A note on a generalized hypersingular integral

by

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Abstract. We study integral transforms related to hypersingular integrals and certain Marcinkiewicz integrals, but which have less stringent homogeneity requirements. We prove the results in the context of the Lebesgue spaces with mixed homogeneity of C. Sadosky and M. Collar.

Introduction. In this note, we will extend the results of [7] to transforms of the form

$$\text{p.v.} \int |f(x - z) - f(x)| d\mu(z),$$

where $\mu$ is a complex measure satisfying the condition $\int |d\mu(z)| = O(\delta^{-n})$, $\delta > 0$, and $f$ belongs to the Lebesgue (Sobolev) space $L^2_\alpha$, $0 < \alpha < 1$. An example of such a measure is $d\mu(z) = \frac{Q(z')}{|x|^{n+1+\alpha}} dz$, where $z \in \mathbb{R}^n$, $|z'| = 1$ and $Q$ is integrable over $|z'| = 1$.

Such an extension might be possible is clearly indicated by results in the paper [3] of E. H. Ostrow and B. M. Stein which correspond essentially to the case $n = 1$ and $\alpha = 1$. The method can easily be adapted to the Lebesgue spaces with mixed homogeneity introduced in [5], and we shall present the results in this form. We will also derive a theorem on certain Marcinkiewicz-type integrals as a corollary of the method.

Preliminaries. Let $x = (x_1, \ldots, x_n)$ denote a point in $n$-dimensional Euclidean space $\mathbb{R}^n$ and $|x| = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}$. Given a fixed vector $a = (a_1, \ldots, a_n)$ of rational numbers $a_i > 1$, $a_1 = 1$, we call $k(x)$ quasi-homogeneous of degree $\sigma$ if $k(x^a_1, \ldots, x^a_n) = x^{\lambda} k(x)$ for $\lambda > 0$. In particular, letting $m$ be the smallest integer divisible by $2a_i$, $i = 1, \ldots, n$, the function $g(x) = |x| = \left( \sum x_i^{2m} \right)^{1/2m}$ is quasi-homogeneous of degree 1. It is easy to see that $g$ is a metric. Moreover, for $g(x) \neq 0$, the point $\left( x_1^{2m}, \ldots, x_n^{2m} \right)$ belongs to $\sum = g^{-1}(1)$ and $dz = g^{m-1} dg dy$, where $|a| = a_1 + \ldots + a_n$ and $dy$ is the element of area on $\Sigma$. (See [3], [4].)

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Following [5], we say \( f \in L_p^2 \), \( 1 \leq p < \infty \), if \( f'(a) = (1+|z|) - \kappa \psi \)(a) for some \( \psi \in L_1^2 \), and we put \( ||f||_{L_p^2} = ||f||_p \). For simplicity, we consider only \( 0 < a < 1 \) throughout the paper. It then turns out (see [5]) that \( f = f_x \psi \) where \( \delta_x \psi \in L_1^2 \) and

\[
|DF \delta_x \psi(x)| \leq c|x|^{-\gamma} \quad \text{if} \quad \alpha \gamma > \alpha + \gamma - 1
\]

and

\[
|DF \delta_x \psi(x)| \leq c|x|^{-\alpha - \beta} \quad \text{if} \quad \alpha > 1
\]

Of course the case \( \alpha = \ldots = \alpha = 1, \Gamma = 0 \) and \( |z| = |z| \) corresponds to the ordinary homogeneous Bessel potentials.

We list here a few simple specific facts about \( |z|, \theta, \) and \( a \) which we will refer to later. \( c \) denotes a positive constant depending on \( n \) and \( a \).

1. (a) If \( |z| \geq 1 \) then \( |z| \leq c|z| \).

(b) If \( |z| \leq 1 \) then \( |z| \geq c|z| \).

2. If \( |z| \geq 1 \) then \( \theta \leq c|z| \).

(b) follows from \( 1 \geq |z| \geq \theta \).

