where $C_{XY}$ is a connecting copula of $X$ and $Y$ (which may be chosen arbitrarily and may depend on $X$ and $Y$).

Next, since $T \neq \min$, there exist numbers $a$ and $b$, $0 < a, b < 1$ such that $T(a, b) \neq \min(a, b)$. Let $X$ be a random variable assuming the value 0 with probability $a$ and the value 1 with probability $1 - a$. Then $F_X = G_a$. Similarly, let $Y$ be such that $F_Y = G_b$. Then (3.16) yields:

$$\tau_T(G_a, G_b) = c_{ab}(G_a, G_b),$$

where $C_{ab}$ is some connecting copula for $X$ and $Y$. Now, using (3.14) and (3.15), we have:

$$\tau_T(G_a, G_b)(1/2) = T(a, b) = c_{ab}(G_a, G_b)(1/2) = C_{ab}(a, b),$$

and

$$\tau_T(G_a, G_b)(3/2) = \max(a, b) = c_{ab}(G_a, G_b)(3/2) = a + b - C_{ab}(a, b).$$

Thus,

$$\min(a, b) = T(a, b) = C_{ab}(a, b) = a + b - \max(a, b) = \min(a, b).$$

This is a contradiction, whence $g$ cannot exist, and the theorem is proved.

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Brownian motion, approximation of functions, and Fourier analysis

by

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**Abstract.** Quantitative approximation theory, initiated by Kolmogorov, is used to show almost-sure properties of all mappings $F : X \to Y$. Here $X$ is Brownian motion, and $F$ is a diffeomorphism of class $Lip^p$, for example. The problems considered touch on Hausdorff dimension, Koebe sets, Salem sets, and Diophantine approximation. In some cases a critical exponent of smoothness can be found by the category method.

**Introduction.** In this paper we apply the quantitative approximation theory of Kolmogorov to certain questions on Fourier-Stieljes transforms and Brownian motion. For example, let $E$ be a compact subset of $(0, +\infty)$ of positive Hausdorff dimension; Kahane proved that $X(E)$ is an $M_0$ set for almost all Brownian paths X. Therefore the same is true of $X(f(E))$ whenever $f$ is a $C^0$-diffeomorphism of $(0, +\infty)$ into itself. How large a class $S$ of diffeomorphisms $f$ can be named, so that $X(f(E))$ is an $M_0$ set for all $f$ in $S$, almost surely? An answer is contained in the first chapter. A similar question for transforms $f \cdot X(E)$ is considered next; for these sets we obtain strong bounds on certain Fourier transforms. Here matters become distinctly non-linear, but we obtain some precise estimates by simple devices.

In the course of the paper we refer to constructions and inequalities in scattered sources; we list some of these now, as a guide to the flavor of the work.

(a) Lipschitz spaces $A^p$ and $X^p$, and Kolmogorov's estimates of the sizes of sets in these spaces, under the name "a-entropy" [13] and [16, 17, 18 Ch. 10].

(b) Hausdorff measures, Hausdorff dimension, and construction of special "dyadic" sets [6 I, II].

(c) Gaussian processes and Brownian motion [4 XI, XIV].

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(d) Special sets introduced in abstract harmonic analysis, in particular, Kronecker sets [3], [5, VIII, 14].
(e) Classical theorems on sets of uniqueness and sets of multiplicity [6, VI], [33, IX (6, 7, 8)].
(f) Banach space methods in the construction of examples to (d) and (e) [5, VII, 8, 9, 12].

1. Theorem 1. Let $E$ be a compact subset of $(0, +\infty)$ of positive Haussdorff $\beta$-measure $0 < \beta < 1$, and let $S$ be the class of diffusions of $(0, +\infty)$ into itself of class $A^{1,0}$. Then it is almost sure that all sets $X_\beta f_t(E)$ are $M_2$-sets, $f \in S$.

To each $\beta$ in $(0,1)$ there is a compact set $E$ of dimension $\beta$, with this property: for almost all paths $X_t$, there is a randomly mapping $f_t$ of class $C^{1,0}$, such that $X_\beta f_t(E)$ is a Kronecker set.

In explanation of Theorem 1, we recall Frostman's theorem on Hausdorff measures [6, I, II]; in consequence of this theorem $E$ carries a probability measure $\mu$ whose primitive belongs to $A^{1,0}$, $\mu(a, a+h) \leq h^\beta$ for all intervals $(a, a+h)$ as the conclusion is rather trivial if $E$ has positive Lebesgue measure, we suppose the opposite. The class $S$ is the union of sets $E_t$, each bounded in the space $A^{1,0}$ on an interval $[0, L]$ containing $E_t$; moreover $f_t^* > h^{-1}$ on $[0, L]$ for each $f_t$ in $E_t$.

1. For each $f_t$ mapping $(0, +\infty)$ into itself, there is a probability measure $\sigma = \sigma(X_t, f_t$) carried by $X_\beta f_t(E)$; its transform is

$$\phi(u) = \int \exp(-2\pi i u \mu(f_t(t))) dt, \quad -\infty < u < \infty.$$ 

We shall prove that $\phi(u) \to 0$ as $u \to +\infty$ along the sequence $1, \sqrt{2}, \ldots, \sqrt{u}$, uniformly for all $f_t$ in $E_t$. For this we require a very precise bound for the moments of $\phi(u)$ [4, p. 189]

$$E(|\phi(u)|^q) \leq C_q u^{-\alpha u^\delta},$$

where $q = 1, 2, 3, \ldots$ and $C_q$ is valid for all $f_t$ in $E_t$. To obtain this estimate from the one cited, we have merely to observe that the probability measures $\mu(f_t^*)$, with $f_t^* > h^{-1}$, are subject to a uniform Lipschitz condition in dimension $\beta$. For fixed $\delta > 0$, inequality (1) leads to

$$P(|\phi(u)| > \delta) \leq \exp(-C_\delta u^{-\alpha u^\delta}).$$

2. To each $\eta$ in $(0,1)$ we choose a finite set $T_\eta \subseteq E_t$, such that each function $f_t$ in $E_t$ has distance $\eta$ from $T_\eta$. In the uniform metric over $E$, from Kolmogorov's estimates [13], [18, Ch. 10] it is known that the cardinality $|T_\eta|$ can be brought down to $\exp(C_\eta u^{-\alpha u^\delta})$; but $E$ has measure $0$, so the observations of Veech [22] allow a bound of the form

$$\exp(-\eta^\beta \phi_u(\eta), \phi_u(0+)) = 0.$$ 

Thus we can define a function $q(\eta)$ such that $q(\eta) = o(\eta)$ as $\eta \to 0+$, while $\log|T_\eta| = o(q(\eta))$. Let us write $T_\eta(q(\eta)) = T_\eta(q(\eta))$.

