On the hyperbolic metric on Harnack parts

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Abstract. If $B$ is a $C^*$-algebra with unit element $e$, $S$ a subspace of $B$ which contains $e$, and $H$ a Hilbert space, then a compact Hausdorff space $\mathcal{G}$ and a subspace $\mathcal{A} \subseteq C_{0}(\mathcal{G})$ are constructed in such manner that there is one-to-one correspondence between completely positive maps from $S$ into $L(H)$ and positive functionals on $\mathcal{G}$. Using this "realization", a new proof of Arveson's extension theorem [1] is given and a relation between Harnack parts [14] and Gleason parts [3] is established. Hyperbolic metric on Harnack parts is introduced and a characterization of convergence in this metric is given.

Introduction. Our initial goal was to find a "Bear schema" (see [5]) in which the Harnack equivalence for the completely positive linear maps on a subspace of $C^*$-algebra $B$ into $L(H)$ (see [14]) turns into the Gleason equivalence for the positive functionals on a suitable function space. Looking for this "realization", we remarked that it permits us to study completely positive maps on a subspace of $B$ as positive functionals on a linear space of real valued continuous functions on a compact Hausdorff space. As the first step we present in Section 1 a new proof for the Arveson extension theorem ([1], [2]) (in a slightly more general form). More about implications of Choquet theory in the study of boundary representations, Silov boundary for noncommutative case and other problems related to Arveson's papers [2], [3] we will take up in a subsequent paper.

In Section 2 we introduce the Harnack equivalence and establish its relation with Gleason equivalence. In analogy with scalar case we define in Section 3 the hyperbolic metric on Harnack parts and prove the equivalence between metric convergence and norm convergence of the corresponding (generalized) Radon-Nikodym derivatives.

Finally we make some remarks on continuous selection of mutually absolutely continuous spectral dilations and integral kernels, but the main facts in this direction remain for the further studies.

1. Let $B$ be a $C^*$-algebra and $S \subseteq B$ a linear subspace such that the identity $e$ of $B$ belongs to $S$. For $H$, a Hilbert space, $\mathcal{A}(B; H)$ (respectively $\mathcal{A}(S; H)$) will denote the vector space of all bounded linear maps
of $B$ (respectively $S$) into $L(H)$. For $r > 0$, let $\mathcal{B}(B; H)$ denote the closed ball of radius $r$ in $\mathcal{B}(B; H)$.

We shall endow $\mathcal{B}(B; H)$ with the $BW$-topology. The $BW$-topology is the strongest locally convex topology on $\mathcal{B}(B; H)$ which coincides on each ball $\mathcal{B}(B; H), r > 0$, with the topology given by the following convergence: a net $\varphi_\tau \in \mathcal{B}(B; H)$ converges to $\varphi \in \mathcal{B}(B; H)$ if $\varphi_\tau(b) \to \varphi(b)$ in the weak operator topology, for every $b \in B$.

By standard arguments $\mathcal{B}(B; H)$ is $BW$-compact for any $r > 0$ (see [11], [2]).

An element $\mu \in \mathcal{B}(B; H)$ is positive if $\mu(b)$ is a positive operator on $H$ for any positive $b \in B$.

Let $n$ be a positive integer. $M_n$ will denote the $C^*$-algebra of all scalar $n \times n$ matrices. $\mathcal{B}(M_n)$ denotes the $C^*$-algebra of all $n \times n$ matrices over $B$ with the usual involution $(u_{ij})^* = (u_{ji})^*$. For a matrix $(u_{ij})$ over $B$ we put

$$Re(u_{ij}) = \frac{1}{2}(u_{ij} + (u_{ij})^*).$$

The matrix $(u_{ij})$ over $B$ is positive, if there exists a matrix $(v_{ij})$ over $B$ such that $(u_{ij}) = (v_{ij}) (v_{ij})^*.$