3. If \( 0 < a < 1, \theta, \alpha \) and \( \left| \frac{\partial}{\partial z} \delta_x \psi \right| \leq c|z|^{-\alpha - \beta} \).

This follows from the two estimates on \( \delta_x \psi \) cited above. For \( |z| \leq 1 \) it is exactly the second estimate, and for \( |z| \geq 1 \), it follows from the first by choosing \( r \) large and using (1a).

4. For \( 0 < a < 1 \) and \( |z| > 2 \), \( \theta, \alpha > 1, \theta, a_1 \leq c|z|^{-\gamma} \).

For

\[
|G_x \psi (z) - G_x \psi (a) - G_z \psi (z) - G_z \psi (a)| = \left| \sum \xi_i \left( \frac{\partial}{\partial \xi_i} \right)^{-1} \right| \leq c|z|^{-\gamma - \beta - 1}
\]

by (3), since \( |z| > 2 \), \( |z| > 2 \). But since \( |z| \leq 1, |z| \geq 1, |z| \leq |z| \leq |z| \).

5. If \( \alpha \) is any complex measure on \( K \), whose total variation satisfies

\[
\int |d\mu(x)| = O(\delta^{-\beta}), \beta > 0, \text{then (cf. Lemma 1 of [3]):}
\]

(a) \( \int |z|^\gamma |d\mu(z)| = O(\delta^{-\beta}) \) for \( \beta < \alpha \)

and

(b) \( \int |z|^\gamma |d\mu(z)| = O(\delta^{-\beta}) \) for \( \beta > \alpha \).

To show (b) for example, note that

\[
\int |z|^\gamma |d\mu(z)| = \sum_{k=0}^{\infty} \int \int |z|^\gamma |d\mu(z)|
\]

\[
\leq c \sum_{k=0}^{\infty} (2^{-\delta})^k \int |d\mu(z)|
\]

\[
\leq c \sum_{k=0}^{\infty} (2^{-\delta})^k (2^{-2\delta - 1})^{\gamma} = O(\delta^{-\beta})
\]

if \( \beta > \alpha \).

\section{In this section we prove the following theorem.}

\textbf{Theorem 1.} For \( f \in L_2^p \), \( 1 < p < \infty \), \( 0 < a < 1 \), \( f\psi \in L_1^p \), \( \mu \) a complex measure satisfying \( \int |d\mu(z)| = O(\delta^{-\beta}), \beta > 0 \). Then for \( 1 < p < \infty \), \( \int |f\psi| \leq c\|f\|_p, \psi \leq c\|f\|_p \), and \( f\psi \) converges in \( L_p \) norm as \( \epsilon \to 0 \). Moreover, \( \|f\psi\|_{L_p} \leq c\|f\|_p \), \( \psi \leq c\|f\|_p \), \( \epsilon > 0 \).

The method of proving this is so well-known that we will only give the main points. We first suppose \( f \in L_2 \) so that

\[
f = f(z) = f(z)B(z), \quad B(z) = \int \gamma(z) - 1 |d\mu(z)|.
\]

If \( |z| \geq 1 \),

\[
|B(z)| \leq \int |d\mu(z)| = O(\delta^{-\beta}).
\]

If \( |z| < 1 \),

\[
|B(z)| \leq \int |d\mu(z)| + \int |d\mu(z)| = O(\delta^{-\beta})
\]

by (3) and (5). Hence \( |f\psi| \leq c(1 + |z|)\psi \leq c|z|\psi \), and \( \|f\psi\|_p \leq c\|f\|_p \) by Plancherel's formula.

