On $\sigma^*$-basic sequences and their applications to the study of Banach spaces

by

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Abstract. $\sigma^*$-basic sequences in conjugate Banach spaces are investigated and existence theorems are obtained for them, analogous to the Besaga-Peczylowski existence theorem for basic sequences. Immediate consequences are that every separable Banach space has a quotient with a basis and every separable conjugate space contains a boundedly complete basic sequence. Other examples of immediate applications of $\sigma^*$-basic sequences are that if $X^*$ has a subspace with a separable dual, $X$ has a quotient space with a boundedly complete basis, and if $X^{**}$ is separable, then both $X$ and $X^*$ have reflexive subspaces. $\sigma^*$-basic sequences and previously known theorems are also applied to show, for separable $X$, that if $X^*$ contains a subspace isomorphic to $\ell_1$ (respectively $L_1$), then $X$ has a quotient isomorphic to $c_0$ (respectively, $C[0,1]$); the technique for obtaining the existence theorems is used to show that every separable $X$ has a subspace $Y$ such that $Y$ and $X/Y$ both have finite-dimensional decompositions.

1. Introduction. In this paper we give a proof of a theorem stated by Milman [12]:

If $X^{**}$ is separable then both $X$ and $X^*$ have reflexive subspaces. (Throughout this paper $X$, $Y$, and $Z$ refer to infinite dimensional Banach spaces. "Subspace" means "infinite dimensional closed linear submanifold.")

Our Theorem IV.2 yields immediately that if $X^{**}$ is separable, then both $X$ and $X^*$ are somewhat reflexive. (A space $Y$ is called somewhat reflexive (see [5]) provided every subspace of $Y$ contains a reflexive subspace). This result has also been obtained by Davis and Singer [3] under the additional hypothesis that $X^{**}$ satisfies the approximation property. They also have given a counter example to a key lemma in [12], and thus the argument in [12] is incorrect.

The techniques developed to prove the above theorem stated by Milman led to new results concerning bases in Banach spaces and their quotients. Some of these results have consequences which may be stated without reference to bases.

* The research for the second named author was partially supported by NSF-GP 12997.
For example, in Section IV we prove

**Theorem IV.1.** (i) If $X$ is separable then $X$ has a quotient space with a basis.

(ii) If $X$ is separable and there is a subspace $Y$ of $X^*$ such that $Y$ is isomorphic to a subspace of a separable conjugate space, then $X$ has a quotient space with a shrinking basis; moreover this basis may be chosen with its biorthogonal functionals lying in $Y$, hence $X$ has a weak* closed subspace with a boundedly complete basis.

(iii) If there is a subspace $Y$ of $X^*$ such that $X$ is separable and $X$ has a quotient space with a boundedly complete basis. Consequently $X^*$ contains a subspace isomorphic to a second conjugate space.

(i) solves the separable version of a problem which of Pelczyński’s [13]: Does every Banach space have a quotient space which has a basis? (ii) yields that every subspace of a separable conjugate space contains a weak* closed subspace, and also that it contains a boundedly complete basic sequence, solving a problem raised in [3]; (iii) yields that if $X^*$ has a subspace with a separable dual, $X$ has a quotient space isomorphic to a separable conjugate space; the converse statement is obvious. (By quotient space of $X$, we mean a space of the form $X/Z$ where $Z$ is subspace of $X$ with infinite codimension in $X$.)

In Theorem IV.3, we show that for separable $X$, if $X^*$ contains a subspace isomorphic to $l_1$ then $X$ has quotient space isomorphic to $c_0$, while if $X^*$ has a subspace isomorphic to $L_1(0,1)$ then $X$ has a quotient space isomorphic to $c_0$. (This extends a result of Pelczyński’s [13].)

The concept underlying the proofs of the above results is that of a *basic sequence* (see the definition in Section II). Our main lemma, Theorem III.1, is the operator analogues of the theorems of Besaga and Pelczyński [1] on the existence of basic sequences.

Our final result, Theorem IV.4, shows that if $X$ is separable then there is a $Y \subseteq X$ such that $Y$ and $X/Y$ both have finite dimensional decompositions (the relevant definitions are given immediately preceding the statement of Theorem IV.4). Although this result is not a direct application of *basic sequences*, its proof has some common features with the proof of Theorem III.1.

II. Notation, definitions, and preliminary results. As mentioned in the introduction, $X$, $Y$, $Z$, etc. refer to infinite dimensional Banach spaces and “subspace” (resp. “quotient space”) means “closed, infinite dimensional linear subspace,” “quotient manifold” (resp. “quotient manifold”). “Operator” means “bounded linear operator” and “isomorphism” means “linear homeomorphism.” If $T$ is an operator on $X$ and $E \subseteq X$, then $T|E$ denotes the restriction of $T$ to $E$. If $x \in X$ and $A \subseteq X$, $\delta(x,A) = \inf \{||x-a||: a \in A\}$. For $A \subseteq X$, $B \subseteq X$, $A+B$ denotes $\{a+b: a \in A, b \in B\}$.

If $A \subseteq X$, $A^\perp$ is the annihilator of $A$ in $X^*$, i.e., $A^\perp = \{x^* \in X^*: x^*(a) = 0 \text{ for every } a \in A\}$. If $A \subseteq X^*$, $A^\perp$ is the annihilator of $A$ in $X$, i.e., $A^\perp = \{x \in X: \delta(x,A) = 0 \text{ for every } a \in A\}$. For $A \subseteq X$, $\bar A$ is the norm closure of $A$ in $X$ and for $A \subseteq X^*$, $\bar A$ is the weak* closure of $A$ in $X^*$. Thus if $A$ is a linear manifold in $X^*$, then $\bar A = A^\perp$.

Sequences are denoted by using parentheses and $[a_n]$ denotes the norm-closed linear span of the sequence $(a_n)$. If $\{a_n\} \subseteq X_n$, $\{x_n\} \subseteq X^*$, we write $a_n \rightarrow x$ (respectively, $x_n \rightarrow x$) if $(a_n)$ converges to $x$ in the norm (respectively weak) topology on $X$. If $[a_n] \subseteq X_n$, $[x_n] \subseteq X^*$, we write $a_n \rightarrow x$ (respectively, $x_n \rightarrow x$) when $(a_n)$ converges to $x$ in the weak* topology on $X$.