We say that an element $\varphi \in \mathcal{B}(S; H)$ is completely positive if for each integer $n \geq 1$, for each $n \times n$ matrix $(u_{ij})$ over $S$ for which the matrix $Re(u_{ij})$ is positive and for each $n$-tuple $h_1, \ldots, h_n$ of elements of $H$ we have

$$Re \sum_{ij} \langle \varphi(u_{ij})h_i, h_j \rangle \geq 0.$$ 

If $S$ is self-adjoint then $\varphi$ is completely positive if and only if for each $n \geq 1$, for each positive $n \times n$ matrix $(u_{ij})$ over $S$ and for each $n$-tuple $h_1, \ldots, h_n$ of element of $H$ we have

$$\sum_{ij} \langle \varphi(u_{ij})h_i, h_j \rangle \geq 0.$$ 

This is the definition of complete positivity used by W.F. Stinespring in [13] and W.B. Arveson in [11], [2].

Let $\mu$ be an element of $\mathcal{B}(B; H)$. Stinespring theorem [13] says that $\mu$ is completely positive if and only if there is a Hilbert space $K$, a bounded linear operator $V : H \to K$, and a representation $\pi$ of $B$ on $K$ such that $\mu(b) = V^* \pi(b) V$ for every $b \in B$. For every completely positive $\mu$ we have:

$$[\mu(b)] = [V^* \pi(b) V] \leq [V^* V] [\mu(b)],$$

hence $[\mu(b)] = [\mu(b)]$.

In what follows we denote by $\Omega = \Omega(B; H)$ the set of all completely positive linear maps $\mu : B \to L(H)$ such that $[\mu(e)] = 1$.

It is easy to see that $\Omega$ is $BW$-closed, and because $\Omega \subseteq \mathcal{B}(B; H)$ it follows that $\Omega$ is $BW$-compact.

Let $\mathcal{M}$ be the set of all functions $w : \Omega \to \mathbb{R}$ for which there exist an $n \times n$ matrix $(u_{ij}) \in \mathcal{B}(M_n)$ and $h_1, \ldots, h_n \in H$ such that for every $\mu \in \Omega$ we have

$$w(\mu) = Re \sum_{ij} \langle u_{ij} h_i, h_j \rangle.$$ 

It is clear that any function $w$ of the form (1.1) is a real valued continuous function on $\Omega$.

**Lemma 1.** We have

(i) $\mathcal{M}$ is a linear subspace of $C_0(\Omega)$.

(ii) There is $w \in \mathcal{M}$ such that for any $\mu \in \Omega, w(\mu) > 0$.

(iii) For each $w \in \mathcal{M}$ there exist $n \geq 1$, a matrix $(u_{ij}) \in \mathcal{B}(M_n)$, and an orthonormal $m$-tuple $e_1, e_2, \ldots, e_m \in H$ such that

$$w(\mu) = \sum_{i=1}^m \langle u_{ij}(e_i), e_j \rangle \quad (\mu \in \Omega).$$

**Proof.** (i) For $w_1, w_2 \in \mathcal{M}$ of the form

$$w_1(\mu) = \sum_{i=1}^m \langle u_{1i}(e_i), h_i \rangle, \quad w_2(\mu) = \sum_{i=1}^m \langle u_{2i}(e_i), h_i \rangle$$

and $\alpha, \beta \in \mathbb{R}$ the function $w = \alpha w_1 + \beta w_2$ can be written as

$$w(\mu) = \sum_{i=1}^m \langle u_{ij}(e_i), h_i \rangle$$

where the matrix $(u_{ij})$ is

$$u_{ij} = \begin{cases} \alpha u_{ij}, & 1 \leq i, j \leq m, \\ \beta u_{i-m,j-m}, & m+1 \leq i, j \leq m+p, \\ 0, & \text{otherwise} \end{cases}$$

and $h_i = \begin{cases} h_i, & 1 \leq i \leq m, \\ h_{i-m}, & m+1 \leq i \leq m+p. \end{cases}$

Hence $w_1 \in \mathcal{M}$, i.e. $\mathcal{M}$ is a linear subspace of $C_0(\Omega)$.