The next step in the proof is to establish the weak-type statement of the theorem. For this we may suppose \( f = f_x \psi \) with \( \psi \geq 0 \), and recall (see e.g. [3]) that given \( \epsilon > 0 \) there are non-overlapping rectangles \( I_\epsilon \) with dimensions \( \delta \psi, \epsilon \), \( \delta \psi \) so that \( \sum |I_\epsilon| \leq c^{-\epsilon} \|f\|_p \), and a decomposition \( \psi = \psi_1 + \psi_2 \) with \( \|\psi_1\|_p \leq c\|f\|_p \psi_2 \leq c\|f\|_p \text{ almost everywhere}, \) and \( \int \psi \psi_2 = 0 \).

Since \( \psi \in L_2^p \) with \( \|\psi\|_p \leq c\|\psi\|_p \), it follows immediately from our \( L_2^p \) result (1)

(1) We use \( L_2^p \) here only because we have already proved that case. If we had a prior knowledge for some other \( L_2^p \) we could use that information instead. We will need this fact later.
that \( (x, |f^* y|)^p_\alpha (x) > s \) has measure at most \( c s^{-1/2} |\alpha| \). Hence, letting \( I^\alpha_k \) denote the rectangle concentric with \( I_k \) whose dimensions are \((2d_\alpha)^n, \ldots, (2d_\alpha)^n\) for a large fixed \( \lambda \) independent of \( k \), it is enough to assume \( \varphi = 0 \) and prove \( \int \sum_{k} |f^* (x)|^p s_k \int_{I_k} (f^* (x) > s) \leq c |\varphi| |\alpha| \), or that

(1.1)

\[
\int \sum_{k} |f^* (x)|^p s_k \int_{I_k} |f^* (x)| > s \leq c |\varphi| |\alpha|.
\]

Recall that the integral of \( \varphi \) over \( I_k \) is zero, and note that \( (x - y_k) > d, \) for \( x \in I^\alpha_k \), if \( y_k \) denotes the center of \( I_k \). Using Fubini's theorem and a change of variables it is then easy to see that (1.1) follows from the statement

(1.2)

\[
\int |f^* (x)|^p s_k \int_{I_k} |G_k (x - y - z) - G_k (x - y) - G_k (x - z) + G_k (x)| d\mu (z) \leq c
\]

for \( |y| < d \). For \( |y| < d \), we will in fact prove the stronger result that

(1.2) \( \int \sum_{k} \int |G_k (x - y - z) - G_k (x - y) - G_k (x - z) + G_k (x)| d\mu (z) \)

\( \leq c \)

for \( |y| < d \). For (1.2) is majorised by

\[
\int |f^* (x)|^p s_k \int_{I_k} |G_k (x - y - z) - G_k (x - y) - G_k (x - z) + G_k (x)| d\mu (z) \leq c \int |f^* (x)|^p s_k \int_{I_k} |G_k (x - y - z) - G_k (x - y) - G_k (x - z) + G_k (x)| d\mu (z) + \int |f^* (x)|^p s_k \int_{I_k} |G_k (x - y - z) - G_k (x - y) - G_k (x - z) + G_k (x)| d\mu (z) \leq A + B + C.
\]

To estimate \( A \), note that for the indicated ranges of \( x \) and \( z \), \( |G_k (x - y - z) - G_k (x - y) - G_k (x - z) + G_k (x)| \) is small compared to \( |x| \). Hence

\[
A \leq c \int |f^* (x)|^p s_k \int_{I_k} |G_k (x - y - z) - G_k (x - y) - G_k (x - z) + G_k (x)| d\mu (z) = O(\alpha^{-1} \mu^{-1}) = O(1)
\]

by 5 (b).