Comparing the size of $T_\eta^*(u^-)$ with our estimate for $P(|\phi(u)| > \delta)$ we can conclude that almost surely we have

$$\sup^* \left| \int \exp(-2\pi i k u^\delta X(f(t))) dt \right| \to 0,$$

where $\sup^*$ means the supremum for $f$ in $T_\eta^*(u^-)$.

3. In the last step of the proof we need controls on the oscillation of $X_t$: these do not follow from Lévy's celebrated work. We let $g$ for an element of $T_\eta^*(u^-)$ and $f$ for an element $s_t$, such that $|f(t) - g(t)| \leq q(u^-)$ uniformly in $E$. We seek a bound for

$$\left| \int \min(1, |uX(f(t)) - uX(g(t))|) \right| \mu(dt).$$

Before passing to this final part of the proof, we notice that a bound $o(1)$, valid uniformly with respect $g$ in $T_\eta^*(u^-)$, as $u \to +\infty$ through values $k^\beta$, will prove the first assertion in Theorem 1.

Let us fix $g$ and estimate the $\mu$-measure of the set on the $t$-axis $B_1$: $X$ oscillates more than $u^{-1} \delta$ on the interval $[g(t) - q(u^-), g(t) + q(u^-)]$.

This is nothing but the $\mu = \mu(1) = \mu_g$ measure of the set on the $t$-axis $B_1$: $X$ oscillates more than $u^{-1} \delta$ on the interval $[g(t) - q(u^-), g(t) + q(u^-)]$.

Let us divide the $s$-axis into adjacent intervals $(I_p) = \delta$ of length exactly $q(u^-)$ and let $X^\delta$ be the sum $\sum (I_p) = X^\delta$ oscillates by more than $u^{-1} \delta$ over $I_p$. Now $X^\delta$ is a sum of independent random variables $y_t$, such that $0 \leq y_t \leq \mu(1)$ and $P(y_t < 0) = o(u^-)$, because $q(u^-) = o(u^-)$. Let $\delta$ be the positive number defined by the inequality $\lambda \mu(1) = 1$. Elementary inequalities yield $E(\exp(\lambda X^\delta)) \leq \exp(\lambda \mu(1))$, $P(X^\delta > 0) \leq \exp(\lambda \mu(1)) - 2\delta$. To finish the argument we recall that all the measures $\mu = \mu(1)$ satisfy a uniform Lipschitz condition in exponent $\delta$, so $\mu(I_p) \leq \exp(-\alpha u^{-\alpha u^\delta})$ for large $u$, while $\log|T_\eta^*(u^-)| = o(u^{-\alpha u^\delta})$. Setting $u = k^\beta$, we obtain a convergent sequence for each $\delta > 0$. The same argument applies of course to the oscillation of $X(t) + q(u^-)$ and $X(t) - q(u^-)$. With the aid of a simple diagram, we see that the $\mu$ measures of sets $B_1$ tend to $0$, uniformly with respect to $g$ in $T_\eta^*(u^-)$.

A moment's reflection shows that it would be sufficient to obtain the bounds of this paragraph along the sequence $u = k^\beta$, and for this we need much weaker bounds on $\mu$, for example $\mu(a, a+h) \leq \exp(\log(k^\beta)^{-1})$ for $0 < \delta < e^{-\delta}$.

4. The set $P$ is a "dyadic" set determined by a strictly increasing sequence $M = (m_n)$ of positive integers: $P$ is the set of all sums $\sum_c z_n^m$, $z_n = 0$ or 1. The sequence $M$ must have two properties
(i) \( m_k < \beta^{-1} k + o(k) \) for all \( k \geq 1 \),
(ii) \( m_{k+1} > \beta^{-1} k + 10 \log k \), for all \( k \) in an infinite sequence \( N \) of positive integers.

Property (i) ensures that \( f \) has Hausdorff dimension \( \beta \), and indeed \( f \) cannot be expressed as a countable union of sets \( F_k \) with \( \dim F_k < \beta \).

Let \( Y \) be the Banach space \( L^1 [0,1] \), or \( Y = C[0,1] \) in case \( \beta = r \), an integer. Let \( Y^+ \) be the open subspace of \( Y \) containing positive functions; \( Y^+ \) is separable. The second part of Theorem 1 is a consequence of this statement.

For all paths \( X \) except a set of probability 0, the set of functions \( f \in Y^+ \) such that \( X \mathcal{C}(f) \) is a Kronecker set, is a dense \( G_\delta \) subset in \( Y \).

Observing that \( Y^+ \) contains an open set \( Y^+_{\epsilon} \), defined by the inequalities \( f > 0, \ f' > 0 \) on \( [0,1] \), and that each element of \( Y^+_{\epsilon} \) admits an extension to a diffeomorphism of class \( C^2 \) on \( [0, +\infty) \), we obtain the asserted properties of \( y \).

Let \( V \) be an open set in \( Y^+, \) a continuous real function on \( [0,1] \) and \( \epsilon > 0 \). We shall prove that for almost all \( X \), there is a function \( f \) in \( V \), and a number \( u > 0 \), such that \( \|X \mathcal{C}(f(t)) - g(t)\| \leq \epsilon \) (modulo 1) for all \( t \) in \( F \). Since \( C[0,1] \) and \( \mathcal{C}^2 \) are separable, this leads by a familiar path to the result stated above [8, 9, 12].

5. Let \( k \in \mathbb{N} \) so that \( m_{k+1} > \beta^{-1} k + 10 \log k > m_k + 1 \), and let \( F = \bigcup F_k \) be the splitting of \( F \) determined by \( \varphi_1, \ldots, \varphi_k \). Each \( F_k \) has diameter \( \leq \varphi_k = 2^{-1} 2^{-m_k} \) and the sets \( F_k \) have mutual distances \( \geq b_k = 2^{-1} 2^{-m_k} \). For large \( k \) we find \( b_k \geq a_k^2 k^2 \). We choose a function \( f_k \) in \( Y^+ \) and elements \( \varphi_k \) of \( F_k \).