We now recall some familiar facts about bases which will be used in the sequel without further reference. A sequence $(a_n, x_n)$ with $(a_n) \subseteq X$, $(x_n) \subseteq X^*$ is called biorthogonal provided $\langle a_n, x_n \rangle = \delta_{mn}$ for all $m, n = 1, 2, \ldots$. A sequence $(a_n) \subseteq X$ is a basis for $X$ that $X$ provided that for each $x \in X$ there is a unique sequence $(x_n(a))$ of scalars for which $\sum_{n=1}^{\infty} x_n(a)x_n(a) = x$. The functionals $a_n$ are linear and continuous, $(a_n, x_n)$ is biorthogonal, and $(a_n)$ forms a basis for $[a_n]$. A basis $(a_n)$ for $X$ is called shrinking provided the functionals $(a_n^*)$ biorthogonal to $(a_n)$ form a basis for $X^*$.

A basis $(a_n)$ is called boundedly complete provided that $\sum_{n=1}^{\infty} a_n(a)\|a_n\|$ is convergent whenever it is bounded. We will use repeatedly two results due to R.C. James: A Banach space with a basis is reflexive if and only if the basis is shrinking and boundedly complete. If $(a_n)$ is a basis with biorthogonal functionals $(a_n^*)$, then $(a_n)$ is shrinking (respectively, boundedly complete) if and only if $(a_n^*)$ is a boundedly complete (respectively, shrinking) basis for $[a_n]$.

A sequence $(a_n) \subseteq X$ is called basic provided that $(a_n)$ forms a basis for $[a_n]$. A basic sequence $(a_n)$ is called shrinking (respectively boundedly complete) provided that $(a_n)$ is a shrinking (respectively, boundedly complete) basis for $[a_n]$. Two basic sequences $(a_n)$ and $(b_n)$ are called equivalent provided that for any sequence $(z_n)$ of scalars, $\sum_{n=1}^{\infty} a_n z_n a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n z_n b_n$ converges. It follows from the closed graph theorem that $(a_n)$ is equivalent to $(b_n)$ if and only if there is an isomorphism $T$ from $[a_n]$ onto $[b_n]$ such that $T(a_n) = b_n$ for $n = 1, 2, \ldots$. If $(a_n)$ and $(b_n)$ are bases with associated biorthogonal functionals $(a_n^*)$ and $(b_n^*)$, respectively, then $(a_n)$ is equivalent to $(b_n)$ if and only if $(a_n^*)$ is equivalent to $(b_n^*)$.

We refer the reader to [18] for proofs of the above and for a comprehensive study of bases and basic sequences in Banach spaces.

We now come to the concept which underlies most of our results:
DEFINITION II.1. A sequence \( (y_n) \subset X^* \) is called \( \alpha^* \)-basic provided that there is a sequence \( (a_n) \subset X \) so that \( (a_n, y_n) \) is biorthogonal and for each \( y \in (\sum_{n=1}^\infty a_n y_n) \), \( \sum_{n=1}^\infty y(a_n)y_n = y \). (Note that although \( (a_n) \) is not uniquely determined by \( (y_n) \), if \( (a_n, y_n) \) and \( (a_n, y_n) \) are both biorthogonal then for each \( y \in \left( \sum_{n=1}^\infty a_n y_n \right) \), \( y(a_n) = y(a_n) \) for \( n = 1, 2, \ldots \).)

If \( (y_n) \in X^* \), then \( (y_n) \) can be identified with \( (X/\langle y_n \rangle)^* \) via the mapping \( T \), where \( T: X \to X/\langle y_n \rangle \) is the quotient mapping, because \( T \) is an isometry and a weak* isomorphism. From this it follows that \( (y_n) \) is \( \alpha^* \)-basic if and only if \( (T^{-1}y_n) \) is a \( \alpha^* \)-Schauder basis for \( (X/\langle y_n \rangle)^* \) according to Singer's definition in [17] (see also [18], p. 144 ff.).

Proposition I summarizes some of the elementary properties of \( \alpha^* \)-basic sequences:

PROPOSITION I.1. Suppose that \( (y_n) \subset X^* \) and let \( T: X \to X/\langle y_n \rangle \) be the quotient map. Then

(a) \( (y_n) \) is \( \alpha^* \)-basic if and only if \( X/\langle y_n \rangle \) has a basis \( (a_n) \) with associated biorthogonal functionals \( (\xi_n^*) \) such that \( T^* (\xi_n^*) = y_n \) for \( n = 1, 2, \ldots \). Thus if \( (y_n) \) is \( \alpha^* \)-basic, then \( (y_n) \) is basic.

(b) The following are equivalent:

(i) \( (y_n) \) is a boundedly complete \( \alpha^* \)-basic sequence;

(ii) \( (y_n) \) is \( \alpha^* \)-basic and \( (y_n) \) is a basis

(iii) \( X/\langle y_n \rangle \) has a shrinking basis \( (a_n) \) with associated biorthogonal functionals \( (\xi_n^*) \) such that \( T^* (\xi_n^*) = y_n \) for \( n = 1, 2, \ldots \).

(c) \( (y_n) \) is a shrinking \( \alpha^* \)-basic sequence if and only if \( X/\langle y_n \rangle \) has a boundedly complete basis \( (a_n) \) with associated biorthogonal functionals \( (\xi_n^*) \) such that \( T^* (\xi_n^*) = y_n \) for \( n = 1, 2, \ldots \).

Proof. (a) follows from the comment preceding the statement of Proposition I and Singer's work on \( \alpha^* \)-Schauder bases (see [18], p. 155). (c) follows from (a) and the duality between shrinking and boundedly complete bases proven by James. The equivalence of (i) and (iii) in (b) also follows from Singer's and James' work. It remains to be seen that (i) and (ii) in (b) are equivalent.

Suppose that \( (a_n) \subset X \) is such that \( (a_n, y_n) \) is biorthogonal. Assume that (ii) holds and let \( (a_n) \) be a sequence of scalars for which \( \left( \sum_{n=1}^\infty a_n y_n \right) \) is bounded. Then \( \left( \sum_{n=1}^\infty a_n y_n \right) \) is a boundedly \( \alpha^* \)-basic cluster point, say \( y \), and of course \( y \in (\sum_{n=1}^\infty a_n y_n) \). Thus by (ii), \( y \in (y_n) \). Since by (a) \( (y_n) \) is basic, \( \sum_{n=1}^\infty y(a_n)y_n = y \).

