The hermitian operators on some Banach spaces

by

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Abstract. The hermitian operators on certain types of Banach spaces are described. It is shown that the hermitian operators on an $p$-direct sum ($1 < p < \infty, p \neq 2$) of a sequence of Banach spaces are precisely the direct sums of hermitian operators on the summand spaces. The spaces $AC[0, 1]$, $C^1[0, 1]$, and $lip-a, 0 < a < 1$, admit only trivial hermitian operators, i.e., real multiples of the identity operator. It is further shown that the set of hermitian operators on the dual space of a $C^*$-algebra $A$ is the closure in the strong operator topology of the set of all adjoints of hermitian operators on $A$.

1. Introduction. Let $X$ be a Banach space (we use complex scalars throughout), and let $T$ be a bounded linear operator mapping $X$ into $X$. $T$ is said to be hermitian if and only if $\|\exp(itT)\| = 1$ for all real $t$. For the background and basic features of the notion of hermitian operator, due to G. Lumer and I. Vidav, the reader is referred to [3]. Let $\mathcal{H}(X)$ denote the algebra of bounded operators on $X$, and let $\mathcal{F}(X)$ be the set of hermitian operators on $X$. In this paper we characterize $\mathcal{F}(X)$ for some special spaces $X$ specifically, for $p$-direct sums of Banach spaces (§ 2), for the spaces $AC[0, 1]$, $C^1[0, 1]$, and $lip-a, 0 < a < 1$ (§ 3), and for the dual space of a $C^*$-algebra (§ 4). It turns out that all the spaces considered in § 3 admit only trivial bounded hermitian operators, i.e., real multiples of the identity operator $I$.

2. Direct sums. Denote by $X^*$ the dual space of the arbitrary Banach space $X$. For $x \in X$, let $\mathcal{S}(x)$ be the set $\{x^* \in X^*: \|x^*\| = 1, x^*(x) = \|x\|\}$. For $T \in \mathcal{H}(X)$ it is well-known [3, p. 84] that $T \in \mathcal{F}(X)$ if and only if $x^* T x$ is real whenever $x \in X$ and $x^* \in \mathcal{S}(x)$. We shall make frequent use of this fact.

In what follows, the $p$-direct sum ($1 \leq p < \infty$) of a sequence $\{X_n\}$ of Banach spaces will be the space of all sequences $\alpha = (\alpha_n)$ in $\prod X_n$ such that $\sum_n \|\alpha_n\|^p < \infty$, with $\|\alpha\| = \left( \sum_n \|\alpha_n\|^p \right)^{1/p}$. The $p$-direct sum will be

* The work of this author was supported by a National Science Foundation grant.
denoted by \( \mathcal{H}(X_\star) \). We denote by \( \mathcal{H}(X_\star) \) the space of all sequences \( x = \{a_n\} \) in \( X_\star \), in that it is easy to see that for each real \( t \), \( \exp(it) \) is an isometry, and hence \( T_x \mathcal{H}(X_\star) \).

Suppose first that \( p > 1 \), and let \( q \) be the index conjugate to \( p \). Identify \( X^\star \) with \( \ell_q(X^\star) \) under the natural isometry. Let us note that (the case \( p = 2 \) included) for each non-zero vector \( x = \{a_n\} \) in \( X \mathcal{H}(X_\star) \) consists of all sequences \( (a_n^\star) \in \ell_q(X^\star) \) such that \( \sum |a_n^\star|^q |a_n|^p \leq ||x||^p \mathcal{H}(X_\star) \) for each \( n \). It is easy to see that such a sequence belongs to \( \mathcal{H}(X_\star) \) (we shall not need the converse of this fact for the proof of the theorem, but we include it for the sake of completeness). Suppose \( s^\star = (a_n^\star) \) belongs to \( \mathcal{H}(X_\star) \). Then we have

\[
||x|| = ||s^\star(x)|| = \sum |a_n^\star| \leq \sum |a_n^\star| \leq ||x||^p \mathcal{H}(X_\star) ||s^\star|| = ||x||.
\]

Clearly, for each \( n, a_n^\star(x^\star) = ||a_n^\star|| \) and \( a_n^\star(x^\star) = 0 \) if and only if \( ||a_n^\star|| = 0 \). Moreover, by virtue of \( (15, p.17) \), there is a constant \( c > 0 \) such that \( ||a_n^\star|| = c a_n^\star \) for all \( n \), it must be \( ||a_n^\star|| > 0 \), and it follows that \( ||a_n^\star||^p = ||a_n^\star||^p ||x||^p \mathcal{H}(X_\star) \).