We assume now that the oscillation of \( X(t) \) over each interval \( [t, t + 1) \) is \( X_{t+1} - X_t \leq k^{-1} b_k^{2^t} \). Later we show that this holds for sufficient large \( k \) and \( \epsilon \), almost surely. Setting \( u = \varphi_k \), we can then choose numbers \( f_k \) such that \( \|X_{t+1} - X_t - f_k\| \leq \varphi_k \), while \( X \mathcal{C}(f_k) = g(X_t) \) (modulo 1), \( 1 < s < 2^t \). Now let \( f \) be \( \beta^{-1} k + 10 \log k \), defined so that it takes the values \( f_k \) and \( f_{k+1} \) is constant over each \( F_k \). This can be done with a function \( f \) such that \( \|f - f_k\| \leq C_k (1 + b_k^{2^t}) = c(1) [8] \); thus for large \( k \) \( f > 0 \) and \( \mathcal{C}(f) \) is defined on \( F \), while \( f \in Y^+ \). Moreover, \( f \) oscillates no more than \( f_k \) on \( F_k \), thus \( f_k \) oscillates \( \leq C(f_k) \varphi_k \).

Returning to the oscillation of \( X(t) \) over an interval of length \( O_k \), we must estimate the probability of the event \( |X(t)| \leq k^{-1} b_k^{2^t} \). This inequality implies \( \|X(m_k^2 \lambda - X(m_k^2 + k^{-1})) < 2k^{-1} \), for integers \( m \geq 0 \), \( m \leq k^{-1} - 1 \). Each of these inequalities has \( P = C_k < 1 \), so their intersection has \( P < \exp -C_k k^2 \). Moreover, the number of intervals in question is \( 2^t \), and \( \sum 2^t \exp -C_k k^2 < \infty \), so the necessary oscillations are obtained almost surely for all large \( k \).

There is an odd variant of the theorem above whose proof is almost identical with the one just completed: everything can be accomplished in the set \( Y^+ \) defined by \( f > 0 \) and \( f' = 1 \) on \( F \). This shows that \( Y^+ \) is almost as massive as \( Y^+ \), as a subset of \( \mathcal{C}(f) \).

II. Let us say that a set \( E \) has \( \mathcal{C}^2 \)-multiplicity if \( X \mathcal{C}(f) \) is an \( \mathcal{A}_0 \) set for every \( f \) in any space \( A_k \), \( k > 1 \), with \( f > 0, f' > 0 \) on \([0,1]\).

Theorem 2. Let \( E \) be the dyadic set based on a sequence \( \mathcal{M} \) such that \( m_{k+1} = m_k + \alpha (m_k \log m_k) \). Then \( E \) is \( \mathcal{C}^2 \)-multiplicity.

The cited condition on the sequence \( \mathcal{M} \) is simply what is required to balance an inequality in the course of the proof; its interest is in examples leading to dyadic sets of Hausdorff dimension 0, for example, \( m_k = k \).

The method of proof also applies on the sequence \( \mathcal{M} \) with positive density \( d' \): \( \lim k^{-1} m_k = d' < \infty \).

For large numbers \( u > 1 \) we define \( k = k(u) \) by the inequalities \( 2^{-n+1} \leq u \leq 2^{-n} \). Thus, if \( n \) is fixed in \([0,1] \) then, since \( m_{k+1}/m_k - 1 \), \( m_k \log m_k \) and the hypotheses on \( m_k \) yield \( k(u) - k(u') = \log m_k (k(u')) \). To apply these calculations we take a bounded subset \( S \subset [0,1] \), and choose \( \beta \) so that \( 1 < \alpha' < \alpha \).

The product measure on \( E \) admits a factorization for each \( \alpha > 1 \): in \( \mu_k \), we group the factors corresponding to indices \( k \) in \([1, k(u')] \), and in \( \mu_k \) we take indices \( k > k(u') \). An interval of length \( u^2 \), has \( \mu_k \)-measure \( 2^{k(u')} - 2^{k(u')} \), and the support of \( \mu_k \) has length \( L(u) = 2 \cdot 2^{k(u') - 1} < 2u^{-1} \).

The integral
\[
\int \exp -2\pi i u X \mathcal{C}(f(t)) d(t)
\]
can be bounded by the maximum of integrals
\[
\int \exp -2\pi i u X \mathcal{C}(g(t)) d(t) = I(g, u),
\]
where the functions \( g \) are subject to inequalities
\[
|g| \leq B, \quad |g|' \leq B, \quad |g'(a)| - |g'(b)| \leq B |a - b|^{-1},
\]
and \( g' \geq C > 0 \) as well if we impose the last inequality on the functions in \( S \). On \([0, L] \) we have by the mean-value theorem, and \( \alpha' > 1 \),
\[
g(t) = g(0) + t g'(0) + O(t') = g(0) + t g'(0) + O(u^{-1} u'),
\]
\[
X \mathcal{C}(g(t)) = X(g(0) + t g'(0) + O(u^{-1} u')), \quad \text{almost surely},
\]
by Lévy's bounds on the modulus of continuity. Moreover, if \(|g(0) - g'(0)| \leq w^2 \) and \(|g(0) - g'(0)| < w^2 \), then \(g - g' = O(w^2) + O(w^2u^{-1})\) on \([0, L]\). Thus to bound the supremum with respect to \(g \in S\), we require \(u\)'s inequalities on integrals of \(I(g, u)\).

Now the \(\mu_E\)-measure of intervals of length \(u \leq 1\), is at most \(2^{-\ell} = k(u) - k(w)\). Moreover, \(s \log \log (k(u)) \rightarrow +\infty\), or \(s \log \log (k(u)) \rightarrow +\infty\), so \(u^2\) \(= o(\log k^{-1})\). Using Kahane's inequality for the moments of \(I(g, u)\) [4, p. 165], much as in the proof of Theorem 1, we find for any \(\delta > 0\) and any \(A > 1\),

\[ P(I(g, u) > \delta) < u^{-\delta}, \quad u = u(A, \delta, g \in S). \]

As we required only \(u\)'s inequalities on individual integrals \(I(g, u)\), the proof is complete.

There is a much more powerful method for uniform approximation; unfortunately in this situation it leads to exactly the same term, namely \(o(m_3/\log m_3)\). The idea is to find a covering of \(E\) by intervals \(I\) of length \(u^2\), whose number \(N\) is estimated by means of the integral \(k(u^{-1})\). On each interval \(I\) we replace \(f \in S\) by its secant, obtaining a function \(f_I\), and then \(|f - f_I| = O(u^{-1})\). Now the functions \(f_I\) belong to a subspace of \(C(E)\), of dimension \(2N\), and here we can use an inequality between widths and entropy [18, p. 163] in Banach spaces; the resulting inequality is obtained for a in (0, 1) by Yossgin [22]. A better estimate, in \(L^1(\mu)\) instead of \(C(E)\), would allow an improvement in the term \(m_3/\log m_3\).