(ii) Let $\mu_0 \in \Omega$. Since $[\mu_0(e)] = 1$, there exists $h_1 \in H$ so that $[\mu_0(e), h_1 \neq 0$. The element $\omega_0 \in \mathcal{M}$ defined by $\omega_0(\mu) = [\mu(e), h_1] > 0$ is positive since $\mu$ is completely positive. We have $\omega_0(\mu_0) = [\mu_0(e), h_1] > 0$. Since $\Omega$ is $BW$-compact, we can find a finite system $w_1, w_2, \ldots, w_n$ of positive functions in $\mathcal{M}$ such that for any $\mu \in \Omega$ there exists $w_k, 1 \leq k \leq n$, satisfying $w_k(\mu) > 0$. If we put $w = \sum w_k$, then $w(\mu) > 0$ for any $\mu \in \Omega$. 

(iii) Let \( w \in \mathcal{M} \) be of the form (1.1) and let \( e_1, \ldots, e_n \) be an orthonormal base for a linear subspace spanned by \( h_1, \ldots, h_n \) in \( H \).

Then

\[
    h_i = \sum_{j=1}^{n} e_i h_j, \quad i = 1, \ldots, n
\]

and we have:

\[
    w(\mu) = \text{Re} \sum_{i,j} \left( \mu(w_{ij}) h_j, h_i \right) = \text{Re} \sum_{i,j} \left( \mu(w_{ij}) \sum_{p=1}^{n} e_p h_p, \sum_{q=1}^{n} e_q h_q \right)
\]

\[
    = \text{Re} \sum_{i,j} \left( \sum_{p,q=1}^{n} e_p e_q \mu(w_{ij}) e_p e_q, e_q \right)
\]

\[
    = \text{Re} \sum_{i,j} \left( \sum_{p=1}^{n} e_p \mu(w_{ij}) e_p, e_q \right)
\]

\[
    = \text{Re} \sum_{i,j} \left( \mu(w_{ij}) e_i, e_q \right).
\]

In the proof of the following lemma, which is the key lemma in our considerations, we have been inspired by some arguments used in Arveson’s proof of this extension theorem [2].

**Lemma 2.** Let \( w \) be an element of \( \mathcal{M} \):

\[
    w(\mu) = \text{Re} \sum_{i,j} \left( \mu(w_{ij}) h_j, h_i \right),
\]

with orthonormal \( e_1, \ldots, e_n \in H \). Then \( w \geq 0 \) if and only if the matrix \( \text{Re}(w_{ij}) \) is positive.

**Proof.** We consider \( B \) embedded in \( L(K) \), where \( K \) is a Hilbert space. Let \( h_1, \ldots, h_n \) be a \( n \)-tuple of elements of \( K \). Since \( e_1, \ldots, e_n \) are linearly independent vectors in \( H \), there exists a bounded linear operator \( V : H \rightarrow K \) defined by \( V e_i = h_i, 1 \leq i \leq n \), and \( V = 0 \) on the orthogonal complement of the linear subspace spanned by \( e_1, \ldots, e_n \) in \( H \).

We consider \( \mu_V : B \rightarrow L(K) \) defined by

\[
    \mu_V(b) = \frac{1}{\|V\|^2} V^* b V \quad (b \in B \subset L(K)).
\]

The map \( \mu_V \in \mathcal{M} \). Indeed \( \|\mu_V(e)\| = 1 \), and for each positive matrix \( (b_{ij}) \in B \otimes M_n \) and each \( h_1, \ldots, h_n \in H \) we have

\[
    \sum_{i,j=1}^{n} \left( \mu_V(b_{ij}) h_j, h_i \right) = \frac{1}{\|V\|^2} \sum_{i,j=1}^{n} \left( V^* b_{ij} V h_j, h_i \right)
\]

\[
    = \frac{1}{\|V\|^2} \sum_{i,j=1}^{n} \left( b_{ij} V h_j, V h_i \right) \geq 0.
\]