Now for arbitrary \( k \) and \( n \geq k \), \( \sum_{n=k}^\infty a_n y_n = a_k \), so \( y(a_n) = a_k \) because \( y \) is a weak* cluster point of \( \left( \sum_{n=1}^\infty a_n y_n \right) \). Hence \( \left( \sum_{n=k}^\infty a_n y_n \right) \) is a splitting basic point of \( \left( \sum_{n=1}^\infty a_n y_n \right) \). Hence \( \sum_{n=k}^\infty a_n y_n \to y \) and (i) follows.

Conversely, if (i) is satisfied and \( y \in (y_n) \), then \( \sum_{n=1}^\infty y(a_n)y_n \to y \) and hence \( \left( \sum_{n=1}^\infty y(a_n)y_n \right) \) is a splitting basic point of \( \left( \sum_{n=1}^\infty a_n y_n \right) \). Hence \( (y_n) \) is a shrinking \( \alpha^* \)-basis, \( \sum_{n=1}^\infty y(a_n)y_n \) is norm convergent, necessarily to \( y \). Hence \( y \in (y_n) \).

The next result is an immediate consequence of the known results about bases mentioned above and Proposition II.1.

PROPOSITION II.2. Let \( (b_n) \) be a basis in the Banach space \( X \) with biorthogonal functionals \( (\xi_n^*) \), let \( (y_n) \subset X^* \) be \( \alpha^* \)-basic and suppose that \( (y_n) \) is equivalent to \( (b_n) \). Then \( X/\langle y_n \rangle \) is isomorphic to \( X/\langle b_n \rangle \) and \( (y_n) \) is \( \alpha^* \)-equivalent to \( (b_n) \); i.e., for any sequence of scalars \( (a_n) \), \( \sum a_n y_n \) converges \( \alpha^* \) if and only if \( \sum a_n b_n \) does.

Remark II.1. It is easily seen that every basis sequence is equivalent to some other basic sequence which is also \( \alpha^* \)-basic. For let \( (a_n) \) be a basic sequence in \( X \) with biorthogonal functionals \( (\xi_n^*) \) and let \( (f_n) \) be the functionals in \( \langle \xi_n^* \rangle \) biorthogonal to \( (\xi_n^*) \). Then \( (f_n) \) is a \( \alpha^* \)-basic sequence and \( (a_n) \) and \( (f_n) \) are equivalent.

PROPOSITION II.3. If \( X \) is separable, \( (a_n) \subset X \), \( a_n \to 0 \), and \( \lim \sup |a_n| > 0 \), then \( (a_n) \) has a shrinking basic subsequence.

Proof. By normalizing a suitable subsequence, we may assume \( |a_n| = 1 \) for \( n = 1, 2, \ldots \). By passing to another subsequence, we may assume by a result of Bessaga and Pelczynski [1] that \( (a_n) \) is basic (see also our proof of II.1). Now by a well-known stability theorem (cf., e.g., [18], p. 93) there is a \( \lambda > 0 \) such that if \( (y_n) \subset X \) and \( \sum |y_n - a_n| < \lambda \), then \( (y_n) \) is a basic sequence equivalent to \( (a_n) \).

Let \( (d_n) \) be a countable dense subset of \( X^* \) with \( d_n = 0 \). By induction we may choose a sequence \( n_1 < n_2 < \cdots \) of positive integers and elements \( (a_n) \in X \) to satisfy, for \( i = 1, 2, \ldots \):

(i) \( |a_i| (d_k^n) \leq 1 \)

(ii) \( |a_i - a_k| < 2^{-i} \).

Indeed, let \( a_1 = a_1, n_1 = 1 \). Suppose \( a_1, \ldots, a_{n-1} \) and \( n_1 < \cdots < n_{k-1} \) have been defined. Since \( a_n = 0 \) we may choose \( n_k > n_{k-1} \) large enough so that

\[
d(a_k, d_k^n) = \sup |d(a_k)|: \ d \in \{d_k^n\}, ||d|| \leq 1 \leq 2^{-i}.
\]

Then choose \( a_k \) to satisfy (i) and (ii).

Since \( \sum |y_n - a_n| < \lambda \), \( (a_k) \) is a basic sequence equivalent to \( (a_n) \). Suppose \( f \in \langle \xi_n^* \rangle \) and let \( f \) be a Hahn–Banach extension of \( f \) to an element in \( X^* \). Then there is a subsequence \( (a_{n_k}) \) of \( (a_k) \) such that \( ||a_{n_k} - f|| \to 0 \).
Now $d_n|_0|_{(y_{n,m})_m} = 0$ hence $\lim_{m \to \infty} \|f|_{(y_{n,m})_m}\| = 0$ whence also $\|f|_{(y_{n,m})_m}\| \to 0$. Thus $(y_n)$, and consequently $(x_n)$, is a shrinking basic sequence.

Remark. II.2. Of course it follows immediately that if $X^*$ is separable, $X$ contains a shrinking basic sequence. This latter result is due to Dean, Singer, and Sternbach [3].

III. Extracting $\sigma^*$-basic sequences. In this section we prove a $\sigma^*$-basic analogue to the following theorem of Beasaga and Pełczyński [1] on the existence of basic sequences: if $(x_n) \subset X, x_n \to 0$, and $\limsup \|y_n\| > 0$, then $(x_n)$ has a basic subsequence. We show that for $\sigma^*$ separable, if $(y_n) \subset X^*, y_n \to 0$, and $\limsup \|y_n\| > 0$, then $(y_n)$ contains a $\sigma^*$-basic subsequence (Theorem III.1), and if also $X^*$ is separable, then $(y_n)$ contains a boundedly complete $\sigma^*$-basic subsequence (Theorem III.2). Another variation of the Beasaga–Pełczyński theorem is that if $(y_n) \subset X^*, (y_n)^* \subset X^*$ is separable, $y_n \to 0$, and $\limsup \|y_n\| > 0$, then $(y_n)$ contains a shrinking $\sigma^*$-basic subsequence (Theorem III.3).