However, most results about approximation of functions in \(S^1\), in the usual spaces \(L^p(0, 1)\), suggest that no improvement can be expected in passing from \(C(E)\) to \(L^p(\mu)\) [1, 15].

III. The construction in this chapter is complementary to the foregoing, leading to random exceptional functions in Banach spaces of \(C^\infty\) functions of very high smoothness. In defining these spaces we follow the classical method of Denjoy–Carleman [7, Y, 20 ch. 19, 19] so that each element of any such space will be considered to be determined by its sequence of derivatives at an arbitrary real number. Let \(1 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_n < \infty\, \lim_{x \to e^{-1}} = e^{-1}\), let \(M_\alpha = \lambda_1, \ldots, \lambda_n\), and let \(Y\) be the Banach space of \(C^\infty\) functions defined by \(|f| = \sup \sup M_\alpha |f|^{\alpha}(t)\). To obtain a separable subspace, we define \(Y_\alpha\) to be the closed linear span of integrable functions \(f\) such that \(f\) has compact support. Finally, we require

(i) \[ \sum \lambda_i^{-1} < \infty \]

(ii) \[ s \log n < \lambda_n \quad (n > 10). \]

The first of these ensures the quasi-analyticity of the class \(Y [7, 20]\), while the second is imposed for technical reasons.

Theorem 3. Suppose that in the sequence \(m_n\), we have \(m_{\nu+1} > m_\nu^{1/2 + \nu}\) infinitely often. Then for almost all paths \(X\), there is a mapping \(f\) in \(Y_s\), such that \(f > 0\) and \(f > 0\) on the dyadic set \(F\), and \(X(f(F))\) is almost surely an \(M_\nu\)-set [4, p. 165].

1. Before assembling all the elements, we list some technical points necessary in the proof.

a) Given an interval \([0, L]\) in \([0, \infty)\) and an integer \(k \geq 1\) we consider a condition on the oscillation of \(X(t)\) over \([gk^{-1}L, gk^{-1}L + k^{-1}L]\), \(g = 0, 1, 2, \ldots\). We require that

\[ |X(gk^{-1}L + k^{-1}L) - X(gk^{-1}L)| > k^{-1}L^{1/2}, \]

and

\[ |X(s) - X(t)| < k^{-1}(L)^{1/2} |s - t|^{1/2} \]

for all numbers \(s\) and \(t\) in our interval of length \(k^{-1}L\).

The probability of each part tends to 1 as \(k\) increases, and for each \(g\) is independent of \(L\). Hence the probability that the condition holds for three consecutive values of \(g = 0, 1, \ldots, k - 1\), exceeds \(1 - \varepsilon(k)\), \(\varepsilon(k) \to 0\).

We apply this to find a middle interval \(I\) over which \(X(t)\) fills an interval of length \(k^{-1}L^{1/2}\), and then majorize the oscillation of \(X\) over small intervals that meet \(I\) — these intervals of course are contained in the union of the three intervals, and the oscillation can be controlled by tripling the bounds above.

b) The function of a real variable \(p(x) = \sup |p(x)|\) can be estimated \(p(x) < \exp(\delta(\log |x|))\) for large \(x\). This becomes clear once we verify that the supremum is attained with \(n\) of magnitude near \(x/\log x\).

c) The function \(G_p(x) = \sin^2(2\pi x)\), \((p = 1, 2, \ldots)\) has derivative \(D^p G_p \leq g^p\) in absolute value. This follows from the representation of \(G_p\) as a Fourier integral.

2. Now we follow as closely as possible the second part of Theorem 1, introducing again the splitting \(F = \bigcup F_r\) and the \(2^r\) intervals \(I_r = (a_n)\) of \(F\). The event described in (a) holds for each \(r\), when \(k\) is large. Thus the interval of length \(L = 4^rN^{1/2} + 1\) about \(f(a_n)\) contains a subinterval of length \(k^{-1} L = 4^rN^{-1} + 1\) on which \(X(t)\) oscillates at least \(k^{2N^{1/2}}\), and on each interval of length \(2^rN^{1/2} + 1\), intersecting \(I_r\), \(X(t)\) oscillates at most \(2^rN^{1/2} + 1\). Therefore we choose a frequency \(u = u_0 = k^{-1/2} + 1\), and corresponding displacements \(\xi_0\) of absolute value \(\leq k^{-1/2} + 1\).

For the displacement of \(f_0\) we choose \(g = L \xi_0 T_0 |T(\xi_0)|\) with \(p = \log k\) and \(T = A^{2N^2}\), \(A\) being a large constant.
To bound $g$ we first investigate the sum $\sum [g_{i,j}(\tau(x-a))]$, where the dash means that we omit the index $x$ at which $|x-a|$ attains its smallest value. The numbers $\tau(x-a)$ generated by the remaining indices $x$ can be estimated from below by the sequence $1, 2, \ldots$, taken twice. As $[g_{i,j}(y)] < |y|^p$ and $p - \log b$, the resulting sum has order of magnitude $b^{-1}$, with $A(1) = \log A - \log 2$. (Therefore we choose $A$ so that $\log A > 30$, say.) This shows that the supremum of $|g|$ is comparable with max $\{\delta, \xi\}$, hence it tends to 0. We also require a strong inequality on the error $g - \xi$ on $F$. Now $p$ has diameter $2 \leq \delta^m$, so $[g_{i,j}(\tau(x-a))] < |p| \leq \delta^{-m} \leq \delta^{-m+1/2}$. Adding to this the bounds obtained before, we obtain an inequality $|g - \xi| < 2^{m+1}$ on $F$.

For the norm of $g$ in $Y$, we have an upper bound $2^{m+1} \max \{\xi, \delta^{-m} \} \times \max \{\delta^m, \delta^{-m} \} \leq \delta^{-m+1/2} \cdot \max \{\delta^m, \delta^{-m} \}$, following (b) and (c). Now $|p| = \delta^{2m} \log \delta \cdot \log p < \delta^{2m} \log \delta \cdot \log \delta = o(\delta^m)$. In view of an upper bound for $\max \{\delta^m, \delta^{-m} \}$, we have $2^{m+1} \max \{\xi, \delta^{-m+1/2} \} < \exp - \delta^{m+1/2}$ (for $\delta$ near $\log 2$), while $\log \delta g \leq \delta \exp \log p = o(\delta^m)$. This accomplishes the estimation of norms.

To show now the efficacy of the displaced function $f_0 + g = f_0 + g \xi$, we recall that on $F$, it differs from $f_0 + \xi$, the correct value, by $o(\delta^m)$. Moreover $f_0 \xi$, so we are considering the oscillation of $X(t)$ over a certain interval of length $\delta^m$, the diameter of $F$. But $f_0(x) + \xi$, has the special property that $X(t)$ oscillates at most $O(\delta^m \log \delta)$, so that the oscillation of $X(t)$ is thus $O(\delta^m \log \delta)$, the construction is complete.

