Convolution of singular measures

by

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Abstract. On a compact abelian group $G$, we prove that if $\mu \neq 0$ is any continuous (possibly singular) measure on $G$, then there is a real singular measure $\nu$ such that the convolution $\mu * \nu$ has an absolutely convergent Fourier series. As a consequence we prove that a multiplier from singular measures to singular measures is necessarily given by convolution with a discrete measure. Also, we prove that on the circle group $T$, if $S$ is an uncountable Borel set of measure zero, then there is a Borel null set $S'$ such that $S + S' = T$.

In this paper $G$ is a compact abelian group with dual $\Gamma$. $M_s = M_s(G)$ is the Banach space of singular measures on $G$. Every $\mu \in M_s$ has a unique decomposition into a continuous part $\mu_c$ and a discrete part $\mu_d$: $\mu = \mu_c + \mu_d$.

If $\nu \in M_s$ is supported by the null set $S'$, then $\mu * \nu$ is supported by an enumerable union of translates of $S'$ and hence $\mu * \nu$ is singular. The counterpart of this statement is the following theorem:

Theorem 1. Let $G$ be a compact abelian group. Let $\mu \neq 0$ be any continuous (possibly singular) measure on $G$. Then there is a real singular measure $\nu$ such that the convolution $\mu * \nu$ has a non-vanishing absolutely convergent Fourier series.

Lemma 1. Let $\mu$ be a continuous measure such that $\hat{\mu}(0) = 1$. Then there is a sequence of characters $\gamma_n \in \Gamma$ such that

\[ |\hat{\mu}(\pm \gamma_n)| < 8^{-n} \]

and such that for every $n \geq 1$ and every sequence $c_1, \ldots, c_n, c_i = 0, \pm 1, \pm 2, \pm 3$, we have

\[ |\hat{\mu}(\pm \gamma_{n+1} + c_1 \gamma_1 + \ldots + c_n \gamma_n)| < 8^{-(n+1)} \]

and consequently the $\gamma_n$ are distinct.

Proof. We use the following theorem about a continuous measure $\mu$: "To every $\varepsilon > 0$ there corresponds a symmetric neighborhood $V$ of 0 in $\mathcal{M}$ such that for any continuous positive definite function $f$, and $f(0) \neq 0$, we have

\[ \sum_{\gamma \in \Gamma} |\hat{\mu}(\gamma)|^2 < \varepsilon. \]

See e.g. Rudin [3], p. 118.
The function $g$ defined by $g(x) = 2^{-1} |f(x) + f(-x)|$ has the same properties as $f$ and $g'(y) = 2^{-1} |g'(y) + f'(-y)|$. Hence

$$\sum_{\gamma} f'(\gamma) |\mu(\gamma)|^2 + |\mu(-\gamma)|^2 = 2 \sum_{\gamma} g'(\gamma) |\mu(\gamma)|^2 < 2e.$$ 

Choose $2e = 8^{-1}$; since $\sum_{\gamma} f'(\gamma) = f(0) = 1$ there is a $\gamma_1 \in \Gamma$ such that $|\mu(\gamma_1)|^2 + |\mu(-\gamma_1)|^2 < 8^{-1}$, i.e.

$$|\mu(\pm \gamma_1)| < 8^{-1}.$$ 

Assume the first $k$ elements of the desired sequence have been chosen satisfying (1) and, in case $k > 1$, also (2) for $n < k - 1$. We show how to choose $\gamma_{k+1}$ so that (2) holds for $n = k$.

Denote by $F_k$ the finite set of elements of the form $c_1 \gamma_1 + \cdots + c_k \gamma_k$, where $c_1, \ldots, c_k = 0, \pm 1, \pm 2, \pm 3$. $F_k$ has at most $7^k$ elements, and $F_k$ is symmetric: $F_k = -F_k$.

Consider the finite set of continuous measures $\delta_{F_k}$. Then $(\delta_{F_k})'(\gamma) = \mu(\gamma)$ for $\gamma \in \Gamma$.