Most of the results of this paper follow easily from the argument for Theorem III.1 and previously known facts. This argument is a variation of the “product” technique for producing basic sequences, due to Mazur (cf. the proof of Theorem (14)).

THEOREM III.1. Suppose that $X$ is separable, $(y_n) \subset X^*, y_n \to 0$, and $\limsup \|y_n\| > 0$. Then $(y_n)$ has a $\sigma^*$-basic subsequence, $(y_{n_k})$. Furthermore $(y_{n_k})$ may be chosen so that $(y_{n_k}) \subset X$ is selected with $(x_{n_k}) \subset X$ biorthogonal and $S_m : (y_{n_k}) \to (y_{n_k})$ is defined (for $m = 1, 2, \ldots$) by $S_m y = \sum_{n \in m} y(x_{n-k})y_{n_k}$ for all $y \in (y_{n_k})$, then $\|S_m\| \to 1$.

Proof. By passing to a subsequence and normalizing, we may assume that $\|y_n\| = 1$ for all $n$. Let $(x_n)$ be a sequence of positive numbers less than one such that $\sum_{n=1}^{\infty} x_n < \infty$; consequently $\prod_{n=1}^{\infty} (1 - x_n) < \infty$.

Now using Helly’s theorem and the compactness of the unit ball of a finite-dimensional space together with the separability of $X$, we may choose an increasing sequence $k_1 < k_2 < \ldots$ of positive integers and finite subsets $F_1 \subset F_2 \subset \cdots$ of the set of elements of $X$ of norm one with the linear span of $\bigcup F_i$ dense in $X$, such that for each $n = 1, 2, \ldots$

(i) for each $\phi \in \mathcal{H}(y_{n_k})^*$ with $\|\phi\| = 1$, there is an $\varepsilon \in F_n$ such that $|y(x) - f(y)| \leq \varepsilon \|y\|$ for all $y \in \mathcal{H}(y_{n_k})^*$.

(ii) $|y_{n_k}(x)| < \varepsilon_n$ for all $\varepsilon \in F_n$.

We claim that $(y_{n_k})$ is the desired $\sigma^*$-basic sequence.

Fix $n$, let scalars $a_1, \ldots, a_n$ be given such that $\|\sum_{k=1}^{n} a_k y_{n_k}\| = 1$, choose $f \in \mathcal{H}(y_{n_k})^*$ such that $f(\sum_{k=1}^{n} a_k y_{n_k}) = 1 = \|f\|$, and choose $\varepsilon \in F_n$ satisfying (i) for this $f$. It follows that $\|\sum_{k=1}^{n} a_k y_{n_k}(x)\| \leq 1 - \varepsilon_n$ for all $x \in X$.

For any scalar $a$,

$$\|\sum_{k=1}^{n} a_k y_{n_k} + \lambda y_{n_{k+1}}\| \geq \|\sum_{k=1}^{n} a_k y_{n_k}(x) + \lambda y_{n_{k+1}}(x)\|$$

$$= 1 - \varepsilon_n - \varepsilon_{n+1} = 1 - \varepsilon_{n+1} \quad \text{if} \quad |\lambda| \leq 2,$$

$$\geq 1 \quad \text{otherwise}.$$

Thus $\sum_{k=1}^{n} a_k y_{n_k} \leq \frac{1}{1 - \varepsilon_n} \sum_{k=1}^{n} a_k y_{n_k}$ holds for all scalars $a_1, \ldots, a_n$.

But then for any $k$ and scalars $a_1, \ldots, a_k$,

$$\|\sum_{k=1}^{n} a_k y_{n_k}\| \leq \frac{1}{1 - \varepsilon_n} \|\sum_{k=1}^{n} a_k y_{n_k}\|,$$

from which it follows easily that $(y_{n_k})$ is a basic sequence. (This is the Mazur argument for producing basic sequences.) Moreover letting $(f_i)$ denote the functionals in $(y_{n_k})^*$ biorthogonal to $(y_{n_k})$ and defining $P_n f_i = [f_i]$ by $P_n f_i = \sum_{k=1}^{n} f(y_{n_k})$, then $\|P_n\| \leq \prod_{k=1}^{n} (1 - \varepsilon_k)$ and hence $\|P_n\| \to 1$ as $n \to \infty$.

Now define $T : X \to [y_{n_k}]^*$ by $(Tx)(y) = y(x)$ for all $y \in [y_{n_k}]$. To complete the proof it suffices to show that $T(X) = [f_i]$ (which yields that $(y_{n_k})$ is a $\sigma^*$-basic sequence) and that $T$ is an isometry (which shows that $\|P_n\| = \|S_n\|$ for all $n$ and hence $\|S_n\| \to 1$ as $n \to \infty$).

Now if $\varepsilon \in F_n$ for some $n$, then $\sum_{k=1}^{n} y_{n_k}(x) < \infty$, whence $T(x) = \sum_{k=1}^{n} y_{n_k}(x)f_i \in [f_i]$ and thus $T(X) \subset [f_i]$.

We shall now show that

(*) for all $g$ in the linear span of $(f_i)$ with $\|g\| = 1$ and $\varepsilon > 0$, there exists an $x \in X$ with $\|x\| = 1$ and $\|x - g\| < \varepsilon$.

Let $0 < \varepsilon < 1$, choose $N$ such that $\sum_{k=N}^{\infty} x_k < \varepsilon$ and $\|P_n\| \leq 1 + \varepsilon$ for all $n > N$ and fix $n > N$. For the sake of convenience, define $\|f_i\| = \|f_i|_{(y_{n_k})^*}\|$ for $x \in (f_i)^*$; it follows that $\|f_i\| \leq 1 + \varepsilon \|P_n\| / \|f_i\| < 1 + \varepsilon$ for all such $f$. Now fix $g \in (f_i)^*$ with $\|g\| = 1$ and put $g = g|_{(y_{n_k})^*}$. Then choosing $\varepsilon \in F_n$ satisfying (i) for $f_i$, we have that $\sum_{k=1}^{\infty} y_{n_k}(x)(f_i - f) \leq \varepsilon_n$.

