p-trivial Banach spaces

by

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Abstract. A pair of Banach spaces $X$ and $Y$ is said to be $p$-trivial if every bounded linear operator $T : X \to Y$ is $p$-absolutely summing. Continuing a study initiated by Lindenstrauss and Pelczynski we study various implications of the statement $\langle X, Y \rangle$ is $p$-trivial. We also study the dual notion of strongly $p$-trivial.

As a by-product of this study we obtain an apparently new characterisation of the isomorphs of Hilbert space.

§ 6. INTRODUCTION

This work is based on the papers [17], [18] of Lindenstrauss, Pelczyński and Rosenthal. In particular we present a detailed study of Proposition 8.1 of [17]. We will frequently use results and ideas from the somewhat neglected paper [12] of Grothendieck. In addition we will make use of two of the most profound theorems in modern functional analysis: the $\varepsilon$-isometry theorem of Dvoretzky [9] and the modification of the principle of local reflexivity [18] found in the remarkable paper of Johnson, Rosenthal and Zippin [14]. Since many of the concepts used in this work are fairly recent, we list below the definitions and results we need.

Classical concepts. All spaces considered are Banach spaces. The word operator will mean a bounded linear transformation. We shall denote by $\mathcal{L}(E, F)$ the operators from $E$ to $F$. By an isomorphism, we mean a one-to-one operator that is open. A projection $P$ is a member of $\mathcal{L}(E, E)$ such that $P^2 = P$. If $A$ is a subspace (= closed linear manifold) of $E$ then $A$ is complemented in $E$ if there is a projection $P : \mathcal{L}(E, E)$ with $P(E) = A$.

If $\{x_n\} \subseteq E$ then by $[x_n]$ we denote the closed linear span of $\{x_n\}$ in $E$.

* Portions of this work appear in the dissertation of the first named author written at Florida State University under the direction of Professor C. W. McArthur.

** Research supported by NSF-GP-20844.
By a biorthogonal system \((x_n, f_n)\) in \(E\) we mean sequences \((x_n) \subseteq E\), 
\((f_n) \subseteq E'\) such that 
\[ f_n(x_n) = \delta_{n1}. \]

The expression \(\sum_{n=1}^{\infty} f_n(x_n)\) is the formal expansion of \(\sum x_n E\) with respect to the biorthogonal system \((x_n, f_n)\). A (Schauder) basis for \(E\) is a biorthogonal system \((x_n, f_n)\) such that the formal expansion of each \(\sum x_n E\) converges to \(x\) in the norm topology of \(E\).

A sequence \((x_n) \subseteq E\) is a basic sequence if \((x_n)\) is a basis for \([x_1]\). The Grislyam constant, \(K\), of a basic sequence \((x_n)\) is defined by
\[ K = \sup_n \left\| \sum_{i=1}^{n} f_i(x_i) x_i \right\| \]
where \(x \subseteq [x_1]\) and \((f_n)\) is biorthogonal to \((x_n)\).

We will have occasion to use the following easily proved fact.

(0.1) If \((x_n)\) is a basic sequence in \(E\) with coefficient functionals \((g_n) \subseteq [x_1]'\) and if \(f_n\) is a norm preserving extension of \(g_n\) to \(E\) then
\[ \frac{1}{\|x_n\|} \leq \|f_n\| \leq \frac{2K}{\|x_n\|}, \]
where \(K\) is the Grislyam constant.

For \(1 \leq p < \infty\) we denote by \(l_p\) the Banach space of scalar sequences \(a = (a_n)\) with
\[ \|a\| = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}, \quad \text{if} \quad 1 \leq p < \infty, \]
\[ \|a\| = \sup |a_n|, \quad \text{if} \quad p = \infty. \]

By \(l_c^n\) we denote the space of \(n\)-tuples with the above norm. Also by \(c_0\), we mean the closed subspace of \(l_c^n\) consisting of those sequences which tend to 0.