Choose a symmetric neighborhood $V$ of 0 in $G$, and a continuous positive definite function $f$ whose compact support lies in $V$ and such that $f(0) = 1$ and

$$\sum_{\gamma} f'(\gamma) |\mu(\gamma + \delta)|^2 + |\mu(-\gamma + \delta)|^2 < 7^{-k} 8^{-2(k+1)}$$ 

for every $\delta \in F_k$.

Then

$$\sum_{\delta \in F_k} \sum_{\gamma} f'(\gamma) |\mu(\gamma + \delta)|^2 + |\mu(-\gamma + \delta)|^2 < 8^{-2(k+1)}$$

that is

$$\sum_{\gamma} f'(\gamma) \sum_{\delta \in F_k} [\mu(\gamma + \delta)]^2 + [\mu(-\gamma + \delta)]^2 < 8^{-2(k+1)}.$$ 

Since $\sum_{\gamma} f'(\gamma) = 1$ there is $\gamma_{k+1} \in \Gamma$ such that

$$\sum_{\delta \in F_k} [\mu(\gamma_{k+1} + \delta)]^2 + [\mu(-\gamma_{k+1} + \delta)]^2 < 8^{-2(k+1)},$$

that is

$$|\mu(\pm \gamma_{k+1} + \delta)| < 8^{-2(k+1)}, \quad \delta \in F_k$$

which is (2) for $n = k$.

**Lemma 2.** Suppose $\{\gamma_n\}$ is an infinite sequence of distinct elements in $\Gamma$. Then there is a real singular measure $\nu$ such that (a) $\nu(0) > 1$, (b) $\nu(\gamma) = 0$ if $\gamma$ is not of the form $\pm \gamma_{k+1} + c_1 \gamma_1 + \cdots + c_k \gamma_k$, where $c_i = 0, \pm 1, \pm 2, \pm 3$, with at most seven exceptional values of such $\gamma_i$.

Proof. It is evident that if $\{\delta_n\}$ is a subsequence of $\{\gamma_n\}$ and if $\nu$ is a measure satisfying the conditions of Lemma 2, with $\gamma_n$ replaced by $\delta_n$, then $\nu$ will also satisfy these conditions for the original sequence $\{\gamma_n\}$. Hence there is no problem, if, considering a subsequence of $\{\gamma_n\}$ we still call it $\{\gamma_n\}$.

We consider two cases:

**Case I.** The set $\{2 \gamma_n\}$ is infinite.

Then we can extract from $\gamma_n$ a subsequence, still denoted $\{\gamma_n\}$, such that, for $n = 1, 2, \ldots$

$$2 \gamma_{n+1} \neq c_1 \gamma_1 + \cdots + c_k \gamma_k; \quad c_i = 0, \pm 1, \pm 2, \pm 3.$$ 

This is possible, since for any fixed $n$, the set

$$\{c_1 \gamma_1 + \cdots + c_k \gamma_k; \quad c_i = 0, \pm 1, \pm 2, \pm 3\}$$

is finite.

Put

$$P_N(x) = \prod_{k=1}^N (1 + \frac{1}{2} (\gamma_k + \frac{1}{2} (x, -\gamma_k))).$$ 

Then $||P_N||_1 = 1$. Since, for every $N$, $P_N'(\gamma) = 0$ except for an enumerable (even finite) set of values of $\gamma$, then a subsequence of $\{P_N\}$ converges in the weak *-topology of $M(\Theta)$, to a measure $\nu = \nu_{\Theta}$.$M(\Theta)$.