Hence $\|\sum_{k=1}^{n} y_{n_k}(x) f_i - f_i\| \leq 2 \varepsilon_n / 3 < (2\varepsilon / 3)$; moreover $\|f_i - f|_{(y_{n_k})^*}\| < (2\varepsilon / 3)$ for all $f_i$, $f$. Thus

$$\|\sum_{k=1}^{n} y_{n_k}(x) f_i - f\| \leq \sum_{k=1}^{n} \varepsilon_k < (4\varepsilon / 3).$$

Thus
Suppose that \( y \| y_n \). Then \( S_{y_n} \xrightarrow{\chi} y \) because \( (y_n) \) is \( \omega^* \)-basic and thus \( \liminf \| y_n \| \geq \| y \| \). But from (a) it follows that \( \limsup \| y_n \| < \| y \| \), hence \( S_{y_n} \xrightarrow{\chi} y \) and \( y \in [y_n] \). Thus \( (y_n) \) is \( \omega^* \)-basic, and \( (y_n) \) is boundedly complete by Proposition II.1(b).

We complete this section with a result that gives information in the case where \( X \) is non-separable.

**Theorem III.3.** Suppose that \( (y_n) \) is a \( \omega^* \)-basic sequence. By the proof of Theorem III.1 (see the remark immediately following the proof of III.1), we may choose a basic sequence \( (y_n) \) of \( (y_n) \) such that if \( (f) \) is the sequence biorthogonal to \( (y_n) \) and \( T \xrightarrow{\chi} [y_n] \) is defined by \( (Ty) = \chi(y) \) for all \( y \in [y_n] \) and \( \sigma \in X \), then \( TX \xrightarrow{\chi} [f] \). Indeed the separability of \( X \) was used only to pick \( (F) \) so that \( \bigcap F = X \), from which it followed that \( T(X) \xrightarrow{\chi} [f] \). When \( X \) is non-separable and \( (y_n) \) is as above, we can still choose \( (F) \) and \( (K) \) to satisfy (i) and (ii). Proceeding exactly as in the proof of III.1, one obtains that \( \chi \) holds; a standard iteration argument based on \( \chi \) alone, yields that \( T(X) \xrightarrow{\chi} [f] \). Thus the result of Beesaga and Pelczynski mentioned above also follows from the proof of III.1.

**Remark III.2.** Theorem III.1 gives, in some sense, the best possible result for \( (X) \) separable, because if \( (y_n) \) is \( \omega^* \)-basic, and \( \| y_n \| \) is bounded, then \( y_n \xrightarrow{\chi} 0 \). Indeed, otherwise \( (y_n) \) has a weak* cluster point, \( y \), with \( y \neq 0 \). If \( (x_n) \) is a \( \sigma \)-weak* biorthogonal, then evidently \( y(x_n) \) for \( n = 1, 2, \ldots \), but \( y \left( \sum_{n=1}^{\infty} x_n \right) \), hence 0, contradicting the fact that \( y = 0 \).

Before proceeding to the next theorem, we recall a renorming theorem due independently to Kadec [6] and Klee [7]: if \( X \) is separable then there is a norm \( ||| \cdot ||| \) on \( X \) equivalent to the original norm on \( X \) such that for any sequence \( (x_n) \in X^* \) and \( x^* \in X^* \) if \( x_n \rangle \to x^* \) and \( ||| x_n ||| \to ||| x^* ||| \) then \( x_n \Rightarrow x^* \). (For a proof of the theorem of Kadec-Klee see [18], p. 480).

**Theorem III.2.** Suppose that \( X^* \) is separable, \( (y_n) \subset X^* \), \( y_n \xrightarrow{w^*} 0 \), and \( \limsup || y_n || > 0 \). Then \( (y_n) \) has a boundedly complete \( \omega^* \)-basic subsequence \( (y_n) \).

Proof. By the aforementioned renorming theorem, we may assume that for any sequence \( (x_n) \subset X^* \) and \( \sigma^* \subset X^* \), \( x_n \rangle \sigma^* \Rightarrow \sigma^* \) and \( || x_n || \to || \sigma^* || \) then \( x_n \to \sigma^* \). By Theorem III.1 we may extract a \( \omega^* \)-basic subsequence \( (y_{n_k}) \) of \( (y_n) \) and find \( (z_k) \subset X \) so that \( (z_k, y_{n_k}) \) is biorthogonal and if \( S_{z_k^*} [y_{n_k}] \to \tilde{y}_{n_k} \) is defined for \( n = 1, 2, \ldots \) by \( S_{z_k^*} [y_{n_k}] \to \tilde{y}_{n_k} \) then \( S_{z_k^*} \to \tilde{y} \).

(a) \[ \lim || S_{n_k} || = 1 \].
Finally, the first conclusion in (iii) follows from Theorem III.3 and Proposition II.1 (c), because if \( X^* \) is separable then \( X \) contains a sequence \( (y_n) \) with \( y_n \rightarrow 0 \) and \( |y_n| = 1 \) for \( n = 1, 2, \ldots \). Now if \( Z \) is a quotient space of \( X \) then \( Z^* \) is isomorphic to a subspace of \( X^* \). If also \( Z \) has a boundedly complete basis \( \{x_n\} \) with biorthogonal functionals \( \{x^*_n\} \), then \( Z \) is isomorphic to \( [x^*_n] \), whence \( [x^*_n]^* \) is isomorphic to a subspace of \( X^* \).

Remark IV.1. We do not know if every separable space \( X \) has a quotient space with a shrinking basis; by IV.1 (ii), this is equivalent to whether \( X^* \) has a boundedly complete basic sequence. However, the conclusion of (ii) concerning \( Y \) requires critically the assumptions of (ii). For example suppose that \( Y \) has no reflexive subspaces and \( X = Y^* \). Then regarding \( Y \) as canonically imbedded in \( X^* = Y^{**} \), \( Y \) has no weak* closed subspaces. If \( Y = Y^* \), \( Y \) is a separable conjugate space yet \( X \) is non-separable; if \( Y = c_0 \), \( X \) is separable but \( Y \) is not isomorphic to a subspace of a separable conjugate space; it is well known that neither \( c_0 \) nor \( l^1 \) have reflexive subspaces.

Our next result shows that if \( Y \) has no reflexive subspaces, then either \( Y^* \) is non-separable or \( Y \) cannot be imbedded in a separable conjugate space.

**Theorem IV.2.** Suppose that \( X^* \) is separable; \( Y \subseteq X^* \), and \( Y^* \) is separable. Then \( Y \) is somewhat reflexive.