Next we show that if \( T_x \mathcal{H}(X_\star) \), then \( T_x \mathcal{H}(X) \), and \( \mathcal{H}(X) \), respectively. For each \( i, j \), \( T_{ij} = (P_i P_j) / X_j \) is the \( j \)-th coordinate projection to \( X_j \). Thus, \( P_i P_j = \delta_{ij} \), \( T_{ij} = \delta_{ij} \), \( P_i \), \( P_j \) are the \( i \)-th coordinate projection to \( X_i \), \( a_n^\star \) a non-zero vector in \( X \), \( x^\star = \{a_n^\star\} \) and \( \mathcal{H}(X_\star) \) if and only if \( a_n^\star = 0 \) for \( n \neq k \), \( \mathcal{H}(X_\star) \). Then \( x^\star = (a_n^\star) \mathcal{H}(X_\star) \) if and only if \( a_n^\star = 0 \) for \( n \neq k \), \( a_n^\star \mathcal{H}(X_\star) \). Thus, \( s^\star = (a_n^\star) \mathcal{H}(X_\star) \) if and only if \( a_n^\star = 0 \) for \( n \neq k \), \( a_n^\star \mathcal{H}(X_\star) \). This, in other words, \( s^\star = (a_n^\star) \mathcal{H}(X_\star) \) if and only if \( a_n^\star = 0 \) for \( n \neq k \), \( a_n^\star \mathcal{H}(X_\star) \). Then \( a_n^\star = 0 \) for \( n \neq k \), \( a_m^\star \mathcal{H}(X_\star) \). For arbitrary \( y_k^\star \mathcal{H}(X_\star) \) and \( y_m^\star \mathcal{H}(X_\star) \) define \( s^\star = (a_n^\star) \mathcal{H}(X_\star) \) if and only if \( a_n^\star = 0 \) for \( n \neq k, m \), and \( a_n^\star = 0 \) for \( n \neq k, m \). We have:

\[
\sum a_n^\star(x^\star) = a_k^\star T_{kk} a_k + a_m^\star T_{kk} a_m + a_k^\star T_{mm} a_m + a_m^\star T_{mm} a_k.
\]

We shall now proceed with the proof of the main result.

Theorem. Let \( X \) be a Banach space, and let \( X \) be the \( r \)-direct sum, \( X = X^r \). Let \( T_x \) be the element of \( \mathcal{B}(X_\star) \) whose matrix (relative to the
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given direct sum decomposition of $X$ is \( \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \). Then $T \in \mathcal{H}(X)$ if and only if $Y$ is a Hilbert space.

Proof. The "if" part of the assertion is obvious. Conversely, suppose $T \in \mathcal{H}(X)$. We note that for each ordered pair $\langle y_1, y_2 \rangle$ of elements of $X$, $T(y_1, y_2) = \langle y_1, y_2 \rangle$. For each $y \in Y$, let $\psi(y)$ be the set $\{ y' | (y', y) = 1 \}$. Since $T \in \mathcal{H}(X)$, it is easy to see that $\langle y_1, y_2 \rangle = \langle y_1, y_2 \rangle$ is real for $y_1, y_2$ in $Y$, $g_1 \in \psi(y_1), g_2 \in \psi(y_2)$. In particular, if $f_1$ and $f_2$ are in $\psi(y_1), \psi(y_2)$, then $(f_1^* f_2) \langle y_1, y_2 \rangle$ is real for all $y_1, y_2$ in $Y$. Thus each $\psi(y)$ for $y \in Y$ is a singleton. Let $\varphi_y$ denote the unique element of $\psi(y)$, and observe that for a complex number and $y$ in $Y$, $\varphi_y = \varphi_y$ (where the bar denotes complex conjugation). On the Cartesian product $Y \times Y$ define the function $\langle [], \rangle$ by setting $\langle [x, y] \rangle = \varphi_x [y] \circ [x] \; \text{is a "semiminer-product" for $Y$ [3, § 9]}. \text{Clearly,}\langle [x, y] \rangle \; \text{is left linear, and, for all $y \in Y$, $(y', y) = |y'|^2$. To complete the proof it suffices to show that $\; [x, y] \; \text{is conjugate commutative. For $y \in \mathcal{H}(X)$, we showed above that $\langle [x, y] \rangle = [y, x]$ is real. Replacing $x$ by $iy$ in this last expression gives the conclusion that $\langle [x, y] \rangle = [y, x]$ is real. It follows readily that $\; [y, y] = [y, y]$.}$