Denote by $A$ the set of finite sums of the form

$$c_1 \gamma_1 + \cdots + c_k \gamma_k; \quad a_i = 0, \pm 1, \quad n = 1, 2, \ldots$$

Clearly

$$\lambda(\gamma_k) \geq \frac{1}{2}$$

while

$$\lambda(\gamma) = 0 \quad \text{if} \quad \gamma \notin A.$$ 

Put

$$\lambda = \gamma \lambda.$$ 

Then $\lambda(\gamma) = 1(\gamma + \gamma_k)$. Again, for every $\lambda$, $\lambda'(\gamma) = 0$ except for an enumerable set of values of $\gamma$. Hence a subsequence of $\lambda'$ converges in the weak *-topology of $M(\Theta)$ to a measure $\nu_{\Theta} = \mu_{\Theta}$.$M(\Theta)$. Using a device due to Helson (see [1]), we see that $\nu_{\Theta}$ is singular.

In fact, let $\lambda = \lambda' + \lambda''$ be the Lebesgue decomposition of $\lambda$, where $\lambda''$ is singular (with respect to Haar measure). The Fourier coefficients of $\lambda'$ vanish at infinity; hence the absolutely continuous translates $\gamma \lambda'$ converge weak * to zero. Therefore $\nu_{\Theta}$ is a weak * limit of the singular translates $\gamma \lambda''$. For every $f$ continuous on $G$ we have

$$|\int f d\nu_{\Theta}| = |\int f d|\lambda''|$$
which shows that $\nu$ is absolutely continuous with respect to $\lambda''$; since $\lambda''$ is singular so is $\nu$.

Since $\hat{\lambda}(\gamma) \geq \frac{1}{2}$, then $\hat{\nu}_k(0) \geq \frac{1}{2}$ and therefore

\[(a)\]

Now, assume $\gamma \neq 0$, $\gamma \neq \pm \gamma_1$, and $\gamma$ is not of the form

\[(4)\]

\[\gamma = \pm \gamma_{n+1} + \cdots + \pm \gamma_n, \quad c_i = 0, \pm 1, \pm 2, \pm 3,\]

If, for some $k > 1$ we have

\[\gamma + \gamma_b = a_1\gamma_1 + \cdots + a_n\gamma_n + A, \quad a_i = 0, \pm 1,\]

then we may suppose $a_b = 0$, for the relation $\gamma + \gamma_b = 0$ is impossible by assumption (see (4)). Also $\kappa = k_b$, otherwise $\gamma$ would be of the excluded form (4). Moreover the relation $a_b = a_{b-1} = 1$ would contradict the assumption on $\gamma$. Thus $a_{b-1} = 1$ and therefore

\[(5)\]

\[\gamma = a_1\gamma_1 + \cdots - 2\gamma_b, \quad a_i = 0, \pm 1.\]

If for some $k > k_1$ we have also

\[(6)\]

\[\gamma + \gamma_b = a_1\gamma_1 + \cdots + a_n\gamma_n + A, \quad a_i = 0, \pm 1,\]

then

\[(7)\]

\[\gamma = a_1\gamma_1 + \cdots - 2\gamma_b.\]

Relations (5) and (6) are incompatible with (3). Therefore, there is at most one integer $k$ such that $\gamma + \gamma_k + A$. Hence, $\hat{\lambda}(\gamma + \gamma_k) = 0$ for large $k$ and therefore $\nu_k(\gamma) = 0$.

Now $\gamma$ is not necessarily real. Put $\gamma = \frac{1}{2}(\nu + \nu^*)$; $\gamma$ is real; if $\gamma \neq 0$, $\gamma \neq \pm \gamma_1$, and $\gamma$ is not of the form (4), then $\nu_k(\gamma) = \nu_k(-\gamma)$ and therefore $\nu_k(\gamma) = 0$. The singular measure $\nu$ now has all the required properties.

Case II. The set $\{2\nu_k\}$ is finite.

Then there is an infinite subset of $\{\gamma_n\}$, still denoted $\{\gamma_n\}$, such that $2\nu_k = 2\nu_{k+1}$ for all $n$. (The case $2\nu_k = 0$ is excluded.)