We have:

\[
    \text{Re} \sum_{i,j} \left( w_{ij} h_j, h_i \right) = \text{Re} \sum_{i,j} \left( \mu(w_{ij}) h_j, h_i \right) = \text{Re} \sum_{i,j} \left( V^* w_{ij} V h_j, h_i \right)
\]

\[
    = \text{Re} \|V\|^2 \sum_{i,j} \left( \mu_V(w_{ij}) e_i, e_j \right) = \|V\|^2 \|\mu_V\| \geq 0.
\]

Hence the operator matrix \( \text{Re}(w_{ij}) \) is positive. It follows that \( \text{Re}(w_{ij}) \) is a positive matrix over \( B \).

Conversely, if \( \text{Re}(w_{ij}) \) is a positive matrix then

\[
    w(\mu) = \text{Re} \sum_{i,j} \left( \mu(w_{ij}) e_i, e_j \right) \geq 0,
\]

by the complete positivity of \( \mu \in \mathcal{M} \).

**Theorem 1.** (Arveson [1], [2].) For every completely positive mapping \( \phi : S \rightarrow L(H) \) there exists a completely positive mapping \( \mu : B \rightarrow L(H) \) such that \( \mu(\phi) = \phi \).

**Proof.** Let \( \phi : S \rightarrow L(H) \) be completely positive. For \( w \in \mathcal{M} \) of the form

\[
    w(\mu) = \text{Re} \sum_{i,j} \left( \mu(w_{ij}) h_j, h_i \right), \quad \mu \in \mathcal{M},
\]

let us consider the sum \( \text{Re} \sum_{i,j} \left( \phi_{ij} h_j, h_i \right) \). If \( w \) is positive, then this sum is positive. To see this, choose an orthonormal base \( e_1, \ldots, e_n \in H \) for the linear space spanned by \( h_1, \ldots, h_n \) in \( H \). If \( h_i = \sum_{j=1}^{n} a_{ij} e_j, 1 \leq i, j \leq n \), then we have

\[
    \text{Re} \sum_{i,j} \left( \mu(w_{ij}) h_j, h_i \right) = \text{Re} \sum_{i} \left( \mu_V \left( \sum_{j=1}^{n} a_{ij} e_j \right) h_j, e_i \right)
\]

\[
    = \text{Re} \sum_{i} \left( \mu_V(e_i) h_j, e_i \right)
\]

and

\[
    \text{Re} \sum_{i} \left( \phi_{ij} h_j, h_i \right) = \ldots = \text{Re} \sum_{i} \left( \phi_{ij} h_j, h_i \right).
\]

Since \( w \geq 0 \), using Lemma 2, we conclude that the matrix \( \text{Re}(\phi_{ij}) \) is positive and since \( \phi \) is completely positive, we obtain

\[
    \text{Re} \sum_{i} \left( \phi_{ij} h_j, h_i \right) = 0.
\]

Furthermore, if \( \phi_{ij}, \mu \in \mathcal{M} \),

\[
    w_1(\mu) = \text{Re} \sum_{i,j=1}^{n} \left( \mu(w_{ij}) h_j, h_i \right), \quad w_2(\mu) = \text{Re} \sum_{i,j=1}^{n} \left( \mu(w_{ij}) h_j, h_i \right),
\]

\[
    \phi_{ij} = \text{Re} \sum_{i,j=1}^{n} \left( \mu(w_{ij}) h_j, h_i \right), \quad \mu \in \mathcal{M},
\]

where \( \mu_V \) is a completely positive mapping over \( B \).
and $a, \beta \in \mathbb{R}$, and if we write $w_0 = \alpha w_1 + \beta w_0$ as in (1.2), then we have

\[
\text{Re} \sum_{j=1}^{m} (\varphi_0 (w_0) h_j, h_j) + \text{Re} \sum_{j,l=1}^{m} (\varphi_0 (w_0, \alpha h_j, \beta h_l) h_j h_l).
\]

