Integration of evolution equations in a locally convex space

by

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Abstract. Let $H = H(R^m)$ be the space of all real-valued functions in $C^0(R^m)$ having every partial derivative in $L^2(R^m)$ and topologised by the seminorms defined as follows:

$$p_0(p) = \left( \int_{|x| = 0} |D^0 p(x)|^2 \, dx \right)^{1/2}, \quad p \in H, \quad t = 0, 1, 2, \ldots$$

Let $A$ be an elliptic differential operator with coefficients possessing bounded derivatives of all orders. This paper solves the Cauchy problem for the system:

$$\frac{\partial u(x, t)}{\partial t} = (A u)(x, t), \quad t > 0, x \in R^m,$$

$$u(0, x) = f(x), \quad f \in H, \quad x \in R^m.$$

1. Introduction. The present paper is a follow-up to [2], and its knowledge is assumed here. Let $Q$ be an open subset of a Euclidean space. For convenience we shall denote by $C^m(Q)$ the space of all functions in $Q$, and by $C^m_s(Q)$ the space of all functions in $C^m(Q)$ having compact support in $Q$.

Now let $A$ be the partial differential operator of 2nd order in $m$-dimensional Euclidean space $R^m$ given by

$$A = -(\xi \cdot \eta - \gamma^\alpha_\beta \eta^\alpha \partial_\beta)^m,$$

where the coefficients $\alpha^\alpha_\beta$ belong to $C^0(R^n)$ with bounded partial derivatives of all orders. We assume further that $\alpha^\alpha_\beta = \alpha_{\gamma \beta} \gamma^\gamma$ for $|\gamma| = |\alpha| = m$ and there is a constant $\varepsilon > 0$ such that

$$\sum_{|\gamma| = m} \alpha^\gamma_{\alpha \beta} (x) t^\gamma_1 \cdots t^\gamma_n > \varepsilon \left( \sum_{|\gamma| = m} x^\gamma \right)^m$$

for each $(t_1, \ldots, t_n) \in R^n$; so that $A$ is an elliptic differential operator.
Let \( H = H(\mathbb{R}^n) \) be the space of all functions in \( C^0(\mathbb{R}^n) \) with every partial derivative in \( L^1(\mathbb{R}^n) \) and denote by \( H \) also the topological vector space obtained by imposing the topology determined by the set \( \{ p_\alpha : \alpha \in \mathbb{N}^n \} = \{ p_\alpha : \alpha \in \mathbb{N}^n \} \) of semi-norms on this family of functions, where

\[
p_\alpha(f) = \left( \frac{\partial^\alpha |f|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right)_{x_1 = 0, \ldots, x_n = 0}, \quad \alpha \in \mathbb{N}^n.
\]

Note that \( H \) is a Fréchet Space.

In this paper we prove the following theorem:

1.4 \textbf{Theorem.} The Cauchy problem for the equation

\[
\frac{\partial u}{\partial t} - (\mathcal{A}u)(t, x) = f(t, x), \quad x > 0, \quad t \in \mathbb{R}^n,
\]

is solvable in the following sense: For any given \( f \in H \) the equation (1.5) admits a solution \( u = u(t, x, \xi) \in C^0([0, \infty) \times \mathbb{R}^n) \) satisfying the following:

(i) \( u(\xi, \eta) = u(\xi, \eta) \) for \( \xi > 0, \eta \in \mathbb{R}^n \),

(ii) \( u(\xi, \eta) \in H \) for each \( \xi > 0, \eta \in \mathbb{R}^n \),

(iii) \( \lim_{t \to +1} u(\xi, \eta) = f(\cdot) \) in \( H \).

Moreover, the solution \( u(\xi, \eta) \) satisfying (ii) and (iii) is uniquely determined for \( f \in H \).

2. Preliminaries.

2.1. Notations. For each non-negative integer \( i \),

(1) \( (\varphi, \psi)_i = \sum_{\alpha \leq i} \int_{\mathbb{R}^n} D^\alpha \varphi(x) D^\alpha \psi(x) \, dx \) for all \( \varphi, \psi \in H(\mathbb{R}^n) \)

and \( i = 0, 1, 2, \ldots \).