IV. In this chapter we study sets $f : X(E)$, using a method of estimating Fourier-Stieltjes coefficients slightly less precise than the one used for Theorem 1, but more flexible. Other methods for obtaining estimates, involving differentiable transformations $f$, were introduced in [10], [11].

1. THEOREM 2. Let $\mu$ be a probability measure on $[0, 1]$ whose modulus of continuity of $\sup |\mu| \leq o(\log \delta^{-1})$, and let $f$ be a function of class $O(\delta^{-1})$. Then $f : X(E)$ is almost surely an $m$-set. This remains true if $f$ is monotonic and $f > 0$ almost everywhere.

Proof. Because $X(E)$ is almost surely a bounded set, we can suppose that $f$ is bounded between two constants, $0 < c_i \leq f \leq c_1$, and uniformly continuous. For each $u > 1$ we divide $[0, 1]$ into $k = k(u)$ adjacent intervals and divide the integral

$$\int exp - 2 \sin f : X(t)(\mu)(dt)$$

into the corresponding partial integrals $J_n$, $0 \leq n \leq k-1$. Then we denote by $I_n$ and $I$, the sums of these for $n$ even and odd, respectively. Focusing on $I_n$ for simplicity, we denote by $J_n^*$ the conditional expectation of $J_n$.

relative to the field of $X(t)$, $0 \leq t \leq (n-2)h^{-1}$. By the Markov property, this is the expectation given $X((n-1)h^{-1})$. We shall prove that $|J_n^*| = o(h^{-1/2} \delta^{-1})$ uniformly with respect to $n$, almost everywhere. The last statement in fact does not involve $\mu$ at all; we need only estimate the supremum of all integrals

$$\int exp - 2 \sin f : X(t)(\mu)(dt)$$

for real $b$ and $b \geq u^{-1}$. We obtain this formulation from the conditional distribution of $X(t)$ given $X(0)$, with $t - u^{-1}$. To accomplish the qualitative study of the integrals we show that the functions $\exp - 2 \sin f : X(t)(\mu)(dt)$ for $\gamma > 0$ tend weakly to 0 in $L^2$, uniformly with respect to $\gamma _1$. In fact, if $I$ is an interval over which $f(b, x)$ increases by exactly $u^{-1}$, then $|J_n| \approx u^{-1} I^{-1} = o(\delta^{-1})$. Hence it differs from its secant line over $I$, by $o(1) |I| = o(u^{-1})$. Thus

$$\int exp - 2 \sin f : X(t)(\mu)(dt)$$

and since $|I| \approx u^{-1}$, this proves the weak convergence needed.

In analyzing the sum $\sum J_n^* \approx 2 \sin X$, say, we have first $\sum E(I_n) \approx 2 \sin Y$, and $Y$ is a probability measure, and $b = \sup \delta \approx o(\log \delta^{-1})$, so $E(I_n) = \delta^{-1}$. Thus, if $y > 0$ and $y \delta^m$, we obtain by Taylor's formula $E(y : \sum J_n^*) < 4 \exp A y^{-1}$, with $A$ an absolute constant. For small $\delta > 0$ we get

$$P(\sum Y_n > 0) < 4 \exp - 4 \delta^{-1} 2^{m-1}.$$

Putting in this inequality, $b - b = o(\log \delta^{-1})$ and $u = 1, 2^{1/2}, 2^{3/2}, \ldots$, we obtain a convergent series for every $\delta > 0$. Since we saw before that $|J_n| \approx u^{-1} I^{-1} = o(1)$, the proof is complete.

The extension to functions $f$ not necessarily of class $O(\delta^{-1})$, follows the technique of [10]; the essential point is this: to each $\delta > 0$ there is a $O(\delta^{-1})$ function $f$, with $f > 0$ and $m (f) < \delta$ [22, II, pp. 73-77]. Now $I(t)$ has an absolutely continuous distribution when $\delta > 0$, and so if $S$ denotes a measurable set in $(-\infty, \infty)$ then by Fubini's theorem,

$$E(\mu(t) : X(t), S) \rightarrow 0 \quad \text{when } m(S) \rightarrow 0.$$

Applying this when $S = (f(t) \neq g(t))$, we obtain the theorem for the more general function $f$ mentioned in the last sentence.

2. To obtain quantitative conclusions about integrals (1) containing $f : X(t)$ it is clear that we need bounds on expected values of the form (2). These are given in the following statement.
LEMMA. Suppose that all functions $f$ in a set $S$ are subject to inequalities

$$0 < C_1 < f' < C_2 < \infty, \quad |f''| < C_3 < \infty, \quad 1 \leq n \leq r + 1.$$  

Then $I = \int \exp -2\pi i u f(x) dx - \frac{1}{2 \pi} F^2(x) F^2(x) dx = O((u^{-\epsilon})^r)$ for large $u$, uniformly for $f$ in $S$.

Proof. We can assume that $f(0) = 0$, and write $F$ for the inverse mapping of $(-\infty, \infty)$ onto itself, so the functions satisfy inequalities analogous to the functions $f$; moreover

$$I = \int \exp -2\pi i u F(x) - \frac{1}{2 \pi} F^2(x) F(x) dx.$$  

To obtain the bound $O((u^{-\epsilon}))$ we shall show that the cofactor of $\exp -2\pi i u F(x)$ has $r$ derivatives, uniformly bounded in $E'(-\infty, \infty)$. By Leibniz's formula, we only need $L^1$-estimates for the derivatives of $\exp -\frac{1}{2 \pi} F^2(x)$. It is clear, however, that each derivative can be expressed by $p[F(x), F'(x), \ldots, x \times \exp -\frac{1}{2 \pi} F^2(x)$ for certain polynomials $p$. Since $F$, $\ldots, F^n$ are subject to uniform bounds, $L^1$-estimates follow from the presence of the term $\exp -\frac{1}{2 \pi} F(x)$.

The integral (2) can be treated by the lemma when $u \gg 1$, by the substitution $f(x) = x^{n-1} f(h + 1)$, for the $n$th derivative of $f$ is $x^{n-1}$.

THEOREM 5. Let $\mu$ be a probability measure, in $E$, satisfying a Lipschitz condition in exponent $\beta$, $0 < \beta < 1$, and let $0 \leq \alpha < \beta$. Then we have almost surely

$$\int \exp -2\pi i u F(x) \mu(dx) = O(u^{-\epsilon})$$

for all $f$ in $C^\infty(-\infty, \infty)$ with $f' > 0$.