It is clear now that the map

\[
w \mapsto \text{Re} \sum_{j=1}^{m} (\varphi_0 (w_0) h_j, h_j)
\]

is a well defined positive linear functional on $\mathcal{M}$. Let us write

\[
L_{\varphi_0}(w) = \text{Re} \sum_{j=1}^{m} (\varphi_0 (w_0) h_j, h_j).
\]

Since, by Lemma 1 (ii), $\mathcal{M}$ contains a strictly positive function, it results that $\mathcal{M}$ contains an interior point of the positive cone of $C_0^1 (Q)$. A familiar theorem of M. Krein (cf. [8], Ch. II, § 3, Prop. 6) implies that $L_{\varphi_0}$ has a positive linear extension to $C_0^1 (Q)$ and, consequently, we can extend $L_{\varphi_0}$ to a positive linear functional $\tilde{L}_{\varphi_0}$ on $C_0^1 (Q)$.

For $b \in B$ and $b \in H$ let $w_{b, b, \lambda}$ be a function in $C_0^1 (Q)$ defined by

\[
w_{b, b, \lambda} (x) = \{ \mu (b) h, h \}.
\]

It is easy to verify that for a fixed $b \in B$, $b \mapsto \tilde{L}_{\varphi_0} (w_{b, b, \lambda})$ is a bounded bilinear form. Then there exists an operator $\mu_\varphi (b)$ in $H$ such that

\[
(\mu_\varphi (b) h, h) = \tilde{L}_{\varphi_0} (w_{b, b, \lambda}) \quad (b \in B) .
\]

Now, let $\mu_\varphi$ be the map from $B$ into $L (H)$ defined as $\mu_\varphi : b \mapsto \mu_\varphi (b)$. A simple calculation shows that $\mu_\varphi$ is a linear extension of $\varphi_0$ to $B$.

It remains to prove that $\mu_\varphi$ is completely positive. For this, let $(h_0)_{0 \in B} \otimes M_0$ be a positive matrix and $h_1, \ldots, h_n$ in $H$. Then:

\[
\sum_{j,l=1}^{m} (\mu_\varphi (h_0) h_j, h_l) = \sum_{j,l=1}^{m} \tilde{L}_{\varphi_0} (w_{h_0, h_j, h_l}) = \sum_{j,l=1}^{m} w_{h_0, h_j, h_l} = 0
\]

because $\tilde{L}_{\varphi_0}$ is positive and

\[
\sum_{j,l=1}^{m} w_{h_0, h_j, h_l} (\mu_\varphi (h_0) h_j, h_l) = 0 \quad \text{for every } \mu_\varphi \in \Omega .
\]

The proof of the theorem is complete.
Moreover we will show that in an adequate “Bear schema” attached to $S$ and $B$, Harnack equivalence turns into Gleason equivalence. Let us recall “Bear schema” of Gleason parts in a slightly more general case. Let $\Omega$ be a compact Hausdorff space and $\mathcal{M} \subset C_0(\Omega)$ a linear subspace which separates the points of $\Omega$ and contains a strictly positive function (B cannot contain non-zero constants). Let $\mathcal{M}'$ be the dual of $\mathcal{M}$ and denote by

$$T_{\mathcal{M}} = \{ L \in \mathcal{M}' \mid \| L \| \geq 0, \| L \| = 1 \}.$$ 

The space $\Omega$ can be embedded in $\mathcal{M}$ as a point-evaluation functionals, and then $T_{\mathcal{M}}$ is the compact convex closure (in weak* topology) of $\Omega$. Two elements $L_1, L_2 \in T_{\mathcal{M}}$ are called Gleason equivalent if there exists a constant $\varepsilon > 0$ such that for any positive function $\mu \in \mathcal{M}$ we have

$$c L_2(\mu) \leq L_1(\mu) \leq c^{-1} L_2(\mu)$$

for every positive $\mu$ in $\mathcal{M}$.