(2) \( H_i \) is the pre-Hilbert space formed by \( H \) under the inner product (1).

(3) \( \tilde{H}_i \) is the completion of \( H_i \) with respect to the norm \( \| \cdot \|_i = p_i(\cdot) \).

Indeed the \( p_i \)'s are norms on \( H \). Hence, under the topology induced by \( p_i(\cdot) \), the normed linear space formed by the elements of \( H \) is \( p_i(\cdot) \) and will be replaced by \( H \) in the sequel. Whenever necessary \( f_i (= f) \) will denote an element of \( H_i \), seen as a coset. Note that the operator \( A \) defined by (1.1) with domain \( H \) and range in \( H \) has the property that the linear operator \( A_i : H_i \to H_i \) defined by

\[
A_i f_i = (Af)_i, \quad f_i \in H_i
\]

is well-defined. For, clearly, \( f_i = g_i \) in \( H_i \) \( \implies \) \( f(\cdot) = g(\cdot) \) in \( H \) (1.5).

Integration of evolution equations in locally convex space

\[
0 = \sum_{i=0}^{n} \int_{\mathbb{R}^n} (D^\alpha \mathcal{A}_i f)(\xi) \, d\xi = 0 \Rightarrow (\mathcal{A}_i f)(\xi) = (\mathcal{A}_i g)(\xi)
\]

is \( 0 \). Hence \( \mathcal{A}_i \) is well-defined on \( H_i \). Observe that \( \mathcal{A}_i \) is effective on \( H_i \) acting as on the normed linear space \( H_i \) itself.

We define the adjoint \( A^* \) of \( A \) by

\[
A^* = -(-1)^n \sum_{i=0}^{n} (-1)^{n+i} D^\alpha a_{\alpha}(\xi) D^\alpha.
\]

Note that \( f \in H \) can be approximated by a sequence of functions in \( C^0_{\mathbb{R}^n} \) (cf. [4], page 38). Thus, since \( (a) \) the inner product \( (\cdot, \cdot)_i \) defined by (3.1) on \( H \times H \) is continuous, (b) \( A^* \) is continuous on \( H \) and (c) by partial integration \( (Af, \varphi)_i = (f, A^* \varphi)_i \) for all \( \varphi \in C^0_{\mathbb{R}^n} \), it follows in the limit that

\[
(Af, g)_i = (f, A^* g)_i \quad \text{for all} \quad f, g \in H.
\]

Similarly,

\[
\sum_{i=0}^{n} (-1)^{n+i} D^\alpha A f_i g_i = \int_{\mathbb{R}^n} \sum_{i=0}^{n} (-1)^{n+i} D^\alpha A^* g_i D^\alpha f_i \quad \text{for all} \quad f, g \in H.
\]

We now state a suitable form of Garding's inequality.

2.5. \textbf{Lemma.} Let \( f, g \in H \). Then \( (f, g)_i = 0 \) for all \( \varphi \in C^0_{\mathbb{R}^n} \) implies \( f = g \) in \( H_i \).

1. \textbf{Studies in Mathematics} 13
Then there exist positive constants $c$ and $C$ such that
\begin{equation}
(\mathcal{A} f, f)_H \geq c \| f \|_H^2 - C \| f \|_H^6
\end{equation}
for all $f \in H$.

Proof. By [1], Theorem 7.6,
\begin{equation}
(\mathcal{A} \varphi, \varphi)_H \geq c \| \varphi \|_H^2 - C \| \mathcal{A} \varphi \|_H^2
\end{equation}
for all $\varphi \in C_0^\infty(R^n)$. This class of functions is dense in $H$. Hence, for any $f \in H$, there exists a sequence $\{\varphi_n\} \subset C_0^\infty(R^n)$ such that $\lim_{n \to \infty} \varphi_n = f$ in $H$. Now observe that the operator $\mathcal{A}$, the norms $\| \cdot \|_H$ and $\| \cdot \|_H^6$ are continuous on $H$ and the inner product $(\cdot, \cdot)_H$ is continuous on $H \times H$. Hence, by taking $\varphi = \varphi_n$ in (2.9) and letting $n \to \infty$, we obtain the lemma.