Proof. Let \( Z \) be a subspace of \( Y \). Then also \( Z^* \) is separable, hence contains a shrinking basic sequence \( (y_n) \) with \( |y_n| = 1 \) for \( n = 1, 2, \ldots \), by Proposition II.3. Of necessity \( y_n \rightarrow 0 \) hence \( y_n \rightarrow 0 \), whence by Theorem III.3 \( (y_n) \) has a subsequence \( (y_{n_k}) \) which is boundedly complete. Of course \( (y_{n_k}) \) is also shrinking, hence \( [y_{n_k}] \) is a reflexive subspace of \( Z \).

**Corollary IV.1.** If \( X^{**} \) is separable then both \( X \) and \( X^* \) are somewhat reflexive.

Proof. That \( X^* \) is somewhat reflexive follows immediately from Theorem IV.2. Now suppose \( Y \subseteq X \). \( Y \) can be considered as a subspace of the separable conjugate space \( X^{**} \) and \( Y^* \) is separable, hence \( Y \) contains a reflexive subspace by Theorem IV.3.

Remark IV.2. We do not know if the converse to IV.2 is true. That is, if \( Y \) is separable and somewhat reflexive, is \( Y^* \) separable and \( Y \) isomorphic to a subspace of a separable conjugate space? (The first part of this question is also raised in [2]). It follows from a recent result of Lindenstrauss [9] and the above corollary that there exists a separable somewhat reflexive space \( Y \) which is not isomorphic to a complemented subspace of any conjugate space. (A subspace \( Z \) of \( X \) is said to be complemented in \( X \) if there exists an idempotent operator on \( X \) whose range is \( Z \).) Indeed, Lindenstrauss showed that there exists a \( Y \) with \( Y^* \) separable and \( Y^{**}/Y \) isomorphic to \( c_0 \) (the space of sequences which converge to zero). Our corollary yields that \( Y \) (and also \( X \) and \( X^* \)) are somewhat reflexive. Nevertheless if \( Y \) were isomorphic to a complemented subspace of a conjugate space, \( X \) would be complemented in \( Y^* \), which would imply that \( c_0 \) is isomorphic to a subspace of the separable conjugate space \( Y^{**} \) (impossible (see the remark following the proof of Theorem IV.1)).

Our next result follows easily from our proof of Theorem III.1 and an argument of Lindenstrauss and Pełczyński [10]. Before proceeding to it, we need some preliminaries: \( L_E \) denotes the space of Lebesgue integrable functions on \([0, 1]\) with \( \|f\| = \int f \, dt \). Given a measurable set \( A \subseteq [0, 1] \), \( \lambda_A \) denotes its characteristic function and \( |A| \) its Lebesgue measure. The Haar basis \( (H_{n, m})_{n, m} \) for \( L_E \) (c.f. [18], p. 13) is defined as follows:

\[
H_{n, m} = 1, \\
\lambda_{A_{n, m, l}} = \chi_{[n 10^{-l} + m, (n + 1) 10^{-l} - m]},
\]

\((1 \leq i \leq 3), \; k = 0, 1, 2, \ldots, \).

Let \( (\mathcal{H}_{n, m})_{n, m} \) denote the functionals biorthogonal to \( (H_{n, m}) \), regarded as elements of \( L^*_E[0, 1] \). It is known and easily seen that \( (\mathcal{H}_n) \) is isometric to the space of continuous functions on the Cantor discontinuum, under the supremum norm. Indeed, the linear span of \( (\mathcal{H}_n) \) consists of those functions on \([0, 1]\) which are equal almost everywhere to a step function which has breaks at dyadic rationals. Then letting \( (0, 1]^m \) denote the compact space of all sequences of zeros and ones, there is a unique surjective linear isometry \( T: (\mathcal{H}_n) \rightarrow C([0, 1]^m) \) such that \( T 1 = 1 \) and for all \( n \) and \( j \) with \( 0 \leq j < 2^n \), \( n = 1, 2, \ldots, T(\mathcal{H}_{n, j}) = \chi_{A_n} \), where

\[
A_n = \{(a_i) \in (0, 1]^m: a_i = 1 \text{ for all } 1 \leq i \leq n \text{ and } j = \sum a_i 2^{n-i} \text{ with } c_0 = 0 \text{ or for all } 1 \leq i \leq n \}.
\]

We recall finally the following consequence of the Liapounoff convexity theorem [8]; if \( F \) is a finite subset of \( L^*_E[0, 1] \), and \( B \) is a measurable subset of \([0, 1]\) then there exists a measurable subset \( A \) of \([0, 1]\) with \( |A| = (1/2)|B| \) such that \( \int f \, dt = \int f \, dt \) for all \( f \in F \).

**Theorem IV.3.** Let \( X \) be a separable Banach space and \( Y \) a subspace of \( X^* \).

(a) If \( X \) is isomorphic to \( Y \), there exists an \( w^* \)-basic sequence \( (y_n) \) in \( Y \) such that \( (y_n) \) is equivalent to the usual basis for \( Y \). Consequently \( c_0 \) is isomorphic to a quotient space of \( X \).

(b) If \( Y \) is isomorphic to \( l_1 \), there exists an \( w^* \)-basic sequence \( (y_{n, m})_{n, m} \) in \( Y \) such that \( (y_{n, m})_{n, m} \) is equivalent to the Haar basis for \( l_1 \). Consequently \( C((0, 1]^m) \) is isomorphic to a quotient space of \( X \).
Proof. (a) follows easily from Theorem III.1 and Propositions II.1 and II.2. Let \( (y_n) \subseteq X^* \) be a basic sequence equivalent to the usual unit vector basis for \( P \). Since \( X \) is separable, \( (y_n) \) has a weak* convergent subsequence \( (y_{n_k}) \). Then \( y_{n_k} \to y \) in the weak* topology, which is equivalent to the usual basis for \( P \). Thus by Theorem IV.1, \( (y_{n_k} - y_{n_l}) \) has a subsequence \( (z_j) \) such that \( (z_j) \) converges weak* to \( x \) for every element \( x \) of \( X \). Hence \( x \in X \) and \( (y_{n_k} - y_{n_l}) \) converges weak* to \( x \) for every element \( x \) of \( X \).