The proof of the theorem is complete.

3. As in the scalar case we can define on a Harnack part of $\Omega(S; H)$ the hyperbolic metric, setting for $\varphi_1, \varphi_2$ in the same Harnack part

$$d(\varphi_1, \varphi_2) = \inf \left\{ \log \left( \frac{1}{\varepsilon} \right) ; 0 < \varepsilon < 1, \varepsilon \varphi_1 \leq \varphi_2 \leq \frac{1}{\varepsilon} \varphi_1 \right\}.$$

Because of Theorem 3 we have

$$d(\varphi_1, \varphi_2) = d(L_{\varphi_1}, L_{\varphi_2})$$

where the hyperbolic metric on Gleason parts is defined as in [5].

Many facts about hyperbolic metric on Harnack parts can be obtained by a simple reformulation in this context of the known results relative to hyperbolic metric on Gleason parts.

We point out only the following

**Corollary.** The hyperbolic metric is complete on Harnack part of $\Omega(S; H)$.

On the other hand, one of the most important results from Bear’s theory of hyperbolic metric on Gleason parts, which establish the equivalence between convergence on Gleason part and the convergence in $L^p$-norm of corresponding Radon-Nikodym derivatives, has not an immediate reformulation in the context of Harnack parts. This is what we intend to do in the remainder of this section.

Let $\Omega(S; H)$ and $\Omega(B; H)$ be as above. We can define the Harnack parts on $\Omega(B; H)$ by using the Harnack inequalities for elements in $\Omega(B; H)$. According to Theorem 2 (ii), if $\varphi_1, \varphi_2 \in \Omega(S; H)$ belong to the same Harnack part of $\Omega(S; H)$ then we can extend $\varphi_1, \varphi_2$ to $\mu_1, \mu_2 \in \Omega(B; H)$ which lie in the same Harnack part of $\Omega(B; H)$.

But it is known ([5]) that (even in the scalar case) it is not generally true that for any Harnack part $A$ of $\Omega(S; H)$ there is a Harnack part $A$ of $\Omega(B; H)$ which contains extensions to $B$ for every element of $A$. We first give a description of metric convergence on the Harnack part of $\Omega(B; H)$ and finally we shall return with some comments relative to this question.

Let $\mu_1, \mu_2 \in \Omega(B; H)$ be Harnack equivalent. According to Theorem 2 (iii), there exist the dilations $[K_1, V_1, \pi_1]$ and $[K_2, V_2, \pi_2]$ of $\mu_1$ and $\mu_2$ respectively, and a bounded invertible operator $B: K_1 \to K_1$ such that $SV_2 = V_1$ and $S \pi_1(b) = \pi_2(b)S$, $b \in B$. By Lemma 2 and definition of $I_\mu$, this is equivalent to:

$$c I_{\mu_2}(\omega) \leq I_{\mu_1}(\omega) \leq c^{-1} I_{\mu_2}(\omega)$$

for every positive $\omega$ in $\mathcal{M}$.

The proof of the theorem is complete.
For any $b \in B$ we have

$$
\mu_4(b) = V^*_4 \pi_1(b) V_4 = V^*_4 S^{-1} \pi_1(b) S V_4 = V^*_4 (S^{-1})^* S^{-1} \pi_1(b) V_4.
$$

Write $D = (S^{-1})^* S^{-1}$; then $D$ is a positive operator on $K_b$ belonging to $\pi_1(B)'$, the commutant of $\pi_1(B)$ in $L(K_b)$, and we have

$$
\mu_2(b) = V^*_4 D \pi_4(b) V_4 \quad (b \in B).
$$

Let us call $D$ the Radón–Nikodým derivative of $\mu_2$ with respect to $\mu_1$.