Define $E_k$, $\lambda$, $\nu_k$ as in Case I. Then, as before

\[\hat{\lambda}(\gamma) \geq \frac{1}{2}, \quad \lambda(\gamma) = 0, \quad \text{if } \gamma \neq A,\]

then $\nu_n$ is singular and $\nu_n(0) \geq \frac{1}{2}$.

Assume that $\gamma \neq 0$, $\gamma \neq \pm \gamma_1$, $\gamma \neq \pm 2\gamma_1$, $\gamma \neq \pm 3\gamma_1$ (seven exceptional values for $\gamma$) and that $\gamma$ is not of the form

\[(4)\]

\[\gamma = \pm \gamma_{n+1} + \cdots + \pm \gamma_n, \quad c_i = 0, \pm 1, \pm 2, \pm 3.\]

If $k \geq 2$

\[\gamma + \gamma_b = a_1\gamma_1 + \cdots + a_n\gamma_n + A, \quad a_i = 0, \pm 1,\]

then as before (see (5))

\[\gamma = a_1\gamma_1 + \cdots + a_n\gamma_n + A, \quad a_i = 0, \pm 1,\]

that is

\[\gamma = (a_1 - 2)\gamma_1 + \cdots + a_n\gamma_n + A.\]

This contradicts the assumption about $\gamma$. Therefore

\[\gamma + \gamma_b + A, \quad \lambda(\gamma + \gamma_b) = 0, \quad \nu_k(\gamma) = 0.\]

Again, replace $\nu_k$ by $\gamma = \frac{1}{2}(\nu + \nu^*)$. Then $\nu$ will have all the properties mentioned in the lemma.

The proof of Lemma 2 is now complete.

For results related to Lemma 2, see [2].

Proof of Theorem 1. By translation and multiplication by a scalar we may assume that $\mu(0) = 1$.

Let $\{\gamma_n\}$ be the sequence given by Lemma 1 and let $\nu$ be the real singular measure given by Lemma 2.

Consider now $\mu \ast \nu$.

Suppose $\gamma$ is not one of the seven exceptional values mentioned in Lemma 2. If $\gamma$ is not of the form

\[(4)\]

\[\gamma = \pm \gamma_{n+1} + \cdots + \pm \gamma_n, \quad c_i = 0, \pm 1, \pm 2, \pm 3,\]

then $\mu(\gamma) \nu(\gamma) = 0$. For any fixed $n$, the set $E_n$ of elements of the form (4) has at most $2 \cdot 2^n$ elements and for these

\[\sum_{\gamma \in E_n} |\mu(\gamma) \nu(\gamma)| \leq 2 \cdot 2^n \cdot 2^{-n+1} |\nu| < (2^n)^{|\nu|}.\]

Hence $\sum_{\gamma \in E_n} |\mu(\gamma) \nu(\gamma)| < \infty$. Since $\mu(0) = 1$ and $\nu(0) \geq \frac{1}{2}$ then $\mu \ast \nu$ has a nonvanishing absolutely convergent Fourier series and Theorem 1 is proved.

Corollary 1. Let $\mu$ be any singular measure with nonvanishing continuous part. Then there is a singular measure $\nu$ such that the convolution $\mu \ast \nu$ is not singular.

Proof. Put $\mu = \mu_1 + \mu_2$ where $\mu_1$ is continuous and $\mu_2$ is discrete. Since $\mu_2$ is nonvanishing, there exists, by Theorem 1, a singular measure $\nu$ such that $\mu_2 \ast \nu$ has a non-singular part. On the other hand, $\mu_1 \ast \nu$ is singular. Hence $\mu \ast \nu$ is not singular and the Corollary is proved.

Application 1

Definition. Denote by $M_1 = M_1(G)$ the Banach space of singular measures on $G$. A function $\varphi$ on $G$ is called a singular multiplier if for every $x \in M_1$, the function $\varphi x$ is the transform of some $\lambda \in M_1$: $\varphi x = \hat{\lambda}$. 
THEOREM 2. Let $G$ be a compact abelian group. A function $\phi$ is a singular multiplier if and only if $\phi$ is the transform of a discrete measure $\mu$: $\phi = \hat{\mu}$.