Theorem 5 expresses a property of the sets $f(x)$; each has "Fourier-dimension" $\geq 2n$. As we can choose $E$ to have Hausdorff dimension exactly $\beta$, and then apply the conclusion with a sequence $\alpha_n \rightarrow \beta$, $f(x) \in E$ is a Salem set [21] of dimension $2\beta$. A similar purpose was achieved in [21], but the sets constructed there had a very special structure.

In the proof of Theorem 5 we use the fact that $X(E)$ is almost surely a bounded linear set, so it suffices to make the argument for functions $f$ that are linear outside a finite interval. Indeed, for each $x$ in $C^\infty(-\infty, \infty)$, and each interval $[-L, L]$, we can find a special function $f_0$, coincident with $f$ on $[-L, L]$, and $f_0 > 0$ everywhere if $f' > 0$ on $[-L, L]$. The same is of course true for the spaces $A'$; $s > 1$. Given $a < b$ we choose $b$ so that $a < b < 2$ and $2a < b$, and then an integer $s$ so that $s(b-a)/2 > 2$ and $(s-1)(b-a)/2 > 1$.

Then we apply the method used before, dividing the compact set $E$ into adjacent intervals of length approximately $V \equiv u^{-\epsilon}$. In the remainder of the proof we omit details similar to those encountered in the proof of Theorem 1, merely indicating the estimations necessary to obtain the bound $u^{-\epsilon}$. The expectations in this case take the form mentioned after the lemma, with $\lambda \gg u^{-\epsilon}$, so that $a \ll u^{-\epsilon}$ and $(s-1)(b-a)/2 > 1$.

Thus the expectations are all $O(u^{-\epsilon})$. Now the $\mu$-measure of each interval is $O(u^{-\epsilon})$, so we obtain bounds of the type $E(\exp |y|) < \exp A y^2 u^{-\epsilon}$ for $0 \leq y \leq u^3$. The resulting estimate for $E(|F| > u^{-\epsilon})$ then becomes $O(u^{-2\epsilon})$. To obtain approximation of $\exp -2\pi i u f(x)$ within $u^{-\epsilon}$, say, we require approximation to $f$ within $u^{-\epsilon}$. Since $2a^{-2} < a^{-2}$, and we are operating in $C \approx C^\infty$, the estimates of entropy used before are still adequate. To obtain the rate of decrease $u^{-\epsilon}$ for real numbers $u$ we use the device from harmonic analysis described in [4, p. 165].

The following lemma uses the ideas of "widths" in a Banach space, as set forth in [17], [18, Ch. 9].

3. LEMMA. Let $E$ be a bounded subset of $A'(a, b)$; $1 < a < 2$. Then for all large numbers $y$, and integers $n \geq 1$, we can find a set $S(y) \subseteq E$ containing exp $A N$ logy elements with the following property: to each set $F \subseteq [a, b]$ covered by $N$ intervals of length $y^{-n}$, and each $f$ in $S$, there is a $f_0$ in $S(y)$, such that $|f - f_0| < y^{-n}$ everywhere on $F$.

Proof. First we divide $[a, b]$ into adjacent intervals $I$ of length $y^{-n}$ (and one odd interval at most), so the number of intervals is $\leq (b-a) y^{-1} < (2(b-a)y^{-1}$ for large $y$. Each set $N_I$, covered by $N$ intervals of length $y^{-1}$, meets at most $2N$ intervals $J$ just constructed, and the number of ways in which these $2N$ can be chosen is $< \exp A N \log y$. Thus, if we construct subsets $I_0 \subseteq S$, containing exp $A N$ logy elements, for each selection $S(\subseteq 2N$ intervals $J$, their union is the set $S(y)$ sought. Let $G$ be a fixed union of that type.

For each $f$ in $S$, as we denote by $f_0$, the function, defined on $G$, linear on each constituent $J$ of $G$ and coincident with $f$ on the end-points of those intervals. Then $|f - f_0| < y^{-n}$ by the mean-value theorem. The functions $f_0$ belong to a bounded subset of $C(G)$, namely a ball with radius independent of $y$. Moreover, the functions $f_0$ belong to a subspace of dimension $A N$, so the largest collection of functions $f_0$, with mutual distances $\geq y^{-n}$ -- that is, a maximal $y^{-n}$-disjointable subset -- contains exp $A N$ logy elements. This means that at most exp $A N$ logy elements of $S$ can be $My^{-n}$-disjointable, with an $M$ depending on $S$ alone. Thus, a set $S(y) \subseteq S$, containing exp $A N$ logy elements gives approximation within $My^{-n}$, and a slight adjustment yields the statement of the lemma. (In case $a > 1$ we use a more complicated interpolation than the secant but the main ideas remain similar). Compare [18, p. 163].

THEOREM 6. Let $E$ be the dyadic set constructed in Theorem 2; then it is almost sure that all sets $f(x)$, with $f \in N'$ and $\alpha > 1$, $f' > 0$, are $M'$-sets.
Proof. It is sufficient to make the proof for a fixed \( a > 1 \), and for this we choose \( C \) in the interval \( a^{-2} < C < 1 \). Let \( N \) be the number of intervals of length \( u^{-2} \) required to cover \( E \). Then \( N \) can be estimated by \( 2^2 \), where \( m_{2} \log 2 \geq 2 \log 2 \geq m_{2} \log 2 \). Thus it is almost sure that \( N \) intervals of length \( u^{-1/2} \) suffice to cover \( E \), so we can apply the lemma with \( y = u \), and note that \( \exp \lambda n \log y = \exp \lambda n \log u \). This gives a bound on the number of intervals to be estimated, for a fixed \( u > 1 \).

However, the estimates of probabilities take the form \( \exp -t^{2} \), where \( m_{2} \log 2 \geq 2 \log 2 \geq m_{2} \log 2 \). Thus \( m_{2} \leq C^{-1} m_{2} \), so that \( \log m_{2} = o(k^{-1}) \), whence finally, \( \lambda n \log u = 2 \log u = o(\lambda n) \). This is sufficient: the remainder of the argument follows Theorem 2.

4. Theorem 5a. Let \( \lambda \) be a probability measure in a set \( E \subseteq (0, + \infty) \), satisfying a Lipschitz condition with exponent \( \beta \) in \( (0, \beta) \). Then it is almost sure that all sets \( f \circ X (E) \), with \( f \) of class \( C^{1} \), \( f > 0 \), are \( M_{\beta} \)-sets.