It is easy to show (see the proof of Theorem 2 in [14]) that if $D$ is a positive operator in $\pi_1(B)'$, then, setting $\mu(b) = V^*_4 D \pi_4(b) V_4$, $b \in B$, we obtain an element $\mu \in \Omega(B; H)$ Harmanek equivalent to $\mu_1$.

Let now $A$ be a Harmanek part of $\Omega(B; H)$ and let us fix an element $\mu_a \in A$ and denote by $[K_\mu, V_\mu, \pi_\mu]$ its (minimal, necessarily unique) dilation. For $\mu \in A$ let us denote by $D_{\mu}$ its Radón–Nikodým derivative with respect to $\mu_a$.

**Theorem 4.** A sequence $(\mu_n)$ in $A$ is convergent in the hyperbolic metric on $A$ if and only if the sequence $(D_{\mu_n})$ of corresponding Radón–Nikodým derivatives is convergent in the norm metric on $L(K_\mu)$.

**Proof.** Consider $\Omega' = \Omega(B; H)$ and $\mathcal{M} \in \mathcal{D}(K)$ constructed as in Section 1 with $B$ instead of $S$. According to Theorem 3, if $\mu, \nu \in \mathcal{M}$, we have:

$$
d(\mu, \nu) = \inf \left\{\log \frac{1}{\varepsilon}; \mu \ll \nu \ll \frac{1}{\varepsilon} \mu \right\} = \sup \{\log \omega(\mu) - \log \omega(\nu); \omega \in \mathcal{M}, \omega > 0\}.
$$

Thus if $\mu_n, \nu \in A$ then $d(\mu_n, \mu) \to 0$ if and only if $\omega(\mu_n) \to \omega(\mu)$ uniformly for $\omega \in \mathcal{M}$, $\omega > 0$, $\omega(\mu) \leq 1$.

Suppose that $D_{\mu_n}$ converges to $D_{\mu}$ in the norm metric on $L(K_\mu)$. Since $D_{\mu_a}$ is a positive operator in $\pi_1(B)'$, we have that $\mu$ defined as $\mu(b) = V^*_4 D_{\mu_a} V_4$, $b \in B$, belongs to $A$. We have

$$
|\omega(\mu_n) - \omega(\mu)| = \left|\sum_{k} \left(\frac{1}{\omega(\mu_n)} - \frac{1}{\omega(\mu)}\right) \pi_4(b_k) V_4 h_k + \left(\frac{1}{\omega(\mu)} - \frac{1}{\omega(\mu_n)}\right) \pi_4(b_k) V_4 h_k\right|
$$

$$
= \left|\sum_{k} \left(V^*_4 D_{\mu_n} \pi_4(b_k) V_4 - V^*_4 D_{\mu} \pi_4(b_k) V_4\right) h_k + \left(\frac{1}{\omega(\mu)} - \frac{1}{\omega(\mu_n)}\right) \pi_4(b_k) V_4 h_k\right|
$$

$$
= \left|\sum_{k} \left(V^*_4 (D_{\mu_n} - D_{\mu}) \pi_4(b_k) V_4 h_k + \frac{1}{\omega(\mu)} \pi_4(b_k) V_4 h_k\right)\right|
$$

$$
= \left|\sum_{k} \left(D_{\mu_n} - D_{\mu}\right) \pi_4(b_k) V_4 h_k + \frac{1}{\omega(\mu)} \pi_4(b_k) V_4 h_k\right|
$$

$$
\leq \left|D_{\mu_n} - D_{\mu}\right| \left(\sum_{k} \pi_4(b_k) V_4 h_k\right)^{\frac{1}{2}} + \frac{1}{\omega(\mu)} \left(\sum_{k} \pi_4(b_k) V_4 h_k\right)^{\frac{1}{2}}.
$$