2.10. Corollary. Let $\lambda$ be the differential operator defined by (1.1). There exist positive constants $c_\lambda$ and $C_\lambda$ such that if $\lambda > C_\lambda$ then
\begin{equation}
(\sum_{|\alpha| = \lambda} (-1)^{|\alpha|} D^{\alpha}(\lambda I - \lambda^\alpha) f, f)_H \geq c_\lambda \| f \|_{H^{\lambda+1}}^2
\end{equation}
for all $f \in H$. Further, for each positive $\lambda$, there exists a positive constant $K_{\lambda+1}$ such that
\begin{equation}
(\sum_{|\alpha| = \lambda} (-1)^{|\alpha|} D^{\alpha}(\lambda I - \lambda^\alpha) f, g)_H \leq K_{\lambda+1} \| f \|_{H^{\lambda+1}} \| g \|_{H^{\lambda+1}}
\end{equation}
for all $f, g \in H$.

Proof. The equality in (2.11) is true by (2.4). It is easy to show that the differential operator $(\sum_{|\alpha| = \lambda} (-1)^{|\alpha|} D^{\alpha}(\lambda I - \lambda^\alpha) A f, f)_H \geq c_\lambda \| f \|_{H^{\lambda+1}}^2 - C_\lambda \| f \|_H^6$ for all $f \in H$.

Now
\begin{equation}
(\sum_{|\alpha| = \lambda} (-1)^{|\alpha|} D^{\alpha}(\lambda I - \lambda^\alpha) f, f)_H = \lambda (\sum_{|\alpha| = \lambda} (-1)^{|\alpha|} D^{\alpha} f, f)_H - (\sum_{|\alpha| = \lambda} (-1)^{|\alpha|} D^{\alpha} A f, f)_H
\end{equation}
for all $\lambda > c_\lambda$. Thus
\begin{equation}
(\sum_{|\alpha| = \lambda} (-1)^{|\alpha|} D^{\alpha}(\lambda I - \lambda^\alpha) f, f)_H \geq c_\lambda \| f \|_{H^{\lambda+1}}^2 + (\lambda - C_\lambda) \| f \|_H^6
\end{equation}
for all $f \in H$. Therefore, provided $\lambda > C_\lambda$, we have
\begin{equation}
(\sum_{|\alpha| = \lambda} (-1)^{|\alpha|} D^{\alpha}(\lambda I - \lambda^\alpha) A f, f)_H \geq c_\lambda \| f \|_{H^{\lambda+1}}^2
\end{equation}
for all $f \in H$, and (2.11) is true.

Each $\| f \|^{\lambda+1}_H$ is bounded. Hence (2.12) follows as a consequence of the Schwartz inequality and the corollary is proved.

3. Proof of Theorem 1.4. We would have proved Theorem 1.4 if we had shown that the differential operator $A$ with domain $D(A) = H$ generates an $L_\infty(H)$-operator semi-group of class $(C_2, 1)$. For this purpose we shall employ a variant of a technique of Yosida ([4], pp. 413–416).

It is clear that the linear operator $A$ with $D(A) = H$ is continuous and therefore closed in $H$. It is also clear that $D(A) = H$ is dense in $H$. We have already noted that $A_\lambda = A : H_\lambda \to H_\lambda$ is well-defined. Now to show that $A_\lambda$ with $D(A_\lambda) = H$ generates an $L_\infty(H)$-operator semi-group of class $(C_2, 1)$ we still need, according to (2.3), to establish the following:

(i) For each $i$, $A_\lambda$ is closable in $H_i$.