By (4), again,

\[
B = \int |G_k (x - y) - G_k (x)| d\mu (x) \leq c \int \frac{|y|}{d_\alpha^{n+1-\alpha}} \int_{I_k} d\mu (x) = O(1).
\]

\[
C \leq c \int |f^* (x)|^p s_k \int_{I_k} |G_k (x - z)| d\mu (z) \leq c \int \frac{|y|}{d_\alpha^{n+1-\alpha}} \int_{I_k} d\mu (z) = O(1).
\]

Next, majorize \( C \) by

\[
\int \sum_{k} \int_{I_k} |f^* (x)|^p s_k \int_{I_k} |G_k (x - z)| d\mu (z) \leq c \int \frac{|y|}{d_\alpha^{n+1-\alpha}} \int_{I_k} d\mu (x) = O(1).
\]

By (4),

\[
C_1 \leq c \int \frac{|y|}{d_\alpha^{n+1-\alpha}} \int_{I_k} d\mu (x) \leq c \int \frac{|y|}{d_\alpha^{n+1-\alpha}} \int_{I_k} d\mu (x) = O(1).
\]

Finally, by (3),

\[
C_2 \leq c \int \frac{|y|}{d_\alpha^{n+1-\alpha}} \int_{I_k} d\mu (x) \leq c \int \frac{|y|}{d_\alpha^{n+1-\alpha}} \int_{I_k} d\mu (x) = O(1).
\]

To prove Theorem 1 for \( 1 < p < 2 \) we use the Marcinkiewicz interpolation theorem, and for \( p > 2 \) we use duality, \( L^p \) and \( L^q \), \( p^{-1} + q^{-1} = 1 \), being dual spaces. (See [5], Theorem 2.) That \( f^* \) converges in \( L^p \) for \( f \in L^p \), \( 1 < p < \infty \), follows from the norm inequality of Theorem 1 and the fact that it converges in \( L^p \) for very smooth \( f \). See [7] for details.

2. Maximal operator. We now use a method like that in [3] to prove

THEOREM 2. Let \( f \in L^p \), \( 0 < p < 1 \), and

\[
f^* (x) = \sup_{\kappa \leq x} |f^\kappa (x)| - \sup_{0 \leq x} |f (x) - f^* (x)| \mu (x)
\]

with \( \mu \) as in Theorem 1. Then \( |f^*|^p \mu \leq c \int_{I_k} |f|^p \mu \), for \( 1 < p < \infty \) and \( |x - f^* (x)| \geq r \). In particular, \( f^* \) converges pointwise almost everywhere as \( r \to 0 \) for \( 1 < p < \infty \).

We will need one additional fact stated in the following form(4).

LEMMA. If \( f = f^\kappa \) and \( x > |x| \alpha \alpha \) then

\[
|f (x + z + y) - f (x + z)| \leq c \frac{|y|}{d_{\alpha}^{n+1-\alpha}} \int \frac{|x|^{n+1-\alpha}}{|y|^{n+1-\alpha}} M_{\kappa} (x) |
\]

where

\[
M_{\kappa} (x) = \sup_{d_{\alpha}^{n+1-\alpha}} |x|^{n+1-\alpha} \int_{I_k} d\mu (x).
\]

(4) This particular statement is convenient for present purposes, but is somewhat arbitrary.
Proof. Write
\[
\begin{align*}
    f(x + z + y) - f(x + z) &= \\
    \int_{y < t < x} &+ \int_{t < x < y} \phi(x - t) G_0(x - z + y) - G_0(x + z) \, dt = I + II.
\end{align*}
\]
Applying Holder's inequality and changing variables, we have
\[
[I] \leq \left( \int_{t < x} \| \phi(x - t) \|^{2q} \, dt \right)^{1/q} \left( \int_{t < x} \| G_0(x - z + y) - G_0(x + z) \|^{2q'} \, dt \right)^{1/q'}
\]
for \( \frac{1}{q} + \frac{1}{q'} = 1 \). However by (3) and (4),
\[
\left( \int_{t < x} \| G_0(x - z + y) - G_0(x + z) \|^{2q'} \, dt \right)^{1/q'} \leq C \left( \int_{t < x} \left( \frac{y}{x + |x - t| + 1} \right)^{q'} \, dt \right)^{1/q'} + C \left( \int_{t < x} \left( \frac{1}{|x - t| + 1} \right)^{q'} \, dt \right)^{1/q'}.
\]
If \(|a - b|/q' < |a|, \) or if \( |a| > |a|/q, \) this is \( O\left(\frac{|b|}{|a|^{1/q'}}\right) \). Therefore,
\[
[I] \leq C \left( \int_{y < t < x} \phi(x - t) \| \frac{y}{x + |x - t| + 1} \|^{q} \, dt \right)^{1/q} \left( \int_{t < x} \| G_0(x - z + y) - G_0(x + z) \|^{2q'} \, dt \right)^{1/q'}.
\]
On the other hand, by (4) and Holder's inequality,
\[
[I] \leq c \left( \int_{t < x} \| \phi(x - t) \|^{2q} \, dt \right)^{1/q} 
\]
for any \( q \geq 1 \). A standard result about approximations to the identity (see [4], p. 70) now gives \( I \leq c \| y \| M_q(\phi)(a) \), and the lemma follows.