3. Certain concrete spaces with only trivial hermitian operators.

In this section we shall be concerned with the following Banach spaces:

(i) $C^0[0, 1]$, the space of continuously differentiable complex-valued functions on $[0, 1]$ with $f = ||f||_{Lip} + ||f||_{lip}$.

(ii) Lip[0, 1], the space of all complex-valued functions on $[0, 1]$ satisfying a Lipschitz condition of order 1 with $||f|| = ||f||_{Lip} + \text{essup} ||f||$

(iii) $AC^0[0, 1]$, the space of absolutely continuous functions on $[0, 1]$, with $||f|| = ||f||_{AC} + \text{essup} ||f||$

(iv) lip, $0 < \alpha < 1$, the space of all complex-valued functions on $[0, 1]$ on the real line $R$ of period 1 such that $\sup |f(x + h) - f(x)| = o(h^{\alpha})$, as $h \to 0$.

In the foregoing, essup and $\| \|_{\alpha}$ are, of course, taken with respect to Lebesgue measure.

(3.1) Theorem. If $X$ is one of the spaces $C^0[0, 1]$, Lip[0, 1], $AC^0[0, 1]$, or lip[0, 1], then $\mathcal{H}(X) = \{ \tau : \tau \in R \}$.

Proof. Let $\mathcal{H}(X)$. Then $\exp(it \tau)$, $\tau \in R$, is a one-parameter group of isometries of $X$ onto $X$, continuous with respect to the uniform operator topology. Let $T_\tau = \exp(it \tau)$, $\tau \in R$. We consider the case where $X$ is one of the spaces $C^0[0, 1]$, Lip[0, 1], $AC^0[0, 1]$. By [10, Theorems 2.5, 3.3, and 4.1] for each $\tau \in R$, $T_{\tau}$ has the form $\langle T_{\tau} f \rangle = \lambda_{\tau} \langle f \rangle$, where $\lambda_{\tau}$ is a unimodular complex constant, and $\tau(\cdot)$ is a monotone one-to-one absolutely continuous mapping of $[0, 1]$ onto itself (if $X$ is $C^0[0, 1]$ or Lip[0, 1], then $\tau(x)$ is, in fact, identically $x$ or identically $1 - x$). Let $\varphi_{\tau}$ (resp., $\varphi_{\tau}$) be the element of $X$ defined for each $x \in [0, 1]$ by $\varphi_{\tau}(x) = 1$ (resp., $\varphi_{\tau}(x) = 1$). We observe that $\tau$ depends only on $\tau(0)$, and that $\tau_{\tau} = \lambda_{\tau} \tau$. Thus $\lambda$ and $\tau$ are uniquely determined by $\tau$, and $\lambda$ and $\tau$ are continuous mappings of $R$ into the set of unimodular complex numbers and $X$, respectively. From the uniqueness of representation for the operators of the group $\{ T_\tau \}$, $\tau \in R$, we have $\lambda_{\tau} = \lambda_{\tau}$ for all $\tau \in R$. Thus $\lambda_{\tau}$ is a one-parameter continuous group of unimodular complex numbers, and hence there is a real constant $c$ such that $\lambda_{\tau} = e^{i t c}$ for all $\tau \in R$. It suffices for the proof to assume that $\lambda_{\tau} = 1$ for all $\tau \in R$, and show that $A$ cannot be zero (since the result could then be applied to the group $\{ e^{i t \tau} T_\tau \}$, $\tau \in R$). If $X$ is $C^0[0, 1]$ or Lip[0, 1], then, as noted earlier, each $\tau \in R$ belongs to the Doob set $\{ \tau \in R \}$. Since $\lim_{|\tau| \to \infty} \varphi_{\tau} = 0$, the continuity of $\tau \mapsto \tau_{\tau}$ as a map from $R$ into $X$, there is a real neighborhood $N$ of 0 such that $\varphi_{\tau} = \varphi_{\tau}$ (and hence $T_{\tau} = T_{\tau}$) for all $\tau \in N$. Thus $\lambda_{\tau} = e^{i t c}$ for all $\tau \in N$. This follows readily that $\; [x, y] = [y, x]$. It follows readily that $\; [x, y] = [y, x]$.
Since for each \( t \in R \), \( T_t \bar{\alpha} \) has the constant value \( \lambda_t \), it is clear that \( \lambda_t \) is uniquely determined by \( t \), and that (as a function of \( t \)) \( \lambda_t \) is a continuous character of the additive group of \( R \). As before, this suffices for the proof to assume that \( \lambda_t = 1 \) for all \( t \in R \) and show that \( A = 0 \). We remark in passing that it was necessary to choose (as we have done) a definite value of \( \alpha \) for each \( t \in R \), since it follows from the periodicity of the functions in \( \alpha \) that \( \alpha_t \) could not be uniquely determined by \( t \) and (3.2). Without loss of generality we let \( \alpha_0 = 0 \). In view of the fact that \( iA \alpha_n = \frac{d^2 T_t \alpha_n}{dt} \bigg|_{t=0} \), we have:

\[
(3.3) \quad iA \alpha_n = \left[ \frac{d \exp(2\pi i \alpha_n)}{dt} \right]_{t=0}, \quad n = 0, 1, 2, \ldots
\]

Define the complex constant \( \beta \) by setting \( 2\pi i \beta = \frac{d \exp(2\pi i \alpha_n)}{dt} \bigg|_{t=0} \). Then it follows from (3.3) (for \( n = 1 \)) that for each \( t \in R \), \( T_t \bar{\alpha}_1 = [\exp(2\pi i \beta t)] \bar{\alpha}_1 t \), and consequently \( \alpha_t = \beta t \) is an integer. Thus without loss of generality we can take \( \alpha_t = 0 \) for each \( t \in R \). Now (3.3) gives \( iA \alpha_n = 2\pi n \beta \alpha_n, \quad n = 0, 1, 2, \ldots \) Since \( A \) is bounded, \( \beta \) must be 0. Hence for all \( t \in R \), \( \alpha_t = 0 \), and \( I_t = 0 \). This concludes the proof.

4. Hermitian operators on the dual space of a \( C^* \)-algebra. Throughout this section a \( C^* \)-algebra \( \mathcal{A} \) will be a Banach *-algebra with identity such that \( \|a^* a\| = \|a\|^2 \) for all \( a \in \mathcal{A} \). A \( W^* \)-algebra will be a \( C^* \)-algebra which is (linearly isometric to) the dual space of a Banach space. It will be convenient henceforth to denote dual spaces and adjoints of operators on Banach spaces by prime superscripts.

In the scholium which follows we record a known result in a form convenient for our purposes.

(4.1) SCHRÖDING (A. M. Sinclair). If \( X \) is a \( W^* \)-algebra, then \( \mathcal{H}(X) \) consists of all operators \( T \in \mathcal{B}(X) \) for which there exist self-adjoint elements \( u \) and \( v \) of \( X \) such that \( Tu = u+2v \) for all \( x \in X \).

Proof. By [12, Remark 3.5 and [8], Theorem 1, p. 311].

If \( A \) is a \( C^* \)-algebra, and \( U \) its universal representation, then it is well-known that \( A' \), the dual of the second dual of \( A \), can be identified with the closure in the weak operator topology of \( U(A) \) so as to make \( U \) the canonical embedding of \( A \) in \( A'' \) (7, 12.1.3-(v)). We shall make free use of this fact; in particular, we shall regard \( A' \) as a \( W^* \)-algebra in the sense of this identification. (4.1) allows us to deduce as a corollary an unpublished result of G. Lumer, which we state next for later convenience.

(4.2) COROLLARY (G. Lumer). Let \( A \) be a commutative \( C^* \)-algebra, and let \( L \) be the regular representation of \( A \) (i.e., \( L_a x = ax \) for all \( a, x \in A \)). Then \( \mathcal{H}(A) = (L_a; a \in A, a = a') \).