(ii) For each $i$, there exist positive numbers $c_i, M_i$ such that the resolvent $R(\lambda; A_\lambda)$ of the closure $\mathcal{A}_\lambda$ of $A_\lambda$ in $H_i$ exists for all $\lambda > c_i$ and
\begin{equation}
\| R(\lambda; A_\lambda) \|_{H_i} \leq M_i (\lambda - c_i)^{-1}
\end{equation}
for all $\lambda > c_i$ and $\lambda = 1, 2, \ldots$.

We first take up (3.1)(i).

3.2. Lemma. The linear operator $A_\lambda = A : H_\lambda \to H_\lambda$ is closable in $H_\lambda$.

Proof. Let $\{f_k\} \subset D(A_\lambda) = H_\lambda \subset H_\lambda$ be such that $\lim_{k \to \infty} \| f_k \|_H = 0$ and $\lim_{k \to \infty} \| A_\lambda f_k \|_{H_\lambda} = 0$. It remains to show that $g = 0$ in $H_\lambda$ to prove the lemma. Now for any $\varphi \in C_0^\infty(R^n)$,
\begin{equation}
(A_\lambda f_k, \varphi)_H = (\sum_{|\alpha| = \lambda} (-1)^{|\alpha|} D^{\alpha} A_\lambda f_k, \varphi)_H
\end{equation}
for all $k = 1, 2, \ldots$ and $\lambda > c_i$. Note that the inner products $(\cdot, \cdot)_H$
and \((\gamma, \nu)\) are continuous on \(H_x \times H_t\). Hence, passing to the limit in (3.3), we have
\[
(g, \varphi) = \left(0, \sum_{|\beta| = 0}^\ell (-1)^{|eta|} A^\beta D^\beta \varphi \right) = 0.
\]
Hence, by Lemma 2.5, \(g = 0\) in \(H_x\). This proves the lemma.

To establish (3.1(ii) we need a few preparatory results.

3.4 Lemma. Let a positive number \(\lambda\) be so chosen that Corollary 2.10 is valid for \(\lambda > \lambda_0\). Then, for any \(f \in H_t\), the equation
\[(3.5) \quad \lambda u - Au = f, \quad (\lambda > \lambda_0),
\]
has a solution \(u_{\lambda} \in \tilde{H}_{\lambda + \varepsilon} \cap C^\alpha\) in the sense that
\[(3.6) \quad (\lambda I - A)u_{\lambda} \varphi = (f, \varphi), \quad \text{for all } \varphi \in C^\alpha(R^n).
\]
Moreover, \(u_{\lambda} \in \tilde{H}_{\lambda + \varepsilon} \cap C^\alpha\).

Proof. Observe that \(\lambda I - A\) is strongly elliptic. Define a bilinear functional
\[
B_{\lambda}(u, v) = \left( \sum_{|\beta| = 0}^\ell (-1)^{|eta|} (\lambda I - A^\beta D^\beta) u, v \right)
\]
for all \(u, v \in H\). From Corollary 2.10,
\[
[\lambda I - A]u_{\lambda} \varphi = (f, \varphi) \quad \text{for all } \varphi \in H, \quad \text{and} \quad \|u_{\lambda}\|_{\tilde{H}_{\lambda + \varepsilon} \cap C^\alpha} \leq C_{\lambda} \|f\|_{\tilde{H}_{\lambda + \varepsilon} \cap C^\alpha}.
\]
Hence we may extend \(B_{\lambda}(u, v)\), by continuity, to a bilinear functional \(\tilde{B}_{\lambda}(u, v)\) defined for \(u, v \in \tilde{H}_{\lambda + \varepsilon}\) and such that
\[(3.7) \quad \tilde{B}_{\lambda}(u, v) \leq K_{\lambda} \|u\|_{\tilde{H}_{\lambda + \varepsilon}} \|v\|_{\tilde{H}_{\lambda + \varepsilon}} \quad \text{and} \quad \tilde{B}_{\lambda}(u, v) \geq C_{\lambda} \|u\|_{\tilde{H}_{\lambda + \varepsilon}}.
\]
The linear functional \(F_{\lambda}(u) = (u, f)\) defined on \(\tilde{H}_{\lambda + \varepsilon}\) is bounded since \(\|u\|_{\tilde{H}_{\lambda + \varepsilon}} \leq \|u\|_{\tilde{H}_{\lambda + \varepsilon}} \leq \|u\|_{\tilde{H}_{\lambda + \varepsilon}}\). Hence, by the Biele representation theorem, in the Hilbert space \(\tilde{H}_{\lambda + \varepsilon}\) (see [4], page 90), there exists a uniquely determined \(v = \varphi(f)\) on \(\tilde{H}_{\lambda + \varepsilon}\) such that \(u_{\lambda} = (v, \varphi(f))_{\tilde{H}_{\lambda + \varepsilon}}\) for all \(u \in \tilde{H}_{\lambda + \varepsilon}\).