To prove (b), we combine the proof of Theorem III.1 and the proof of Theorem 4.1 of [10] as follows:

Let \( Y \) be the direct sum \( \bigoplus_{i,j} Y_{i,j} \) and let \( Y \to L_1 \) be a surjective isomorphism.

Let \( (x_n) \subseteq X^* \) be an increasing sequence of positive numbers with \( \sum x_n < 1 \). We may choose a sequence \( (x_n) \subseteq X^* \) of finite subsets of \( (x_n) \subseteq X^* \) of measurable subsets of \( [0, 1] \) (with \( A_0 = [0, 1] \) and a sequence \( (y_n) \subseteq X^* \) of \( [0, 1] \) satisfying the following properties for \( n \in \mathbb{N} \) and \( 0 \leq j < 2^r, r = 0, 1, 2, \ldots \):

(i) for all \( f \in C([0, 1]) \), there is an \( x \in L_1 \) such that \( |y(f) - f(x)| \leq x_n |f| \) for all \( y_n \).

(ii) \( |y_n| \leq r \).

(iii) \( \sum_{j=0}^{2^r} x_{A_{j+1}} \leq x_{A_j} \).

(iv) \( A_{2^r+1} = 2^{i+1} \).

(v) \( A_{2^r+1} \cap A_{2^r+2} = A_{2^r+2} \).

(vi) \( A_{2^r+1} \cap A_{2^r+2} = \emptyset \).

To see that this is possible, for each \( f \in X^* \), let \( f \) denote the unique element of \( X^*[0, 1] \) such that \( \int f(t) dY(t) = y(f) \) for all \( y \in X^* \) and let \( (a_0, a_1, \ldots) \) be a dense subset of \( \bigoplus_{i,j} \). Put \( y_n = x_i^{-1}(1), A_0 = [0, 1], \) and choose \( F_n \) a finite set of elements of \( X \) of norm one satisfying (i) for \( n = 0, \) such that \( a_i \in F_n \).

Suppose that \( y_n \) and \( F_n \) have been defined for all \( n < 2^j + 1 \) and that \( A_n \) has been defined for \( n < 2^j + 1 \), satisfying (i)-(vi). By the Lebesgue decomposition theorem, there exist measurable sets \( A_n \) and \( B_n \) such that \( A_n \cap B_n = \emptyset, |A_n| = |B_n| \), and

\[
\int f dY = \int \mathbb{1}_A f dX + \int \mathbb{1}_B f dX \quad \text{for all } f \in C[a,b].
\]

Define \( A_{2^r+1} = A_n, A_{2^r+2} = B_n, \) and \( y_n \) by (iii). Now using the compactness of the unit ball of a finite dimensional space, choose \( F_n \) satisfying (i) with \( a_i \in F_n \). This completes the inductive definition of these objects.

As observed in [10], it follows that \( (y_n) \subseteq X^* \) is isometrically equivalent to the Haar basis of \( L_1 \), hence \( (y_n) \subseteq X^* \) is equivalent to \( (y_n) \subseteq X^* \). Our proof of Proposition III.1 yields immediately that \( (y_n) \subseteq X^* \) is a basis sequence, and thus by Propositions II.1 and II.2, \( (y_n) \subseteq X^* \) is isomorphic to \( \mathbb{R}^\infty \), which is in turn isometric to \( C(0, 1)^\infty \) by our preliminary remarks.

Remark IV.3. Our proof of the above result and III.1 yields that if \( P \) (resp. \( L_1 \)) is isometric to a subspace of \( X^* \) and \( X \) is separable, then \( (y_n) \subseteq X^* \) is isomorphic to a quotient space of \( X \). The proofs of Theorem IV.1 and Theorem III.1 also yield that if \( E \) is a symmetric function space (as defined in [10]) such that the Haar system \( (h_n) \subseteq X \) is a basis for \( E \), then if \( Y \) is a subspace of \( X^* \) isomorphic to \( E \) and \( X \) is separable, \( Y \) contains a \( \sigma \)-basis sequence \( (y_n) \subseteq X^* \) (and consequently \( \mathbb{R}^\infty \) (in \( C(0, 1)^\infty \)) is isomorphic to a quotient space of \( X \)).

Remark IV.4. The result mentioned in the Introduction is an immediate consequence of a theorem of M"{u}ntch which asserts that \( C(0, 1) \) is isomorphic to \( C(0, 1)^\infty \) (cf. [16]).

Remark IV.5. It is a theorem of Pelczynski [15] that if \( X \) contains a semi-norming subspace isomorphic to \( L_1(\mu) \) for some non-purely atomic measure \( \mu \), then \( X \) contains a subspace isomorphic to \( P \). It has recently been observed by James Hagler that the proof in [15] may be modified so as to yield this result without the “semi-norming” hypothesis. Actually, this result also follows from our Theorem IV.3. For if \( X \) contains a subspace isomorphic to \( L_1(\mu) \) for some non-purely atomic \( \mu \), it also contains a subspace isomorphic to \( L_2 \), and if there is a separable subspace \( Z \) of \( Z \) such that the separable space \( Z \) is isomorphic to a subspace of \( Z \), and hence by Theorem IV.3, \( C(0, 1)^\infty \) is isomorphic to a quotient space of \( Y \).

The final result of this paper, we need the following definitions and notation:

A sequence \( (E_n) \) of finite dimensional subspaces of \( X \) is called a finite dimensional decomposition (f.d.d., in short) for \( X \) provided that each \( f \in X \) can be written uniquely as \( \sum f_n \in E_n \) with \( f_n \in E_n \). The operators \( Q_n : X \to E_n \) defined by \( Q_n = \sum f_n \in E_n \), and \( (Q_n) \subseteq X \) is the inverse. Conversely, if \( Q_n \subseteq X \) is uniformly bounded sequence of finite rank operators on \( X \) for which \( Q_n = Q_m \) and \( (Q_n) \subseteq X \) then \( (Q_n) \subseteq X \) (where \( Q_0 = 0 \) is a f.d.d. for \( X \), and we call \((Q_n) \subseteq X \) the f.d.d. determined by \( Q_n \). If \( (Q_n) \subseteq X \) determines
a f.d.d. for $X^*$-equivalently, if $[Q_n^*X^*] = X^*$ — then the f.d.d. determined by $(Q_n)$ is called shrinking.