Proof. If \( T \in \mathcal{H}(A) \), then \( T'' \in \mathcal{H}(A') \). Since \( U(A) \) is commutative, \( A'' \) is also commutative. By (4.1) there is a self-adjoint element \( a \in \mathcal{A}' \) such that \( T'' = a x \) for all \( x \in A'' \). Thus \( T'' = (U(1)) = U(T) U(A) \), by a standard property of second adjoint operators. Thus \( \mathcal{H}(A) \subseteq (L_a; a \in A, a = a') \). The reverse inclusion is easy.

(4.3) THEOREM. Let \( \mathcal{A} \) be a \( C^* \)-algebra. Then \( \mathcal{H}(\mathcal{A}) \) is the closure in the strong operator topology of \( \{T': T \in \mathcal{H}(\mathcal{A})\} \).

Proof. Since for any Banach space \( X \), \( \mathcal{H}(X) \) is closed in the strong operator topology of \( \mathcal{B}(X) \), and a convex subset of \( \mathcal{B}(X) \) has the same closure in the weak operator topology as in the strong operator topology (1, Lemma 3.3), it suffices for the proof of the theorem to show that if \( T \in \mathcal{H}(A) \), then there is a net \( (T_j) \in \mathcal{H}(\mathcal{A}) \) such that \( T_j \to T \) in the weak topology. By Goldstone’s theorem (5, V.4.5), there are (bounded) nets \( (u_j)_{j \in J}, (v_j)_{j \in J} \) in \( A \) such that \( \{U(u_j)\} \) (resp., \( \{U(v_j)\} \)) converges to \( u \) (resp., \( v \)) in the weak topology of \( A' \). (We remark that \( A' \), being a \( W^* \)-algebra, has a unique weak*-topology [11, p. 30].) The involution \( A'' \) is weak*-continuous in each variable separately [11, Theorem 1.7.8], and so for all \( x \in A'' \) and all \( y \in A' \). Note finally that by (4.1) there are self-adjoint elements \( u \) and \( v \) of \( A' \) such that \( T' = u x + v \) for all \( x \in A'' \). By Goldstone’s theorem (5, V.4.5), there are (bounded) nets \( (u_j)_{j \in J}, (v_j)_{j \in J} \) in \( A' \) such that \( \{U(u_j)\} \) (resp., \( \{U(v_j)\} \)) converges to \( u \) (resp., \( v \)) in the weak*-topology of \( A'' \). (We remark that \( A'' \), being a \( W^* \)-algebra, has a unique weak*-topology [11, p. 30].) Since the involution \( A'' \) is weak*-continuous in each variable separately [11, Theorem 1.7.8], and so for all \( x \in A'' \) and all \( y \in A' \).

(4.4) \((T')_x(y) = \lim \{U(u_{t_j}) x + v U(v_{t_j}) y\}.

Denote by \( \mathcal{A} \) (resp., \( \mathcal{B} \)) the left (resp., right) regular representation of \( A \), i.e., \( A, x \) (resp., \( F_x \)) is \( a x \) (resp., \( a x \)) for all \( a, x \in \mathcal{A} \). Then for all \( a, x \in \mathcal{A}, (A'' \cdot U(x) \) (resp., \( (F_x)^+ \)) is equal to \( U(a) U(x) \) (resp., \( U(x) U(a) \)). Since multiplication in \( A'' \) is weak*-continuous in each variable separately, and \( U(A) \) is weak*-dense in \( A'' \), it is obvious that, on all of \( A'' \), \( (A'' \cdot U(x) \) (resp., \( (F_x)^+ \)) is left (resp., right) multiplication by \( U(a) \). Combining this last observation with (4.4) completes the proof.

We know of no Banach space \( X \) such that \( \mathcal{H}(X) \) is not the closure in the strong operator topology of \( \{T': T \in \mathcal{H}(X)\} \).