Thus, by the Lax-Milgram theorem ([4], page 92),
\[(u, f) = (u, \varphi(f))_{\tilde{H}_{\lambda + \varepsilon}} = B_{\lambda}(u, S_{\lambda} \varphi(f)) \quad \text{for all } u \in \tilde{H}_{\lambda + \varepsilon},
\]
where \(S_{\lambda}\) is a bounded linear operator from \(\tilde{H}_{\lambda + \varepsilon}\) onto \(\tilde{H}_{\lambda + \varepsilon}\). Let \(\{u_k\} \subset H\) be a sequence such that \(\lim_{k \to \infty} S_{\lambda} \varphi(f)_{\tilde{H}_{\lambda + \varepsilon}} = 0\). Then for \(f \in C_0^\alpha(R^n)\),
\[
(3.9) \quad \|u\|_{\tilde{H}_{\lambda + \varepsilon}} \leq (\lambda - \sigma_0)^{-1}\|f\|_{\tilde{H}_{\lambda + \varepsilon}}.
\]

Proof. It is clear from Lemma 3.4 that if \(\lambda > \sigma_0\), then the solution \(u = u_{\lambda}\) exists in the sense of (3.6) and is unique. We now obtain the estimate (3.9). The inequality (2.13) implies
\[
(3.10) \quad \left| \sum_{|\beta| = 0}^\ell (-1)^{|eta|} D^\beta (\lambda I - A) u, u \right| \geq (\lambda - \sigma_0)\|u\|^2_{\tilde{H}_{\lambda + \varepsilon}}
\]
for all \( u \in H \) and \( \lambda > C \). Now, for each \( u \in H \), we have, by the Schwartz inequality,

\[
\left\| \sum_{|\alpha| \leq n} (-1)^{|\alpha|} D^{\alpha}_0 D^{\alpha}_1 (\lambda I - A) u, w \right\|^2 \\
\leq \left( \sum_{|\alpha| \leq n} \int \left| D^{\alpha}_0 (\lambda I - A) u \right|^2 dx \right) \left( \sum_{|\alpha| \leq n} \int \left| D^{\alpha}_1 u \right|^2 dx \right) \\
= \left\| (\lambda I - A) u \right\|^2 \left\| u \right\|^2.
\]

Combining (3.10) and (3.11) gives

\[
\left\| (\lambda I - A) u \right\| \geq (\lambda - C) \left\| u \right\| \quad \text{whenever} \quad u \in H.
\]

Since the solution \( u = u_r = u_{r, \lambda} \in H_{ur} \cap C^\infty \) of (3.5) is approximated in \( \left\| u_r \right\|_r \) by a sequence of functions in \( H \) and since the norm \( \left\| u \right\|_r \) is larger than the norm \( \left\| u \right\|_r \), we obtain, passing to the limit,

\[
\left\| u \right\| \leq (\lambda - \sigma_i)^{-1} \left\| f \right\|, \quad \lambda > \sigma_i,
\]

which concludes the proof.