Returning to Theorem 2, suppose \( f \in L^p_\alpha \), \( 1 < p < \infty \), and let \( f' \) denote the limit in \( L^p \) of \( f_{\alpha} \) (see Theorem 1). Let \( d(a) \) be a smooth, non-negative, decreasing function of \( |a| \) supported in \( |a| < 1 \) with integral 1. Then
\[
A_\alpha(x) = e^{-d(a)} \left[ \frac{a_1}{x^{1/2}}, \ldots, \frac{a_n}{x^{1/2}} \right]
\]
is supported in \( |a| < \epsilon \) and \( \sup |f_{\alpha} A_\alpha| \) has \( L^p \) norm bounded by a constant times \( \| f \|_{L^p} \). By [4], \( \| [f_{\alpha} A_\alpha](a) \| \leq c \| M_q(\phi)(a) \) and \( M_q \) is a bounded operator on \( L^p \), \( 1 < p < \infty \). Hence Theorem 2 for \( 1 < p < \infty \) will follow if we show that \( [f_{\alpha} - f_{\alpha} A_\alpha] \) has \( L^p \) norm less than a constant times \( \| f \|_{L^p} \). Since \( f_{\alpha} = f + f_{\alpha} A_\alpha \), it is enough to show the same for \( f_{\alpha} A_\alpha \) and \( f_{\alpha} A_\alpha \), where
\[
A_\alpha(x) = \left[ \int_{|x| < a} |f_{\alpha}(x) - f_{\alpha}(a)| \, d\mu(x) \right] \left[ \frac{a_1}{x^{1/2}}, \ldots, \frac{a_n}{x^{1/2}} \right]
\]
and
\[
B_\alpha(x) = \left[ \int_{|x| < a} |f_{\alpha}(x) - f_{\alpha}(a)| \, d\mu(x) \right] \left[ \frac{a_1}{x^{1/2}}, \ldots, \frac{a_n}{x^{1/2}} \right].
\]
Consider first \( B_\alpha(x) \).
\[
B_\alpha(x) \leq \int_{|x| < a} |d\mu(x)| \int_{|x| < a} |f_{\alpha}(x) - f_{\alpha}(a)| \, d\mu(x) \left| A_\alpha(x) \right| \, dy.
\]
In the inner integral \( |y| < 2a \), since otherwise \( A_\alpha(x) = 0 \). Moreover,
\[
|A_\alpha(x) - A_\alpha(y)| \leq c \sum_{k \in \mathbb{Z}} |x^{nk}| |D_k A_\alpha(y) - D_k A_\alpha| \leq c \sum_{k \in \mathbb{Z}} \left( \frac{|x|}{|y|} \right)^k - \mu \leq c \left( \frac{|x|}{|y|} \right)^{k_0} \leq c \frac{|y|}{|x|} \left( \frac{|x|}{|y|} \right)^{|k_0|}
\]
since \( |x| > |y| \). Hence,
\[
B_\alpha(x) \leq c e^{-k_0 - \mu} \int_{|x| < a} |d\mu(x)| \int_{|x| < a} |f_{\alpha}(x) - f_{\alpha}(a)| \, d\mu(x) \, dy,
\]
which is the expression just considered.