(a) Since \( 0 < \beta < \frac{1}{2} \), we can assume that \( E \) has Hausdorff dimension exactly \( \beta \); otherwise \( E \) would carry a measure with a higher Lipschitz condition, and then the proof is much easier. We use this condition on \( E \) only to ensure that \( X (E) \) has Lebesgue measure \( 0 \), and thus \( \mu_{\beta} X^{-1} \) is almost surely singular.

(b) For each fixed \( \eta > 0 \), there is a sequence \( (k_{i}) \) of compact sets, such that \( m(k_{i}) < \eta \) for each \( i \), and \( \sup_{i} \lambda(k_{i}) = 1 \) for every singular probability measure \( \lambda \) on \( (0, + \infty) \). Applying this to the measure \( \mu_{\beta} X^{-1} \) for a sequence \( \eta_{1} > \eta_{2} > \cdots > \eta_{n} > \cdots \), we find a subset \( \Omega \) of the probability space \( X \), with \( P(\Omega) = 1 \), and for each \( \eta > 0 \), there is a compact set \( \mathbb{k} \) of \( \Omega \), with \( m(k_{i}) < \eta \) and \( \sup_{i} \mu_{\beta} X^{-1}(k_{i}) > 1 - \eta \) for each \( \mathbb{k} \).

(c) Now we consider the problem of approximating the integrals \( \int f \exp -2\pi i f \cdot X(t)/\mu(\mathbb{E}) \), where the elements \( f \) of a bounded subset of \( \mathbb{C} \), \( \int f \cdot X(t)/\mu(\mathbb{E}) \). To obtain approximation with error \( \eta > 0 \), it is sufficient to approximate \( f \) with an error \( \eta/8 |w| \), except on a set of \( \mu_{\beta} X^{-1} \)-measure \( \eta/2 \). But when \( X \) is \( \Omega \), this can be accomplished by approximating \( f \) on certain compact sets \( \mathbb{k} \), all of Lebesgue measure \( \eta \).

(d) If we divide \( (0, + \infty) \) into intervals of length \( u^{-1/2} \), then the estimates of probabilities become \( \exp -Cu^{3/2} \), since \( u \) satisfies the Lipschitz condition of order \( \beta \). However, if we express the bound \( \exp -Cu^{3/2} \), obtained before in the form \( \exp -Cu^{3/2} /h(u) \), \( h(\infty) = 0 \), then we can employ a division of \( (0, + \infty) \) into intervals of length \( u^{-1/2}/h(u) \), because the probabilities then admit a bound \( \exp -Cu^{3/2} /h(u) \).

The factor \( 1/h(u) \to + \infty \) makes the expected values tend to 0, because \( u^{-1/2}/h(u)^{2} \to + \infty \). Since the estimate of probabilities is small enough in comparison with the number of functions needed in approximation, Theorem 5a is proved.

In the extreme case \( \beta = \frac{1}{2} \), the result seems to hold for \( C^{1/2} \) functions, after some variations in the argument; it is useful to note that \( C^{1/2} \) is a Souslin set in the metric space \( (\infty, \infty) \). The theorem may indeed fail for \( \beta > \frac{1}{2} \); the absolute continuity of the measures \( \mu_{\beta} X^{-1} \) is a difficult problem.

Y. Our aim is to show that the order of smoothness \( (2 \beta )^{-1} \) is best possible. This is not very difficult if we choose a weak condition on \( f \circ X (E) \) that nevertheless prevents \( f \circ X (E) \) from being an \( M_{\beta} \)-set. If, however, we seek properties close to the Lebesgue property, the arguments become much more subtle. Therefore our plan is to find geometric properties of \( X (E) \), for certain dyadic sets \( E \) of Hausdorff dimension arbitrarily close to \( \beta \), and then study transformations of special sets by diffeomorphisms of class \( C^{1/2} \). For definiteness we state a complement to Theorem 5a in a simple form.

Theorem 7. Let \( E \) be a dyadic set constructed over a sequence \( (m_{n}) \), such that \( m_{n} \geq m_{n+1} \geq d \geq 2^{-n} \) for infinitely many integers \( n \). Then for almost all paths \( X \), there is a dense \( \Omega \)-set \( \mathbb{E} \subseteq \lambda_{\infty} \), such that \( f \circ X (E) \) is not an \( M_{\beta} \)-set, for each \( f \) in \( \lambda_{\infty} \).

As before we use a splitting of \( E \) into sets \( E_{p} \), \( 1 < p < 2 \), and choose \( E_{p} \rightarrow \mathbb{E} \). Arguably. Later we shall choose \( k \) to be a special index, but the next statement is valid for all \( k \).

1. Lemma. \( \mathcal{X}(E_{p}) \). Let \( \beta \) be a real number in the interval \( 1 < \beta < 2 \), then \( \beta \) is an integer such that \( (\beta - 1)(\beta - 1) > 1 \). We claim now that with probability near 1 for large \( k \):

any interval of length \( M_{k} = 2^{-k} \) contains at most \( r - 1 \) of the images \( X(E_{p}) \).

To see this we consider increasing \( \mathcal{E}_{p} \), \( \mathcal{E}_{p} \), chosen from the numbers \( \mathcal{E}_{p} \), and the corresponding event: \( \mathcal{X}(E_{p}) \). Because \( (\beta - 1)(\beta - 1) > 1 \), the probability is less than \( 2^{-k} \) for some \( \eta > 0 \), and the claim is established. By Levy’s modulus of continuity, the sets \( X(E_{p}) \).
have length $\varepsilon_0 < 2^{-m_0+1} \cdot m_0+1$ for large $k$, and this length is $o(M_k)$ because $\frac{1}{2^k} > \gamma$.

Let us now summarize the property of $X(E) = \mathbb{F}$ that is used in the remaining steps of the argument: there is a sequence $D_k \to 0$ and corresponding decompositions $X(E) = \bigcup F_k$, with diam $F_k \leq D_k$, such that no interval of length $D_k$ meets more than $r$ of the sets $F_k$, and finally $L_0 = O(D_k^2)$. The last relation follows from the inequalities $1 < \gamma < \delta$.

2. In using these properties of $X(E)$ we need a finer splitting than $\bigcup F_k = \mathbb{F}$. Let $e_k$ be defined by the formula $L_0 = (e_k^n D_k)^{1/2}$, and let $I_1$ be a maximal selection of sets $F_k$, having mutual distances $\geq e_k D_k$. Let $I_2$ be a maximal selection of sets $F_k$ not in $I_1$, having mutual distances $\geq e_k D_k$ etc. Then $I_1 \cup \ldots \cup I_r$ exhausts $(F_k)$ if $e_k$ is small, because any set $F_k$ not selected after $r$ steps, has distance $< e_k D_k$ from $r$ distinct sets $F_k$.

