On the localization property of square partial sums for multiple Fourier series

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To A. Zygmund on the 50th anniversary of his first mathematical publication

Abstract. It seems that localization and convergence of multiple Fourier series are related to the Sobolev spaces $W^s_p$. This paper establishes the existence of such a relation regarding the square partial sums. It is shown that for $f \in W^s_p$, $p > s - 1$, this sort of localization holds for the $s$-torus. For each $p < s - 1$ there is an $f \in W^s_p$ for which localization fails. Examples are given of an everywhere differentiable periodic function of 2 variables for which localization by square partial sums fails and of a function in $W^s_p$ for which localization by rectangular partial sums fails.

1. In the study of Fourier series, a primary feature is the localization property, which has been known to hold in the case of functions of one variable since Riemann. That localization does not generally hold for functions of several variables has also been known for a long time. Our purpose is to obtain precise information regarding the functions of $s$ variables which have this property.

Tonelli, [2], observed that, for $s = 2$, localization holds for those functions now known as the functions whose partial derivatives (in the distribution sense) are measures; this includes the Sobolev space $W^s_1$. An example by Torrigiani, [3], shows that a condition given by Tonelli, which guarantees convergence at a point, and holds almost everywhere for $s = 2$ for functions in $W^s_1$, may hold nowhere for $s = 3$.

In a recent paper, Igar [1], settles the localization problem for the square $(C,1)$ partial sums of a multiple Fourier series. He shows that this sort of localization holds for $f \in L^p$, $p > s - 1$, and fails to hold for $p < s - 1$. For the square partial sums themselves — not the averages — he points out that there are continuous functions for which localization fails.

* Supported by a grant from the NSF.
In the present paper, we return to the main issue. We show that localization holds for square partial sums of the multiple Fourier series for all functions in the Sobolev space $W^p$, $p \geq n-1$. For the converse, we show that for every $p < n-1$ there is a function in $W^p$ for which this sort of localization fails. For $p = n-1$ we construct an everywhere differentiable function for which localization fails. For rectangular partial sums localization fails for $p = n-1$ but we do not know the situation for larger values of $p$.

Thus, the solution of the localization problem is found to lie among the properties of the first derivative.

2. Let $T_n$ be the $n$-dimensional cube which consists of those points $x = (x_1, x_2, \ldots, x_n)$ in $n$-space with $-\pi \leq x_i < \pi$, $i = 1, 2, \ldots, n$, and let $T_n^b$ be the interior of $T_n$. As usual, we denote by $W^p_n(T_n^b)$ (or simply by $W^p_n$) the Sobolev space of those functions in $L^p(T_n^b)$ whose partial derivatives in the sense of distributions are functions and belong to $L^p(T_n^b)$. We shall only consider with the class $W^p_n$ which consists of those functions $f$ in $W^p_n$ with $f(-\pi, \ell) = f(\pi, \ell)$ for a.e. $\ell$, $i = 1, 2, \ldots, n$. This class may be described in several ways. Perhaps the simplest is the completion of the periodic and continuously differentiable functions with respect to the $W^p_n$ norm. We are of course interested in the operators

$$L_p(x, f) = \frac{1}{\pi^n} \int_{\mathbb{T}^n} f(x + t) D_{(\ldots, \ell)}(t) dt,$$

where $D_{(\ldots, \ell)}$ is the appropriate Dirichlet kernel. For a function $f \in W^p_n$, we shall use $f_j$, $i = 1, 2, \ldots, n$, to denote its partial derivative with respect to $x_i$. The norm $\|f\|_{L_p(n)}$ for $f \in W^p_n$, defines

$$\|f\|_{L_p} = \|f\|_p + \sum_{i=1}^n |f_j|_p,$$

where $\|\cdot\|_p$ is the $L^p$ norm.

3. We show that localization holds for $f \in W^p_n$, $p \geq n-1$. We first state a trivial lemma.

**Lemma 1.** Let $Q$ consist of those functions in $W^p_n$ which are piecewise linear. Then the localization property for square partial sums holds for all functions in $Q$.

Let $B(0, r)$ be the open cube of center $(0, \ldots, 0)$ and side $2r > 0$, and let $\overline{B}(0, r)$ be its closure.

**Lemma 2.** If $p \geq n-1$ and $0 < \varepsilon < \delta < \pi$, there is a constant $A = A(\varepsilon, \delta) > 0$ such that

$$\sup_{x \in \mathbb{T}^n} |L_p(x, f)| \leq A \|f\|_{L_p},$$

for any $f \in W^p_n$ with $f = 0$ almost everywhere on $B(0, \delta)$.