3.3. Corollary. The closure \( \bar{A}_I \) of \( A_I \) is \( H_I \)-closed, for \( \lambda > \sigma_i \), the resolvent \( R(\lambda; \bar{A}_I) \) defined on \( H_I \) into \( H_I \), such that

\[
\left\| R(\lambda; \bar{A}_I) \right\| \leq (\lambda - \sigma_i)^{-k}, \quad k = 1, 2, \ldots
\]

Proof. Let \( f \in H_I \). Now \( (\lambda I - \bar{A}_I) f = g \). Note that \( g \in H_I \) is unique. If \( \lambda > \sigma_i \), then as a consequence of this uniqueness and Lemma 3.4, the map \( f \rightarrow g \) is one-to-one from \( H \) onto \( H_I \). Thus if we set \( f = R(\lambda) g \), then \( R(\lambda) \) is a bounded linear operator from \( H_I \) onto \( H_I \). Now

\[
(\lambda I - \bar{A}_I) R(\lambda) g = g, \quad \forall g \in H_I.
\]

Furthermore, consider \( R(\lambda)(\lambda I - A) g, g \in H_I \). We have just seen that \( g \in H_I \) implies that there exists \( h \in H_I \) such that \( R(\lambda) h = g \). Hence we have \( R(\lambda)(\lambda I - A) R(\lambda) h = R(\lambda) h = g \); that is,

\[
R(\lambda)(\lambda I - A) g = g, \quad \forall g \in H_I.
\]

Now by Corollary 3.6

\[
\left\| R(\lambda) g \right\| \leq (\lambda - \sigma_i)^{-k} \left\| g \right\|, \quad \forall g \in H_I.
\]

This \( R(\lambda) \) is continuous linear operator on \( H_I \) into \( H_I \). Hence it is uniquely extendable to a continuous linear operator \( R(\lambda) \) on \( H_I \) into \( H_I \) such that

\[
\left\| R(\lambda) g \right\| \leq (\lambda - \sigma_i)^{-k} \left\| g \right\|, \quad \forall g \in H_I.
\]

By Lemma 3.2, \( A_I = A \) is closable in \( H_I \). Its closure is denoted by \( \bar{A}_I \). Since \( H_I \) is \( \left\| \cdot \right\| - \)dense in \( H_I \) and \( R(\lambda) \) is continuous on \( H_I \), we see that, for any \( g \in H_I \), there exists a sequence \( \{ g_k \} \in H_I \) such that \( \left\| g_k - g \right\| \rightarrow 0 \) and \( \left\| R(\lambda) g_k - R(\lambda) g \right\| \rightarrow 0 \) as \( k \rightarrow \infty \). Clearly \( \{ R(\lambda) g_k \} \in D(\bar{A}_I) \) and \( \bar{A}_I \) being closed in \( H_I \), we have, for any \( g \in H_I \),

\[
(\lambda I - \bar{A}_I) R(\lambda) g = \lim_{k \rightarrow \infty} (\lambda I - \bar{A}_I) R(\lambda) g_k = \lim_{k \rightarrow \infty} (\lambda I - A) R(\lambda) g_k = \lim_{k \rightarrow \infty} g_k = g
\]

in the topology of \( H_I \) (consequence of (3.10)). Similarly,

\[
R(\lambda)(\lambda I - \bar{A}_I) g = g, \quad \forall g \in D(\bar{A}_I).
\]

Thus \( R(\lambda) \) is the resolvent \( R(\lambda; \bar{A}_I) \) of \( \bar{A}_I \) over the space \( H_I \). Moreover, (3.17) gives

\[
\left\| R(\lambda; \bar{A}_I) g \right\| \leq (\lambda - \sigma_i)^{-k} \left\| g \right\|, \quad \forall g \in H_I \text{ and } \lambda > \sigma_i,
\]

from where it follows that

\[
\left\| R(\lambda; \bar{A}_I) g \right\| \leq (\lambda - \sigma_i)^{-k} \left\| g \right\|, \quad \forall g \in H_I \text{ and } \lambda > \sigma_i, \text{ and } k = 1, 2, \ldots
\]

This proves the corollary.

We have thus established (3.1)(ii). It follows that the differential operator \( A : H \rightarrow H \), defined by (1.1), is the infinitesimal generator of an \( L_p(R) \)-operator semi-group of class \( (C_0, 1) \) and thus Theorem 1.4 is established.

References


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