Let \((x_n)\) be a sequence in a Banach space \(E\), \(1 \leq p \leq \infty\) and \(p'\) given by \(\frac{1}{p} + \frac{1}{p'} = 1\). Then

(i) \((x_n)\) is weakly \(p\)-summing, written \((x_n) \subseteq l_p(E)\), if for each \(f \subseteq E'\)
\[ f(x_n) \subseteq l_{p'}. \]

With the norm \(\|x_n\| = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}, \ l_p(E)\) is a Banach space.

(ii) \((x_n)\) is \(p\)-summing, written \((x_n) \subseteq l_p(E)\), if \(\|x_n\| \subseteq l_p\).

With the norm \(\|x_n\| = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}, \ l_p(E)\) is a Banach space; and

(iii) \((x_n)\) is \(p\)-trivial Banach spaces, written \((x_n) \subseteq l_p(E)\), if for each \(f \subseteq l_{p'}(E'), \ l_p(x_n) \subseteq l_{p'}.\)

With the norm \(\|x_n\| = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}, \ l_p(E)\) is a Banach space.

Moreover, we clearly have the set theoretic inclusions
\[ l_p(E) \subseteq l_q(E) \subseteq l_{p'}(E) \]

(Of course above for \(p = \infty\) we understand \(p' = 1\), for \(p = 1\), \(p' = \infty\) and \(\epsilon_0, \ a_0, \ c_0\) given by the appropriate suprema.)

The principal work for weakly \(p\)-summing and \(p\)-summing sequences is [12]; for strongly \(p\)-summing sequences, see [3].

\(p\)-absolutely summing operators. If \(\gamma\) is one of the norms \(\tau_p, a_p, c_p\) above we will abuse the notation and write for finite sets \((x_n)_{n=1}^N \subseteq E, \gamma(x_n)\).

This is of course meaningful if we consider the sequence \((x_n)\), \(\delta_n = x_n\) for \(i \leq N, \delta_i = 0\) for \(i > N\) in the appropriate space above.

Let \(T : L(E, F)\). Then

(i) \(T\) is \(p\)-absolutely summing if there is a constant \(C\) such that
\[ \gamma(Ta) \subseteq C\gamma(a) \]
for all finite sets \((a_n)_{n=1}^N \subseteq E\); and

(ii) \(T\) is \(p\)-strongly summing if there is a constant \(C\) such that
\[ \gamma(Ta) \subseteq C\gamma(a) \]
for all finite sets \((a_n)_{n=1}^N \subseteq E\).

It is clear from the above definitions that \(T\) is \(p\)-absolutely summing if and only if \(T_l([E]) \subseteq l_p(F)\) and \(T\) is \(p\)-strongly summing if and only if \(T_l([E]) \subseteq l_p(F)\).

The \(p\)-absolutely summing operators have been extensively studied in [17] and [21]. The strongly \(p\)-summing operators were introduced in [3].

We denote the \(p\)-summing operators from \(E\) to \(F\) by \(II_p(E, F)\) and the strongly \(p\)-summing operators from \(E\) to \(F\) by \(D_p(E, F)\). If \(\tau_p(T)\) and \(\delta_p(T)\) denote, respectively, the infimum of the constants \(C\) occurring in (i) and (ii) above, then these respective norms \(II_p(E, F)\) and \(D_p(E, F)\) are Banach spaces ([21] and [3]).

Finally we say that a pair of Banach space \(E\) and \(F\) is \(p\)-trivial, written \(\langle E, F, \delta_p \rangle\) is \(p\)-trivial, if \(II_p(E, F) = \delta_p(E, F)\), and strongly \(p\)-trivial, written analogously, if \(D_p(E, F) = \delta_p(E, F)\).

Motivation for this concept is [17]. Our 1-trivial spaces coincide with the "unconditionally trivial" spaces of [17].