Proof. Taking $\nu_0$ to be the unit mass at the origin, we have $\nu_0 = 1$. Hence $\phi = \phi \ast \nu_0$ is the transform of some singular measure $\mu$: $\phi = \hat{\mu}$ where $\mu \ast M_\nu$. Hence

$$\phi \ast \nu = \hat{\mu} \ast \nu = (\hat{\mu} \ast \nu)^*$$

for every $\nu \in M_\nu$.

If now $\mu$ has a nonvanishing continuous part, then, by Corollary 1, there is a singular measure $\nu$ such that $\mu \ast \nu$ is not singular. Hence $\phi \ast \nu$ is not the transform of a singular measure. This contradiction shows that the continuous part of $\mu$ is zero and therefore $\mu$ must be discrete.

APPLICATION 2

We specialize $G$ to the unit circle $T$.

THEOREM 3. Let $S$ be an uncountable analytic (in particular Borel) set in $T$ of measure 0. Then there is a Borel set $S'$ of measure zero such that $S + S' = T$.

Proof. By a theorem of Souslin, see e.g. [4], p. 224, the analytic set $S$ contains a nonempty perfect set $P$. Since $m(P) = 0$ then $P$ is totally disconnected. A classical construction of Lebesgue gives a real nonvanishing continuous singular measure $\mu$ on $P$. By Theorem 1 there is a real singular measure $\nu$, concentrated on a null set $P'$ such that $\mu \ast \nu$ has an absolutely convergent Fourier series (and hence $d(\mu \ast \nu) = \phi(x)dx$ with $\phi$ real continuous). Since $\nu$ is regular, there is a sequence of compact sets $K_n \subset P'$ such that $\nu(K_n) \to \nu((P'))$. Hence, replacing $P'$ by $\bigcup K_n$ we may assume that $P'$ is a Borel set and therefore that $P + P'$ is measurable (Borel).

Let $x_0$ be a point such that $\phi(x_0) = -\alpha \neq 0$. We may assume $\alpha > 0$; hence there is a non-degenerate interval $I$ such that $\phi(x) > 0/2$ for $x \in I$.

We shall show that the set $A = T \setminus (P + P)$ has measure 0.

For assume $m(A) > 0$. We have

$$p \in P, \ p \ast \nu \in P' \Rightarrow p + p' \in A \Rightarrow p \in A - p'$$

that is

$$\mathcal{L}_{A,p}(p) = \int_{A-p}^{A+p} d\nu(p') = \int_{A-p}^{A+p} \mathcal{L}_{A,p}(p) d\nu(p') = 0.$$  

Therefore

$$\mu \ast \nu(A) = \int_{\mathbb{R}} \mu(A-p') d\nu(p') = \int_{\mathbb{R}} \mathcal{L}_{A,p}(p) d\nu(p') = 0.$$  

But

$$\mu \ast \nu(A) = \int_{\mathbb{R}} \phi(x) dx = \int_{\mathbb{R}} \mathcal{L}_{A,p}(p) d\nu(p') = 0.$$  

This contradiction shows that $m(A) = 0$, i.e. that $P + P'$ covers almost all $T$. If $P''$ is a suitable finite union of translates of $P'$, then $P + P''$ covers almost all $T$:

$$m(T \setminus (P + P'')) = 0, \quad m(P'') = 0.$$  

If now $x_1$ is any element of $P$ and if we put

$$S' = P' \cup (T \setminus (P + P')) - x_1$$

then $S'$ is a null set and

$$P + S' \supset P + P' \cup T \setminus (P + P') = T.$$  

A fortiori $S + S' \supset T$ and the theorem is proved.

A direct proof of Theorem 3 would be desirable.

References


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