Let \( \mathcal{D} \) be a compact Hausdorff space, and let \( \mathcal{D}(\mathcal{D}) \), resp., \( \mathcal{B}(\mathcal{D}) \)) be the algebra of all complex-valued continuous (resp., bounded Borel) functions on \( \mathcal{D} \). With the usual involution and with the norm of \( f \) given by \( \|f(x)| = \sup \{\|f(x)| : x \in \mathcal{D} \}, \mathcal{D}(\mathcal{D}) \) and \( \mathcal{B}(\mathcal{D}) \) are \( C^* \)-algebras. By the Riesz representative theorem \( U([\mathcal{D}]) = \mathcal{B}(\mathcal{D}) \), the space of all regular Borel measures on \( \mathcal{D} \). For each \( f \in \mathcal{B}(\mathcal{D}) \) define \( \mathcal{S}_f \), \( \mathcal{B}(\mathcal{D}) \) by \( E_f(a) = \int M(a) \), for all \( a \in \mathcal{D}(\mathcal{D}) \). It is easy to see that \( \mathcal{S}_f \) is an isometric algebra isomorphism.
of $B(L)$ into $H(M(L))$, that $S_f$ is hermitian if and only if $f$ is real-valued, and that for each $g \in C(L)$, $S_g = (Lg)'$ (in the notation of (4.3)). In terms of the foregoing notation (4.3) has the following corollary.

(4.5) Corollary. $H(M(L))$ is the closure in the strong operator topology of $\{S_g; g \in C(L), f \text{ is real-valued}\}$.

Proof. By (4.2) and (4.3).

(4.6) Remarks. If $A$ is a $C'$-algebra, then it follows from (4.1) that each $T \in H(A^{**})$ is weak*-continuous on $A'$, and hence is the adjoint of a (necessarily hermitian) operator on $A'$. The map which assigns $Q'$ to $Q$ is one-to-one from $H(A')$ onto $H(A')$. It follows from this remark and (4.2) that $H(M(L))$ is a commutative subring of $H(M(L))$. This last fact is also clear from (4.5).

Example. We show that $H(M([0, 1]))$ is strictly larger than $\{S_g; g \in C([0, 1]), g \text{ real-valued}\}$. Indeed, the cardinal number of the latter set is $c$, the power of the continuum. We shall demonstrate that the set of idempotent elements in $H(M([0, 1]))$ has cardinality at least $2^c$.

By virtue of (4.2) and the first part of (4.6) this amounts to showing that the maximal ideal space $\Omega_c$ of $C([0, 1])$ has at least $2^c$ open-closed sets. Identify $C([0, 1])$ with $C(\Omega_c)$. For each $\alpha \in [0, 1]$ let $h_\alpha$ be the homomorphism of $C([0, 1])$ onto the complex field given by $h_\alpha(f) = f(\alpha)$. Since $U(C([0, 1]))$ is weak*-dense in $C(\Omega_c)$, it is easy to see that evaluation at $h_\alpha$ is a weak*-continuous homomorphism of $C(\Omega_c)$ onto the complex field. Thus there is a one-to-one map $\alpha \mapsto p_\alpha$ of $[0, 1]$ into $\Omega_c$ such that unit mass at $p_\alpha$ is a normal measure on $\Omega_c$. [Corollary, p. 171]. Thus by [5, Proposition 3] the singleton set $\{p_\alpha\}$ is open-closed. For each subset $\alpha$ of $[0, 1]$ let $T(\alpha)$ be the closure in $\Omega_c$ of $\{p_\beta; \beta \in \alpha\}$ is easy to see, since $\Omega_c$ is stonian, that $T(\alpha)$ is open-closed in $\Omega_c$. Also, it is now easy to see that $T(\cdot)$ is one-to-one.

Remark. In [2, (3.3)] there was defined for an arbitrary Banach space $X$ a notion of orthogonality (relative to a suitable family of idempotent elements of $H(X)$). Let $\Omega$ be, as above, a compact Hausdorff space, and for each Borel set $\gamma$ in $\Omega$, let $h_\gamma$ be the characteristic function of $\gamma$, and put $E(\gamma) = S_{h_\gamma}$. Then in the notation of [2, § 3] is straightforward to see that for any two measures $\mu, \nu$ in $M(\Omega)$, $\mu$ and $\nu$ are mutually singular if and only if $\mu 1_{\gamma} = 0$ in $M(\Omega)$. We omit the details.

References


Received May 20, 1973