We need the following facts:
(0.3) If $\langle E, F \rangle$ is $p$-trivial (strongly $p$-trivial) then there is a constant $M$ such that

$$
\sigma_p(Tx) \leq M \| T \| \sigma_p(x)
$$

for every $T \in \mathcal{L}(E, F)$ and finite set $\{x_i\} \subset E$. (0.2) is immediate from the above definitions and the open mapping theorem.

(0.3) Let $1 \leq p < \infty$. Then $T \in \Pi_p(E, F)$ if and only if the adjoint $T^* \in \Pi_p(F^*, E^*)$.

Let $1 \leq p < \infty$. Then $T \in \Pi_p(E, F)$ if and only if $T^* \in \Pi_p(F^*, E^*)$.

(0.3) is the main result of [3]. (Here, and for the remainder of the paper $p'$ is determined by $\frac{1}{p} + \frac{1}{p'} = 1$.)

The $L_p$-spaces. Our main concern in this paper is the $L_p$-spaces of [17].

If $E$ and $F$ are isomorphic Banach spaces, the distance coefficient of $E$ and $F$, $d(E, F)$, is defined by

$$
d(E, F) = \inf \{ \| T \| \| T^{-1} \| \}
$$

where the infimum is over all isomorphisms from $E$ onto $F$.

Let $1 \leq p \leq \infty$. A Banach space $E$ is an $L_p^\infty$-space if for each finite dimensional subspace $F \subset E$ there is a finite dimensional subspace $B$ with $F \subset B \subset E$ such that

$$
d(B, L_p^\infty) \leq \lambda,
$$

where $n = \dim B$, the dimension of $B$.

A space $E$ is an $L_p^\infty$-space [17] if $E$ is an $L_p^\infty$-space for some $\lambda \geq 1$. These spaces include and generalize the classical $L_p(\mathbb{E}, \mu)$ spaces and $C(K)$-spaces. We frequently use the following results of [17] and [18].

(0.4) The conjugate of an $L_p$-space is an $L_{p'}$-space.

(0.5) If $E$ is an $L_p$-space and $F$ an $L_q$-space then $\langle E, F \rangle$ is $p$-trivial for all $p \geq 1$.

If $E$ is an $L_\infty$-space and $F$ an $L_q$-space then $\langle E, F \rangle$ is $p$-trivial for all $p \geq 1$.

(0.6) For $1 \leq p < \infty$ an $L_p$-space contains a complemented subspace isomorphic to $L_p$.

Local reflexivity and the bounded approximation property. One of the most remarkable results of the past few years is the principle of local reflexivity of Rosenthal and Lindenstrauss [18] whose proof rests on a selection theorem of Klee. We use the version of [14]:

(0.7) (The Principle of Local Reflexivity) Let $X$ be a Banach space (regarded as a subspace of $X^{**}$) and let $U$ and $F$ be finite dimensional subspaces of $X^{**}$ and $X^*$, respectively, and let $\varepsilon > 0$. Then there is a one-to-one operator $T: U \to X$ with $T \varepsilon = \varepsilon$ for $x \in X \cap U$, $f(T) = f(x)$ for $x \in U$ and $f \in F$ and $\| T \| \| T^{-1} \| < 1 + \varepsilon$.

An immediate consequence of (0.7) is

(0.8) Let $X$ and $Y$ be Banach spaces with $\dim X < \infty$. Let $F$ be a finite dimensional subspace of $X^*$, let $B$ be an operator from $X^*$ to $Y$ and $\varepsilon > 0$. Then there is a weak$^*$-continuous operator $S$ from $X^*$ to $X$ such that

(i) $B$ and $S$ agree on $F$; and,

(ii) $\| S \| \leq (1 + \varepsilon)\| B \|$.

A Banach space $X$ has the bounded approximation property (b.a.p.) if there is a constant $C$ such that if $B$ is a finite dimensional subspace of $X$ there is a $T \in \mathcal{L}(X, X)$ with finite dimensional range such that $\| T \| \leq C$ and $T$ restricted to $B$ is the identity.