**Proof.** For any positive integer $j, x = B(0, \varepsilon)$, and $f$ satisfying the condition of the lemma, we have

$$(*) \quad |L_p(x, f)| \leq \frac{1}{\pi^n} \int_{\mathbb{T}^n} f(x + t) D_{(\ldots, \ell)}(t) dt,$$

$$= \frac{1}{\pi^n} \int_{\mathbb{T}^n} f(x + t) D_{(\ldots, \ell)}(t) dt \cdots \sum_{i=1}^n,$$

$$+ \left| \frac{\pi}{\pi^n} \int_{b \in \mathbb{Z}^n} \sum_{i=1}^n \int_{\mathbb{T}^n} f(x + t) D_{(\ldots, \ell)}(t) dt \cdots \sum_{i=1}^n \right|,$$

where $b = (b - \varepsilon)$.

For convenience we let $j = (j_1, \ldots, j_n), y_i = (y_{i1}, \ldots, y_{in+1}, \ldots, y_{in})$ and $d_i = d_{i-1} \cdots d_1 d_{i+1} \cdots d_n$. Consider, for the first integral on the right hand side of $(*)$,

$$\left| \frac{\pi}{\pi^n} \int_{b \in \mathbb{Z}^n} \sum_{i=1}^n \int_{\mathbb{T}^n} f(x_i + t_i, \omega_i) D_{(\ell_1)}(t_i) dt_i D_{(\ell_2)}(t_2) dt_2 \cdots \sum_{i=1}^n \right|,$$

where $b = (b - \varepsilon)$.

Since the partial derivative of $f$ in the distribution sense is a function, $f(x_i + t_i, \omega_i)$ is an absolutely continuous function of $t_i$ for almost all values of $t_i$. Also, it follows from Fubini theorem that there is $t_i$ with $b < t_i < \pi$ such that