Let $g \in \mathcal{R}^{1/2}$, let $y$ be continuous on $F_k$, and $s > 0$. We shall choose a function $g_k$, with small norm in $\mathcal{R}^{1/2}$ such that

\[ |(e_k^{1/2} D_k^{1/2})(g - g_k) - y| < \varepsilon \text{ modulo } 1 \quad \text{on } I_k, \quad 1 \leq k \leq r. \]

First we define $h_k$ on the intervals $F_k$ occurring in $I_k$, so that the inequality above is an equality modulo $1$ at some point in each $F_k$; this can be accomplished with $|h_k| \leq e_k^{1/2} D_k^{1/2}$ and thus $h_k$ has small norm in $\mathcal{R}^{1/2}$, since the intervals in $I_k$ have mutual distances $e_k D_k$. Also, $L_0 = e_k^{1/2} D_k^{1/2} \to 0$, so the inequality actually holds on the intervals $F_k$, whose length is $L_0$ at most. Next we construct $h_k$ so that $g - h_k$ has the necessary properties on each interval $F_k$ in $I_k$. The critical point here is that $|h_k| < e_k^{1/2} D_k^{1/2}$, so the addition of $h_k$ does not interfere substantially with the inequality attained on $I_k$. Continuing in this manner we construct $g_k = h_1 + \ldots + h_r$.

Writing $a_k = e_k^{1/2} D_k^{1/2}$, we note that $a_k \to +\infty$, while $a_k > e^{-1} a_k, \quad 1 \leq k \leq r$. Thus, taking $\varepsilon = 0$, we have a dense $G_0$-set which is $\mathcal{R}^{1/2}$ whose members have the following properties:

To each $s > 0$, we can find numbers $h_k < \ldots < h_{s+1}$, so that $h > e^{-1}$ and $h_k > e^{-1} h_k$, and one of the inequalities

\[ |h_k(s)| < \varepsilon \text{ modulo } 1, \quad s = 1, 2, \ldots, r, \]

holds for each $s$ in $F_k$.

Thus $f(E)$ is a set $\mathcal{R}^{1/2}$ of Pyateckii–Sapiro [23, p. 346; 6, p. 38] and is a system of uniqueness for trigonometric series: $f(E)$ does not carry a (Schwartz) distribution $\varphi \neq 0$, whose Fourier transform $\hat{\varphi}$ is $\mathcal{L}^\infty$ and tends to zero at infinity. Thus $f(E)$ is not an $\mathcal{M}$-set, since finite measures are distributions.

It is very plausible that $\mathcal{R}^{1/2}$ contains a dense $G_0$-set of functions $f$, such that $f(E)$ is a Helson set [5, 14] but the proof of this would involve machinery from harmonic analysis (21, 19, VII.A).

Instead of introducing ideas divergent from our main topics, we shall show how the construction just completed can be improved. Let us fix an m-tuple $\alpha_1, \ldots, \alpha_m$ of continuous functions on $F_k$, and $\varepsilon > 0$. We shall construct $g_k$ with similar functional-analytic properties to $g_k$. To each $j = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, r$ there will be a number $O(s, j)$ so that

\[ |O(s, j) a_k (g + g_k) - \varepsilon| < \varepsilon \text{ modulo } 1 \quad \text{on } I_k. \]

Moreover, $O(s, j)$ is not too large: $1 \leq O(s, j) \leq O(m, s)$.

In fact, functions $g_k$ and numbers $O(s, j)$ are found by a simple device. Let $a_1, \ldots, a_m$ be rationally independent numbers in $[1, 2]$ and let $R$ be so large that the vectors $(p_1, \ldots, p_m), \quad 1 \leq p \leq R$ form an "$m$-net" modulo 1 in the m-cube; each point in $E$ has distance $< \varepsilon$ from some $m$-tuple $(p_1, \ldots, p_m), \quad 1 \leq p \leq R$. Then we define $O(s, j) = a_j$ for all $s, j$, and construct $g_k$ at the $s$th step by making adjustments equal to some number $p_k a_k, \quad 1 \leq p \leq R$. (For details, see [22].)

Applying Bruhat's theorem in this more complicated situation, we find functions $f$ such that $f(E)$ resembles the union of $r$ Kronecker sets; hence we conjecture that $f(E)$ is a Helson set for all $f$ in $\mathcal{R}^{1/2}$ except a set of first category.

References

An inequality for the distribution of a sum of certain Banach space valued random variables

by

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Abstract. We prove an inequality for the distribution of a sum of independent Banach space valued random variables provided they take values in a space having a norm with a smooth second directional derivative and the random variables have \( 2+\delta \) moments. This inequality is applied to obtain the central limit theorem and the law of the iterated logarithm, and it is shown that these results apply to the \( L^p \) spaces, \( 2 < p < \infty \).

1. Introduction. Throughout the paper \( B \) is a real separable Banach space with norm \( \| \cdot \| \), and all measures on \( B \) are assumed to be defined on the Borel subsets of \( B \) generated by the norm open sets. We denote the topological dual of \( B \) by \( B^* \).

A measure \( \mu \) on \( B \) is called a mean zero Gaussian measure if every continuous linear function \( f \) on \( B \) has a mean zero Gaussian distribution with variance \( \int [f(x)]^2 \mu(dx) \). The bilinear function \( T \) defined on \( B^* \) by

\[
T(f, g) = \int f(x)g(x) \mu(dx)
\]

is called the covariance function of \( \mu \). It is well known that a mean zero Gaussian measure on \( B \) is uniquely determined by its covariance function. This is so because \( T \) uniquely determines \( \mu \) on the Borel subsets of \( B \) generated by the weakly open sets, and since \( B \) is separable, the Borel sets generated by the weakly open sets are the same as those generated by the norm open sets.

However, a mean zero Gaussian measure \( \mu \) on \( B \) is also determined by a unique subspace \( H_\omega \) of \( B \) which has a Hilbert space structure. The norm on \( H_\omega \) will be denoted by \( \| \cdot \|_\omega \), and it is well known that the \( B \) norm \( \| \cdot \| \) is weaker than \( \| \cdot \|_\omega \) on \( H_\omega \). In fact, \( \| \cdot \|_\omega \) is a measurable norm on \( H_\omega \) in the sense of [7]. Since \( \| \cdot \|_\omega \) is weaker than \( \| \cdot \|_\omega \), it follows that \( B^* \) can be linearly embedded (by the restriction map) into the dual of \( H_\omega \), call it

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