From a series of papers [15], [14], and [20] the relationship between various approximation properties and other structures (e.g. having a basis) are becoming clear.

While there are no known Banach spaces lacking the b.a.p. it is now known, combining the results above, that a separable $B$ has b.a.p. if and only if $B$ is a quotient of a space with a basis.

The Dvoretzky Theorem; $L_p$- and $P_p$-spaces. Perhaps the most profound result in the isomorphic theory of Banach spaces is the following result of Dvoretzky [6] concerning spherical sections of convex bodies in Banach spaces.

(0.9) For each $\varepsilon > 0$ and each positive integer $n$, there exists a positive integer $n(\varepsilon)$ such that if $E$ is any Banach space and $\dim E > n(\varepsilon)$, then there exists a subspace $F$ of $E$ such that

$$
d(F, L_p^\infty) \leq 1 + \varepsilon.
$$

In particular, in any infinite dimensional Banach space, there are finite dimensional subspaces of arbitrary large dimension, nearly isometric to Euclidean spaces.

This result motivates our next two definitions.

We say that a Banach space $E$ is an $P_p$-space if there is a constant $\lambda > 0$ and sequences of operators $\{p_n\}, \{q_n\}$ such that

$$
p_{p, q_n} \to E \mathcal{L}_{p, q_n} E,
$$

where $p_{p, q_n}$ is the identity on $L_p^\infty$ and $\| p_{p, q_n} \| \leq \lambda$.

In [24] the $P_p$ spaces were called sufficiently Euclidean.
Given $p \geq 1$ and $\lambda \geq 1$, a Banach space $E$ is called a $\mathcal{B}_{p,1}$-space if for each positive integer $n$ there is a subspace $U$ in $E$ with $d(U, E) \leq \lambda$. A space $E$ is a $(\mathcal{S}_p)$-space if it is a $(\mathcal{S}_p,1)$-$\mathcal{B}_{p,1}$-space for some $\lambda \geq 1$.

Some remarks concerning these classes of spaces are in order.

The $\mathcal{S}_p$-spaces are a true isomorphic class and a beautiful theory of these spaces is now emerging.

However, the spaces in class $\mathcal{S}_p$ or $\mathcal{B}_p$ have little structure placed on them. Indeed the Dvoretzky $\varepsilon$-isometry theorem says that every infinite dimensional Banach space in class $\mathcal{S}_p$ is nearly isometric to a sequence space. It is quite probable that every Banach space $E$ is an $\mathcal{S}_p$-space for some $p$. As a technical device, however, these spaces appear useful.

(0.10) (i) For $1 < p < \infty$ $\mathcal{B}_p \subset \mathcal{S}_p \subset \mathcal{B}_p$ and all containments are proper;

(ii) $\mathcal{S}_\infty \subset \mathcal{S}_\infty = \mathcal{B}_\infty$ and the first containment is proper;

(iii) $\mathcal{B}_p \subset \mathcal{S}_p$ for all $p \geq 1$;

(iv) $\mathcal{S}_p \subset \mathcal{B}_p$ for $1 < p < \infty$.

Now (iii) is true since for each $n$ and $s > 0$ there is an $m(n)$ and $E \in \mathcal{S}_m$ such that $d(E, E) < 1 + s$ for (ii) $\mathcal{S}_p \subset \mathcal{B}_p$ is an $\mathcal{S}_p$-space but not an $\mathcal{S}_\infty$-space. For the equality if $X$ is a $\mathcal{S}_\infty$-space then for each $n$ there is an $\mathcal{E}_n \subset X$ and an isomorphism $T$ from $\mathcal{E}_n$ onto $\mathcal{T}_n$ with $||T|| = ||T^{-1}|| < \lambda$. By the Hahn–Banach theorem $T$ has an extension $T'$ from $X$ onto $\mathcal{E}_n$ with $||T'|| = ||T||$. Let $Q = T^{-1}T'$. Then $Q : X \to \mathcal{E}_n$ is a projection and $||Q|| \leq \lambda$. i.e. $X \subset \mathcal{S}_\infty$.

