangle^* \) to \( L^p \). But since \( L(p, \infty) \) is the dual Banach space of \( L(\mathbb{R}^d, 1) \), it follows that the operators \( S \) are uniformly bounded from the Lorentz space \( L(\mathbb{R}^d, 1) \) to \( L^p \).

That is, there exists a constant \( C \) independent of \( E, \varepsilon \) and the sequence \( \{ R_{q,j} \} \) such that

\[
\left| \sum \chi_{\Omega} \right| \leq C|\sum \chi_{\Omega}| \leq C|E|^{1/2}
\]

(This is true because \( \| \sum \chi_{\Omega} \|^2 = |E|^{1/2} \) for every measurable set \( E \), and every \( r, 1 < r < \infty \), see [5].)

The implication (2) \(\Rightarrow\) (1) is straightforward. Q.E.D.

**Remark.** The same linearization technique also allows us to prove the following two results:

1. If \( \mathcal{A} \) differentiates integrals of functions in \( L^1 \) then it has a covering property of exponential type, i.e., there exists a constant \( C > 0 \) such that given a \( \mathcal{A} \)-Vitali covering of the set \( E \), we can find a subcovering \( \{ E_k \} \) satisfying

\[
\left| \exp \left( C \sum \chi_{\Omega}(x) \right) \right| \leq |E|
\]
2° If \( \mathcal{B} \) differentiates integrals of functions in \( L^\log L \) (for example the basis of intervals in \( \mathbb{R}^3 \)), then there exists \( C > 0 \) such that, under the same conditions of \( 1° \), we have
\[
\left\| \exp \left( C \sum I_{x_k}(x) \right) \right\|_1 \leq |E|.
\]

However, these two covering properties are far from being the best possible for the corresponding situations.

C. The halo problem. Let \( \mathcal{B} \) be a differentiation basis in \( \mathbb{R}^n \) (not necessarily invariant by translations) and let \( \psi(u) \) be its halo function, that is
\[
\psi(u) = \sup \left\{ \frac{1}{|A|} \left| \left\{ x : M_{x,A}(x) > u^{-1} \right\} \right|, \text{ A bounded and with positive measures} \right\}, \quad u \geq 1.
\]

We can extend \( \psi \) to \([0, \infty)\) by setting \( \psi(u) = u \) for \( u \in [0,1] \) (see [2]).

Theorem 2 gives us an alternative proof of some results of Hayes and de Guzman.

**Theorem 2.** Suppose that \( \psi(u) = O(u^p) \) as \( u \to \infty \) for some \( 1 < p < \infty \), then \( \mathcal{B} \) differentiates integrals of functions in \( L_{\log}(p,1) \).

**Proof.** We shall show that \( \mathcal{B} \) has the Vitali covering property \( V_{\log}(\text{weak}) \), \( 1/p + 1/q = 1 \). That is, there exists \( C > 0 \) such that given a bounded measurable set \( E, \varepsilon > 0 \), and a Vitali covering of \( E \), we can select a sequence \( \{B_k\} \) satisfying \( \sum_{k} \varepsilon \delta_{B_k} \leq \varepsilon \) and
\[
\left| \left\{ x : \sum_{k} I_{x_k}(x) > \lambda \right\} \right| \leq C \frac{|E|}{\lambda^p} \text{ for every } \lambda > 0.
\]

To see this we select a sequence \( \{B_k\} \) as in Theorem 1 and we consider the linear operators \( T \) and \( T^* \).

Then
\[
|B_k| = \left| \left\{ x : \sum_{k} I_{x_k}(x) > \lambda \right\} \right| 
\leq \frac{2}{\lambda} \int_{B_k} T^* I_{x}(x) dx = \frac{2}{\lambda} \int T I_{x_k}(x) dx = \frac{2}{\lambda} \left( \|x_k\|_{V_{\log}} \right) \left| T I_{x_k}(x) \right| dx
\leq \frac{2}{\lambda} \|x_k\|_{V_{\log}} \left| T I_{x_k}(x) \right|_{L_p} \leq \frac{C_2}{\lambda} \left| T \right|_{L_{\log}} \left| E_k \right|_{L_p} \leq \frac{C_2}{\lambda} \left| T \right|_{L_{\log}} \left| E_k \right|_{L_p} \lambda^p
\]

and therefore \( |B_k| \leq C \frac{|E|}{\lambda^p} \).

References


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