$$(\Delta) \quad \int_{\mathbb{T}^n} |f(t_i, \omega_i)|^p dt_i \leq \frac{1}{\pi^n} \|f\|_p.$$
We write, for almost all $t_1$,
\[
\left| \int_0^n \left( f(x_1 + t_1, x_2 + t_2) - f(\tilde{x}_1, x_2 + t_2) \right) \, d\tilde{x}_1 \right| \\
\leq \left| \int_0^n \left[ f(x_1 + t_1, x_2 + t_2) - f(\tilde{x}_1, x_2 + t_2) \right] \, D_2(t_2) \, dt_2 \right| \\
+ \left| f(\tilde{x}_1, t_1) \right| \left| \int_0^n D_2(t_2) \, dt_2 \right|
\]
\[
\leq \left\{ \int_0^n \left( f(x_1 + t_1, x_2 + t_2) - f(\tilde{x}_1, x_2 + t_2) \right) \, D_2(t_2) \, dt_2 \right\} \\
+ \left\{ \int_0^n \left( f(x_1 + t_1, x_2 + t_2) - f(\tilde{x}_1, x_2 + t_2) \right) \, D_2(t_2) \, dt_2 \right\} + A(j + 1)^{-1} |f(\tilde{x}_1, t_1)|
\]
where $A > 0$ is independent of $j$ and $x \in B(0, \epsilon)$. We note that in the preceding inequality, the last step follows by applying second mean-value theorem to the integral $\int_0^n D_2(t_2) \, dt_2$. For the next step we express $f(x_1 + t_1, x_2 + t_2) - f(\tilde{x}_1, x_2 + t_2)$ as the difference of its positive and negative variation starting from $\tilde{x}_1$ as follows
\[
f(x_1 + t_1, x_2 + t_2) - f(\tilde{x}_1, x_2 + t_2) = p(t_1) - n(t_1),
\]
where $p(\tilde{x}_1) = n(\tilde{x}_1) = 0$. This can be done for almost all $t_1$. Applying the second mean-value theorem, we have
\[
\left| \int_0^n \left[ f(x_1 + t_1, x_2 + t_2) - f(\tilde{x}_1, x_2 + t_2) \right] \, D_2(t_2) \, dt_2 \right| \\
\leq \left| \int_0^n p(t_1) \, \frac{\sin(j + 1)t_1}{2 \sin \frac{1}{2}t_1} \, dt_1 \right| + \left| \int_0^n n(t_1) \, \frac{\sin(j + 1)t_1}{2 \sin \frac{1}{2}t_1} \, dt_1 \right|
\]
\[
= \left| \int_0^n p(t_1) \, \frac{\sin(j + 1)t_1}{2 \sin \frac{1}{2}t_1} \, dt_1 \right| + \left| \int_0^n n(t_1) \, \frac{\sin(j + 1)t_1}{2 \sin \frac{1}{2}t_1} \, dt_1 \right|
\]
\[
= |p(t_1)| \left| \int_0^n \frac{\sin(j + 1)t_1}{2 \sin \frac{1}{2}t_1} \, dt_1 \right| + |n(t_1)| \left| \int_0^n \frac{\sin(j + 1)t_1}{2 \sin \frac{1}{2}t_1} \, dt_1 \right|
\]
\[
\leq A(j + 1)^{-1} \left[ |p(t_1)| + |n(t_1)| \right] \leq A(j + 1)^{-1} \left[ \int_0^n |f(x_1 + t_1, x_2 + t_2)| \, dt_1 \right].
\]
and
\[
\left| \int_0^n \left( f(x_1 + t_1, x_2 + t_2) - f(\tilde{x}_1, x_2 + t_2) \right) D_2(t_2) \, dt_1 \right| \\
\leq A(j + 1)^{-1} \left[ \int_0^n |f(x_1 + t_1, x_2 + t_2)| \, dt_1 \right] \leq A(j + 1)^{-1} \int_0^n |f(x_1 + t_1, x_2 + t_2)| \, dt_1.
\]
It follows that (from Hölder’s inequality and $(\Delta)$)
\[
\left| \int_0^n \left( f(x_1 + t_1, x_2 + t_2) - f(\tilde{x}_1, x_2 + t_2) \right) D_2(t_2) \, dt_1 \right| \\
\leq A(j + 1)^{-1} \left[ \int_0^n \left| f(x_1 + t_1, x_2 + t_2) \right| \, dt_1 \right] \leq A(j + 1)^{-1} \left[ \int_0^n \left| f(\tilde{x}_1, x_2 + t_2) \right| \, dt_1 \right]
\]
\[
\leq 2A(j + 1)^{-1} \left[ \int_0^n \left| f(\tilde{x}_1, x_2 + t_2) \right| \, dt_1 \right] \leq 2A(j + 1)^{-1} \left[ \int_0^n \left| f(\tilde{x}_1, x_2 + t_2) \right| \, dt_1 \right]
\]
\[
\leq D(j + 1)^{-1} \left[ \int_0^n \left| f(\tilde{x}_1, x_2 + t_2) \right| \, dt_1 \right] \leq D(j + 1)^{-1} \left[ \int_0^n \left| f(\tilde{x}_1, x_2 + t_2) \right| \, dt_1 \right]
\]
where $D$ is independent of $j$ and $x \in B(0, \epsilon)$, and $q = \frac{p}{p-1}$.

That similar inequalities hold for the other integrals that appear on the right-hand side of (1) is evident. The claim of the lemma is now clear.

**Theorem 1.** If $p \geq n - 1$, the square partial sums of the multiple Fourier series of a function $f$ of $n$ variables has the localization property for $f \in \mathcal{W}^p$.

**Proof.** It is sufficient to prove that for any pair of positive numbers $\epsilon, \delta$ with $0 < \epsilon < \delta$, and $f \in \mathcal{W}^p$, with $f = 0$ almost everywhere on $B(0, \delta)$, $\lim_{n \to \infty} |f(x, f) - g| = 0$ uniformly for $x \in B(0, \epsilon)$.

Let $f$ be any such function. Choose $A$ as in Lemma 2. For any $\eta > 0$ there is a $g \in Q$ with the same property as $f$ and $|f - g|_{L^p} < \frac{n}{2A}$. By Lemma 2
\[
\sup_{\xi \in \mathbb{C}} |L_\eta(x, f)| \leq \sup_{\xi \in \mathbb{C}} |L_\eta(x, f - g)| + \sup_{\xi \in \mathbb{C}} |L_\eta(x, g)| < \eta + \sup_{\xi \in \mathbb{C}} |L_\eta(x, g)|.
\]
The theorem follows from Lemma 1.
4. We now show that if \( p < n - 1 \) there is a function \( f \) of \( n \) variables, with \( f \in W^p_2 \), the square sums of whose multiple Fourier series do not have the localization property.

For each positive integer \( j > N \), where \( N \) is suitably large, let \( b = b_j \) and \( m = m_j = \text{int} \{ k : b_k \geq \frac{1}{10} \} \) and \( M = M_j = \sup \{ k : b_k < \frac{1}{10} \} \).