Finally, in the first part of \( A_\alpha \),
\[
\| f_{\alpha} A_\alpha(x) - f_{\alpha}(a) \| = \left| \int_{|x| < a} |f_{\alpha}(x) - f_{\alpha}(a)| \, d\mu(x) \right| \leq c \| M_q(\phi)(a) \| \left| \int_{|x| < a} \left( \frac{1}{|x|} + \frac{1}{y} \right)^q \, dy \right|
\]
where
\[
\| f_{\alpha} A_\alpha(x) - f_{\alpha}(a) \| \leq c \| M_q(\phi)(a) \| (e^{-\mu} + \frac{\mu}{q}) \frac{|x|}{|y|} \leq c \| M_q(\phi)(a) \| (e^{-\mu} + \frac{\mu}{q}) \frac{|x|}{|y|}
\]
since \( a > \frac{|a|}{q} > 0 \). Hence the first part of \( A_\alpha(x) \) is at most a constant times
\[
M_q(\phi)(a) \sup_{|x| < a} \left( e^{-\mu} + \frac{\mu}{q} \right) \frac{|x|}{|y|} \left| d\mu(x) \right| \leq c M_q(\phi)(a)
\]
by 5 (a), since \( a > \frac{|a|}{q} \).
We can now easily complete the proof of Theorem 2. We have shown that for $f = \mathcal{F}^q$ and $q > \frac{|a|}{a}$

$$\sup \left| I_a^b \right| \leq c M_q(x)(x).$$

Since $M_0$ is a bounded operator on $L^p$ for $p > 1$, $M_q$ is a bounded operator on $L^p$ for $p > q$. Theorem 2 follows immediately for $p > \frac{|a|}{a}$. But it is also true for $p = 1$, that is, $\|f \mathcal{F}^q(x)\| < c^{-1}\|f\|_{L^p}$. To see this, we simply refer to the proof in Section 1, adding two comments. First, the expression (1.2) used in the proof for the "bad" part $\psi$ does not even depend on $\varepsilon$. Second, instead of the a priori $L^p$ result used in the argument for the "good" part $\psi$, we use the $L^p$ result for $f^*$ for any $p > \frac{|a|}{a}$. (See the footnote on p. 4.) Theorem 2 follows for all $1 < p < \infty$ by interpolation.

$\S\ 3$. In this section we show that results on some Marcinkiewicz-type integrals can be obtained as corollaries of the method of Section 1.

Let $B_1 = \{x: |x| < 1\}$ and let $\nu$ be a non-negative measure on $\mathbb{R}$ satisfying

1. $0 < \nu(B_1) < \infty$, $0 < t < \infty$
2. $\nu(B_t) \leq \nu(B_1)$, independent of $t$.

For example, let $\nu(x) = k(x)dx$ where $k > 0$ is quasi-homogeneous of degree $\beta - |a|$. Let $\nu$ be such a measure. Then $\nu(B_t)$ is a constant times $t^\beta$. (See [6] and [8].)

Let

$$\mathcal{F}(x, t) = \frac{1}{\nu(B_1)} \int_{B_1} |f(x - s) - f(x)|\nu(s)\, ds.$$  (3.2)

For functions $g(x, t)$ we use the norm $X^\beta T^\gamma g = \left\| \left( \int_0^\infty g^\beta(x, t)\, dt \right)^{\gamma/\beta} \right\|_p$.

**Theorem 3.** Let $f \in L^p$, $1 < p < \infty$, $0 < \alpha < 1$. Then

$$X^\alpha T^\gamma \{t^{-\alpha} \mathcal{F}(x, t)\} \leq c \|f\|_{L^p}, \quad 1 < p < \infty,$$

and

$|f| \mathcal{F}^\gamma(x) > \alpha^{-1}\|f\|_{L^p}$.