On the other hand

$$
|\omega(\mu_n) - \omega(\mu)| = \left|\sum_{k} \left(\frac{1}{\omega(\mu_n)} - \frac{1}{\omega(\mu)}\right) \pi_4(b_k) V_4 h_k\right|
$$

$$
= \left|\sum_{k} \left(V^*_4 (D_{\mu_n} - D_{\mu}) \pi_4(b_k) V_4 h_k + \frac{1}{\omega(\mu)} \pi_4(b_k) V_4 h_k\right)\right|
$$

$$
= \left|\sum_{k} \left(D_{\mu_n} - D_{\mu}\right) \pi_4(b_k) V_4 h_k + \frac{1}{\omega(\mu)} \pi_4(b_k) V_4 h_k\right|
$$

$$
\leq \left|D_{\mu_n} - D_{\mu}\right| \left(\sum_{k} \pi_4(b_k) V_4 h_k\right)^{\frac{1}{2}} + \frac{1}{\omega(\mu)} \left(\sum_{k} \pi_4(b_k) V_4 h_k\right)^{\frac{1}{2}}.
$$

If we take $\omega = (1 + \delta)^{-\gamma}$, $0 < \omega \leq 1$, and $\delta > 0$ sufficiently small, then we have $w_0 > 0$, $w_0(\mu) \leq 1$ and

$$
|\omega(\mu_n) - \omega(\mu)| > \varepsilon_0,
$$

which contradicts $d(\mu_n, \mu) \to 0$, $n \to \infty$.

The proof of the theorem is complete.

4. As we already remarked, if $A$ is a Harmanek part in $\Omega(S; B)$, then, in general, it is not possible to find a Harmanek part of $\Omega(B; H)$ containing
extensions of any element \( \varphi \in A \). An example in this matter was given by Bear in [5] (for the scalar case). Like in the scalar case, in this context, interesting problems relative to the selection of mutually absolutely continuous dilations and integral kernels arise.

From this point of view, the case of functional calculus for contraction is of particular interest.

Let \( T \) be the one-dimensional torus in complex plane, \( B = C(T) \) and \( A = \mathbb{C} \), the disc algebra. If \( T \) is a contraction on a Hilbert space \( H \), then the von Neumann functional calculus with function in \( A \) gives up an element \( \varphi_T \in \Omega(A; H) \). We say that two contractions \( T_1, T_2 \) on \( H \) are Harnack equivalent if \( \varphi_{T_1} = \varphi_{T_2} \), lie in the same Harnack part of \( \Omega(A; H) \) (cf. [15]). In [9] C. Pincz proved that the set of all strict contractions on \( H \) forms a Harnack part, the Harnack part of contraction \( 0 \). Let us remark that in this case we have a usual formula for hyperbolic distance, namely:

\[
d(T, 0) = \log \frac{1 + |T|}{1 - |T|}.
\]

Since in this case we have the unique dilation, according to Theorem 2 there is a Harnack part in \( \Omega(C(T); H) \) which contains extension to \( C(T) \) of \( \varphi_T \) for any strict contraction \( T \).

If we take contraction \( 0 \) as a center of the Harnack part of strict contractions, and the bilateral shift of multiplicity \( \dim H \) as a unitary dilation of \( 0 \), then the Radon–Nikodym derivative of a strict contraction will be a positive operator in the commutant of the bilateral shift of \( \dim H \).

**COROLLARY.** Let \( T_n \to T \), be strict contractions on \( H \) and \( D_n \), the corresponding Radon–Nikodym derivatives. Then \( T_n \) converge to \( T \) in the hyperbolic metric if and only if \( D_n \) converge to \( D \) in the norm metric.

The problem of selection of mutually absolutely dilations effectively appear in the case of functional calculus for pairs of commuting contractions [12]. In this case we have unitary dilation (Ando Theorem), but it is no more unique.

In a subsequent paper we shall study in details the topic of selection of mutually absolutely continuous dilations and integral kernels.

References


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