A subset $A$ of $X^*$ is called norm determining over $X$ provided that for each $x \in X$, $\|x\| = \sup \{ |a(x)| : a \in A, \|a\| \leq 1 \}$.

**Theorem IV.4.** If $X$ is separable then there is a $Y \subset X$ such that $Y$ and $X/Y$ both have finite dimensional decompositions. If also $X^*$ is separable, then $Y$ may be chosen so that both $Y$ and $X/Y$ have shrinking finite dimensional decompositions.

Proof. By a result of Gapsinski and Kaldec [4] (see also Lemma 2 of Mackey's paper [1]) $X$ has a biorthogonal sequence $(\sigma_n^*, \sigma_n^*)$ with $[\sigma_n] = X$ and $[\sigma_n^*] = X^*$ norm determining over $X$. It follows easily that we can choose finite sets $\sigma_1 < \sigma_2 < \ldots, A_1 < A_2 < \ldots$ so that $\sigma = \bigcup_{\sigma_n}$ and $\Delta = \bigcup_{\Delta_n}$ are complementary infinite subsets of the positive integers and, for $n = 1, 2, \ldots$,

(i) if $x^* \in [\sigma_n^*, \sigma_n]$ there is $x^* \in [\sigma_n^*, \sigma_n \cup \Delta_n]$ so that $\|x^*\| = 1$ and $\|x^*\| > 1 - 1/n + 1/\|x^*\|$;

(ii) if $x^* \in [\sigma_n^*, \sigma_n]$ there is $x^* \in [\sigma_n^*, \sigma_n \cup \Delta_n]$ so that $\|x^*\| > 1 - 1/n + 1/\|x^*\|$.

For $n = 1, 2, \ldots$, define $S_n : X \to X$ and $T_n : X \to X$ by

- $S_n x = \sum_{i \in \sigma_n} \sigma_i^* a_i x_i$,
- $T_n x = \sum_{i \in \sigma_n} \sigma_i^* a_i x_i$.

We claim that, for $n = 1, 2, \ldots$,

(iii) $\|T_n x\| \leq 1 + \frac{1}{n}$;

(iv) $\|S_n x\| \leq 1 + \frac{1}{n}$.

To see that (iii) holds, suppose that $y \in [\sigma_n^*, \sigma_n \cup \Delta_n]$ and using (i) pick $x^* \in [\sigma_n^*, \sigma_n \cup \Delta_n]$ so that $\|x^*\| = 1$ and $\|T_n x^* y\| > 1 - 1/n + 1/\|x^*\|$. Since $y \in [\sigma_n^*, \sigma_n \cup \Delta_n]$ and hence $\|y\| > 1 - 1/n + 1/\|x^*\|$, so that $\|T_n y\| \leq 1 + 1/n$.

Thus (iii) is true and (iv) follows from (iii) in a similar manner.

We show now that if $y \in [\sigma_n^*, \sigma_n \cup \Delta_n]$ then $T_n x^* y \to y$. By (ii) $T_n y$ is bounded; hence has a weak* cluster point, say, $y^*$. Evidently $T_n x^* y = T_n y$ for $n = 1, 2, \ldots$, hence $x^* - y \in [\sigma_n^*, \sigma_n \cup \Delta_n]$ for each $n = 1, 2, \ldots$, whence $x^* - y \in [\sigma_n^*, \sigma_n \cup \Delta_n]$. Since also $x^* - y \in [\sigma_n^*, \sigma_n \cup \Delta_n]$, $x^* - y = 0$ and $x^* = y$. Thus $T_n x^* y = (\sigma_n \cup \Delta_n)$. Setting $Y = (\sigma_n^*, \sigma_n \cup \Delta_n)$ and using the obvious analogue of Proposition III.1(a) for f.d.d.'s, we have that $X/Y$ has a f.d.d.

Now $Y = (\sigma_n \cup \Delta_n)^* = (\sigma_n^*, \sigma_n \cup \Delta_n)^*$, so from (iv) it follows that $(\sigma_n \cup \Delta_n)$ determines a f.d.d. for $Y$. This completes the proof of the first statement.

Suppose now that $X^*$ is separable. By the aforementioned result of Mackey and Gapsinski—Kaldec, the biorthogonal sequence $(\sigma_n^*, \sigma_n^*)$ may be chosen so that $[\sigma_n^*] = X$ and $[\sigma_n^*] = X^*$. Also, in view of the renorming theorem of Kaldec-Klee mentioned in Section III, we may assume that for any sequence $(y_n) \subset X^*$ and $x^* x \in X^*$, if $y_n \to y$ and $\|y_n\| \to \|y\|$ then $y_n \to y$. It follows from (ii) (cf. the proof of Theorem III.2) that if $x^* \in [\sigma_n^*, \sigma_n^*]$ then $T_n x^* \to x^*$, hence $T_n x^* \to x^*$, whence $([\sigma_n^*, \sigma_n^*]) = ([\sigma_n^*, \sigma_n^*])$. Using the f.d.d. variant of Proposition III.1(b), we have that $X/Y$ has a shrinking f.d.d. Of course, since $[\sigma_n^*] = X^*$, $(\sigma_n^*, \sigma_n^*)$ automatically determines a shrinking f.d.d. for $Y$.

**Remark IV.6.** Kaldec and Pelczynski have proved the following theorem related to Theorem IV.4 (see [18], p. 489): If $(\sigma_n^*, \sigma_n^*)$ is a biorthogonal sequence with $[\sigma_n] = X$ and $[\sigma_n^*] = X^*$ norming over $X$, then the positive integers can be partitioned into disjoint infinite subsets $\sigma$ and $\Delta$ with $\sigma = \bigcup_{\sigma_n}$ and $\Delta = \bigcup_{\Delta_n}$, where $(\sigma_n)$ and $(\Delta_n)$ are disjoint and finite, and such that $[\sigma_n^*, \sigma_n] \cap \bigcup_{\sigma_n}$ (resp. $[\sigma_n, \sigma_n^*] \cap \bigcup_{\sigma_n}$) is a finite dimensional decomposition for $([\sigma_n^*, \sigma_n], \sigma)$ (resp. for $([\sigma_n^*, \sigma_n^*], \Delta)$).

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