For $1 < p < \infty$, $\mathcal{S}_p$-space (by (iii)) but not an $\mathcal{S}_p$-space and $\mathcal{S}_p \subset \mathcal{B}_p$ is not isomorphic but $\mathcal{S}_p$. Statement (iv) is immediate from the discussion on the preceding page. Also, statement (i) follows from (0.6).

It follows from the principle of local reflexivity that $E$ is in $\mathcal{S}_p$ if and only if $E' \subset \mathcal{S}_p$. Most of the statements concerning $\mathcal{S}_p$-spaces in [24] are valid for $\mathcal{S}_p$-spaces. However, arbitrary Banach spaces need not have $\mathcal{S}_p$-subspaces for $p \neq 2$, contrary to [24]:

(0.11) Every infinite dimensional Banach space $E$ contains an infinite dimensional subspace $\mathcal{E}_p \subset \mathcal{S}_p$.

Acknowledgement. The authors would like to thank Drs. C. Stegall and W. Johnson for many helpful suggestions. Also our thanks is expressed to the referee.

§ 1. p-TRIVIAL SPACES

We first show that $p$-triviality is implied by seemingly weaker assumptions. We denote by $K(X, Y)$ the space of compact operators from $X$ into $Y$.

1.1. REMARK. Let $1 < p < \infty$ and let $X$ and $Y$ be Banach spaces. If either $X$ or $Y$ has the b.a.p., then $K(X, Y)$ is contained in $\mathcal{P}_p(X, Y)$ if and only if $\langle X, Y \rangle$ is $p$-trivial.

Proof. Assume that $X$ has the b.a.p. By hypothesis and the closed graph theorem, there is a $C > 0$ such that

$$\mathcal{P}_p(T) \leq C||T|| \quad \text{for all } T : K(X, Y).$$

Let $S \in \mathcal{S}(X, Y)$ and $x_1, \ldots, x_n$ be in $X$. Since $X$ has the b.a.p. there is a $T : X \to Y, T$ compact, $||T|| < M$ and $Tx_j = x_j$ for $j = 1, \ldots, n$. Here the constant $M$ depends only on $X$. Thus, $a_p(||S||) - a_p(||ST||) \leq C||ST||a_p(||x_j||) \leq CM||S||a_p(x_j)$ i.e. $S \in \mathcal{P}_p(X, Y)$.

A similar proof works if $Y$ has the b.a.p.

Since the notion of a strongly $p$-summing operator involves only finite sums we have replacing $a_p$ by $a_p$ and $x_j$ by $a_j$ in 1.1, the following corollary.

1.2. COROLLARY. Let $1 < p < \infty$ and let $X$ or $Y$ have the b.a.p. Then $K(X, Y) \subset \mathcal{P}_p(X, Y)$ if and only if $\langle X, Y \rangle$ is strongly $p$-trivial.

Since the space of all adjoint operators in $\mathcal{L}(Y^*, X^*)$ is closed the same argument used above together with (0.8) proves the following result.

1.3. THEOREM. If every adjoint operator $T' : Y^* \to X^*$ is $p$-absolutely summing (strongly summable) and $Y^*$ has the b.a.p. then $\langle X^*, Y^* \rangle$ is $p$-trivial (strongly $p$-trivial).

For the next corollary we adopt the following notation. For a Banach space $X$, $X_n = X$ and $X_n = X_n^*$. We now introduce the concept of a Hilbertian space $H$.