There are positive numbers \( a \) and \( \beta \) such that \( a_j \leq M - m < \beta j \) for all \( j > N \).

For each \( j > N \) and \( k \) with \( m < k < M - 1 \), let \( I_k = \{ (a_1, \ldots, a_n) : b_k < a_i < (k + 1)b, 0 < a_i < b, i = 2, \ldots, n \} \), and let \( J_k \) be the closed cube concentric with \( I_k \) with sides half the length of the sides of \( I_k \). We define a function \( f_j \) on \( I_n \) by

\[
f_j(x) = \begin{cases} 0 & \text{if } x \in \bigcup_{k=M}^{M-1} J_k, \\ \text{the sign of } D_{(y_{M-1})} & \text{on } J_k \text{ if } x \in J_k, \end{cases}
\]

and elsewhere \( f_j \) is defined in the natural way so that \( f_j \) is quasilinear, of the same sign on each \( I_k \) and so that the partial derivatives of \( f_j \) are bounded in magnitude by \( \frac{1}{b} \).

If \( ||f|| \) is the norm of \( f_j \) in \( W^p_2 \), an elementary calculation yields

\[
||f|| \leq 4 \delta^{1/p} (n+1) b^{n-1} \leq 8 \delta^{1/p} b^{n-1}
\]

where \( \delta \) is a positive constant.

Let \( B \) consist of those \( f \in W^p_2 \) which are 0 almost everywhere on the cube \( -\frac{\pi}{2} \leq a_i < \frac{\pi}{2}, i = 1, \ldots, n, \) \( B \) is a Banach space with norm induced from \( W^p_2 \), Consider the sequence of linear functionals \( L_{ij} \):

\[
L_{ij}(f) = \frac{1}{\pi} \int r_s r_n \frac{d}{dt} f(t) dt = S_j(0, f), \quad i > N.
\]

Then

\[
|L_{ij}(f_j)| \geq \frac{1}{\pi} \sum_{k=M}^{M-1} \int_{J_k} |D_{(y_{M-1})} f(t)| dt = \frac{1}{\pi} \sum_{k=M}^{M-1} \int_{ab}^{ab + 1} \sin(j + \frac{1}{2}) t \left( \frac{2}{2 \sin^2 \frac{1}{2}} \right) \left( \frac{\sin(j + \frac{1}{2}) t}{2 \sin^2 \frac{1}{2}} \right) dt
\]

\[
\geq c \delta^{1/p} b^{n-1} \min_{m < k < M - 1} \int_{ab}^{ab + 1} \frac{\sin(j + \frac{1}{2}) t}{2 \sin^2 \frac{1}{2}} dt \geq c > 0,
\]

where \( c \) is a constant.

It follows from these estimates on \( ||f|| \) and \( |L_{ij}(f_j)| \) that

\[
|L_{ij}(f_j)| \geq \frac{c}{\delta} \delta^{1/p} \frac{b^{n-1}}{\log \frac{j}{j}}
\]

and this tends to \( +\infty \) as \( j \rightarrow \infty \) whenever \( p < n - 1 \). By the uniform boundedness principle there is an \( f \in B \) such that \( \limsup_{j \rightarrow \infty} S_j(0, f) = +\infty \).

**Theorem 2.** For every \( p < n - 1 \), there is a function \( f \in W^p_2 \), of \( n \) variables, the square partial sums of whose multiple Fourier series do not have the localization property.

5. Our purpose now is to give an example of an everywhere differentiable function of two variables such that localization does not hold for the square partial sums of its double Fourier series.

The construction depends on the following elementary lemma.

**Lemma 2.** For any \( a, b, 0 < a < b < \pi, \) any \( m > 0, M > 0, \) and positive integer \( j > b \), there is a \( j > j_0 \) and a continuously differentiable \( f \), whose support is in the vertical strip \( a < x < b \), such that \( |f(x, y)| < m \), for all \( (x, y) \), and \( L_{ij}(f) \geq M \).

**Proof.** For each \( j \) sufficiently large let \( r \) and \( s, r < s \), be positive integers such that \( a < \frac{r}{j} < \frac{r}{j} < b \) and \( s - r < \frac{j}{\log j} \).