We will briefly prove the cases $p = 2$ and $p = 1$ of Theorem 3. Let $f \in L^p$. By Schwarz's inequality,

$$\|f(y) - f(x)\|^{\alpha/\beta} \leq \frac{1}{\nu(B_1)} \int_{B_1} |f(y) - f(x)|^\alpha\nu(s).$$

and therefore

$$\int_y^\infty \left( \frac{dt}{t^{\alpha+1-\gamma}} \right)^{1/\gamma} \left( \int_{B_1} |f(y) - f(x)|^\alpha\nu(s) \right)^{\gamma/\alpha} \, dt.$$

For any $t$, $\int_{B_1} |f(y) - f(x)|^\alpha\nu(s) \leq c \|f\|^\alpha_{L^p}$, while if $t < \frac{1}{|x|}$ then by (3)

$$\int_{B_1} |f(y) - f(x)|^\alpha\nu(s) \leq \|f\|^\alpha_{L^p} \int_{B_1} |x|^2\nu(s) \leq c \|f\|^2_{L^p}.$$

Hence

$$\int_{B_1} |f(y) - f(x)|^\alpha\nu(s) \leq c \|f\|^\alpha_{L^p} \int_{B_1} |x|^2\nu(s) \leq c \|f\|^2_{L^p}.$$

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$$\{X^\alpha T^\gamma \{t^{-\alpha} \mathcal{F}(x, t)\}\} \leq \int_0^\infty \left( \frac{dt}{t^{\alpha+1-\gamma}} \right)^{1/\gamma} \left( \int_{B_1} |f(y) - f(x)|^\alpha\nu(s) \right)^{\gamma/\alpha} \, dt.$$

for any $t$, $\int_{B_1} |f(y) - f(x)|^\alpha\nu(s) \leq c \|f\|^\alpha_{L^p}$, while if $t < \frac{1}{|x|}$ then by (3)

$$\int_{B_1} |f(y) - f(x)|^\alpha\nu(s) \leq \|f\|^\alpha_{L^p} \int_{B_1} |x|^2\nu(s) \leq c \|f\|^2_{L^p}.$$

Hence

$$\int_{B_1} |f(y) - f(x)|^\alpha\nu(s) \leq c \|f\|^\alpha_{L^p} \int_{B_1} |x|^2\nu(s) \leq c \|f\|^2_{L^p}.$$

for $f \in L^p$. But

$$T^\gamma \{t^{-\alpha} \mathcal{F}(x, t)\} \leq \int d\mu(z) \int \theta(y) |G_t(x - z - y) - G_t(x - y)|\, dy,$$

where $d\mu(z) = |w(z)|\, dz$, $w(z) = \left( \int_0^\infty \frac{dt}{t^{\alpha+1-\gamma}} \right)^{1/2}$. It is therefore enough to show that

$$\int_{|z| < d} \int d\mu(z) \leq c \|f\|^\alpha_{L^p}.$$

is bounded for $|y| < d$, which is what we did in Section 1 for measures satisfying $\int |d\mu(z)| \leq c \|f\|_{L^p}$. In the present case, $\mu > 0$ and

$$\int d\mu(z) = \sum_{|z| < d} \int |w(z)|\, dz.$$
Since \( w \) decreases as \([x]\) increases, this is at most
\[
\sum_{i=1}^{m} \left( \int_{0}^{w} \frac{dt}{\delta^{n+1} \rho(B(t))} \right) v(B_{y+w}) \leq c \sum_{i=1}^{m} v(B_{y+w})^{-1} (2 \delta)^{-n} v(B_{y+w})
\]
\[
\leq c \delta^{-d} \sum_{i=1}^{m} 2^{-is} = O(\delta^d).
\]

References


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