Let \( S_j \) be the vertical strip \( \frac{r}{j} < x < \frac{s}{j} \) and let \( f_j \) be the sign of the Dirichlet kernel \( D_{ij} \) on \( S_j \), i.e.

\[
f_j(x, y) = \begin{cases} 1 & \text{if } (x, y) \in S_j \text{ and } D_{ij}(x, y) > 0, \\ -1 & \text{if } (x, y) \in S_j \text{ and } D_{ij}(x, y) < 0, \\ 0 & \text{if } (x, y) \notin S_j. \end{cases}
\]

Then

\[
\int S_j f_j(x, y) D_{ij}(x, y) dy dx = \int S_j f_j(x, y) D_{ij}(x, y) dy dx
\]

\[
\geq \frac{\log j}{j \log j} \frac{j}{\log j} \geq 1
\]

By choosing \( j \) sufficiently large and slightly modifying \( f_j \) to a continuously differentiable \( f \), we obtain the desired result.

The next lemma is also elementary and we omit the proof.

**Lemma 3.** Let \( 0 < a < b < c < d < \pi \) and suppose \( f \) is continuously differentiable with support in the strip \( S_1 = (a \leq x < b), \) \( j_1 \) and \( j_2 \) are positive integers, \( m > 0, \) \( M > 0, \) and \( c > 0. \) There is a continuously differentiable \( g, \) whose support is in \( S_2 = (c \leq x < d), \) and a positive integer \( j_3 > \max \{j_1, j_2\}, \)
such that $|g(x, y)| \leq n$, for every $(x, y)$, $\int g(x, y) D_{\lambda_0, \gamma}(x, y) dxdy \geq M_0$, \[
\int_{\sigma_n} g(x, y) D_{\lambda_0, \gamma}(x, y) dxdy < \infty \quad \text{and} \quad \int_{\tau_n} g(x, y) D_{\lambda_0, \gamma}(x, y) dxdy < \infty.
\]

We now define an everywhere differentiable $f$ for which localization fails. Let $0 < a_1 < b_1 < \ldots < a_n < b_n < \ldots$, where $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = a < \infty$. For each $n$, let $\sigma_n = (a - b_n)^t$. By Lemmas 2 and 3 we may obtain, for each $n$, a continuously differentiable function $f_n$ whose support is in the strip $\tau_n = \{a_n \leq x \leq b_n\}$ such that $|f_n(x, y)| < m_n$ for all $(x, y)$, and $\int f_n(x, y) D_{\lambda_0, \gamma}(x, y) dxdy > n - 1$, for some $\lambda_0$, where $(\lambda_0)$ is increasing, and if $\tau_n = \bigcup_{x \in \tau_n} \tau_n$, then $\left[ f_n(x, y) D_{\lambda_0, \gamma}(x, y) dxdy \right] < 1$.

Let $f$ be defined by $f(x, y) = f_n(x, y)$ for all $(x, y) \in \tau_n$, $n = 1, 2, \ldots$, and $f(x, y) = 0$ otherwise. It is clear that

$$\limsup_{n \to \infty} \int f(x, y) D_{\lambda_0, \gamma}(x, y) dxdy = +\infty,$$

and that $f$ is everywhere differentiable.

**Theorem 3.** There is an everywhere differentiable function $f$, of 2 variables, the square partial sums of whose double Fourier series does not have the localization property.

6. For $n = 2$, Tonelli actually showed that localization holds for rectangular partial sums for $f \in W^1$. We now note that this does not hold for $n > 2$. For $n = 3$, we give an example of a function $f \in W^1$, for which this sort of localization does not hold. The function $f$ is of the form $f(x_1, x_2, x_3) = g(x_3) h(x_1, x_2)$, where $h(x_1, x_2)$ is in $W^1$ and is such that the sequence of square partial sums, $\{x_j h(x, 0, 0)\}$, of $h$ at $(0, 0)$ are unbounded, and where $g(x_3)$ is infinitely differentiable, zero in a neighborhood of $a_1 = 0$, but not identically zero. Then, $f$ is zero in a neighborhood of $(0, 0, 0)$, but there are increasing sequences $\{m_0\}$ and $\{m_1\}$ such that $\{x_{m_0} f(x, 0, 0, 0)\}$ is unbounded.

We indicate an example of an $h(x_1, x_2)$ of the desired type. For each $n$, let $I_n$ be a square of center $(0, 0)$ and side $2n$, with sides parallel to the coordinate axes. Let $h_n$ be continuous, zero off $I_n$, with $h_n(0, 0) = \frac{1}{n}$, and linear on each of the 4 parts into which the lines $x_2 = x_3$ and $x_2 = -x_3$ divide $I_n$. By properly choosing the sequence $\{h_n\}$ it is easy to see that the function $h = \sum h_n$ has the desired properties. A similar construction applies to each $n > 2$ for $f \in W^1$. Thus localization for square partial sums holds for $f \in W^1$, while localization for rectangular partial sums does not hold.