1.4. COROLLARY. Suppose $X_n$ and $Y_n$ have the b.a.p. for each integer $n \geq 0$. The following are equivalent:

(a) $\langle X_n, Y_n \rangle$ is $p$-trivial (strongly $p$-trivial) for all $n \geq 0$; and
(b) $\langle X_n, Y_n \rangle$ is strongly $p$-trivial (strongly $p$-trivial) for all $n \geq 0$.

The proof is immediate from (0.3), 1.2 and 1.3.

As an application of the above results we give an apparently new characterization of the $\mathcal{L}_p$-spaces ($\equiv$ isomorphisms of Hilbert space).

Let $T : \mathcal{L}(X, Y)$. We say that $T$ can be factored through a Banach space $Z$ if there are operators $T_1 : \mathcal{L}(X, Z)$ and $T_2 : \mathcal{L}(Z, Y)$ such that $T = T_2T_1$. An operator $T$ is Hilbertian [17] if it can be factored through a Hilbert space $H$.

We recall the following result (Theorem 5.2, p. 293 of [17]).

1.5. THEOREM. Let $X$ be an $\mathcal{S}_p$-space with $2 < p < \infty$ and let $Y$ be an $\mathcal{S}_p$-space with $1 < p < 2$. Then every $T : \mathcal{L}(X, Y)$ is Hilbertian.

In particular Hilbertian operators can have a $\mathcal{S}_p$-domain and a range totally incomparable [25] to a Hilbert space. For $\mathcal{S}_\infty$-ranges the situation
is different. We first prove a result which is probably known. However, to our knowledge, the result does not appear in print.

1.6. Theorem. Suppose every \( T \in \mathcal{B}(E, F) \) is Hilbertian and \( F \) is an \( \mathcal{L}_\infty \)-space. Then \( E \) is an \( \mathcal{L}_\infty \)-space. And, of course, conversely.

Proof. If \( T \in \mathcal{L}(E, F) \) then by hypothesis there is a Hilbert space \( H \) such that

\[
\begin{array}{c}
E \\
\downarrow T
\end{array}
\begin{array}{c}
F
\end{array}
\]

\[
\begin{array}{c}
H
\end{array}
\]

commutes. Thus we have

\[
\begin{array}{c}
F^* \\
\downarrow T^*
\end{array}
\begin{array}{c}
E^*
\end{array}
\]

\[
\begin{array}{c}
H
\end{array}
\]

Since \( F^* \) is an \( \mathcal{L}_\infty \)-space it follows from (0.5) that \( T^* \) hence \( T \) is absolutely summing. Thus by 1.3 \( \langle F^*, E^* \rangle \) is \( 1 \)-trivial. The proof now proceeds as in Theorem 4.2 of [17]. By Proposition 7.3 of [17] there is a projection \( P \) from \( F^* \) onto \( l_2 \). Let \( Z \) be a separable subspace of \( E^* \) and \( Q \) an operator from \( l_2 \) onto \( Z \). Since \( \langle F^*, E^* \rangle \) is \( 1 \)-trivial, \( PQ \) is absolutely summing and so [21] is Hilbertian. This says that \( Z \) is a quotient of a Hilbert space. Thus by Lemma 3 of [16] \( E^* \) and hence \( E \) is an \( \mathcal{L}_\infty \)-space. We remark that 1.3 applies since an \( \mathcal{L}_\infty \)-space has the b.a.p.

We now show that a considerably stronger result can be proved.

1.7. Theorem. Suppose every \( T \in \mathcal{B}(E, Y) \) is Hilbertian and \( Y \) is a \( \mathcal{S}_\infty \)-space. Then \( E \) is an \( \mathcal{S}_\infty \)-space.

Proof. Let \( H(E, Y) \) denote the class of Hilbertian operators from \( E \) to \( Y \). Under the norm \( \epsilon(T) = \inf \| A \| \| B \| \) where the infimum is taken over all factorizations \( (A, B) \) of \( T \) through a Hilbert space, \( H(E, Y) \) is a Banach space. Thus by our hypothesis and the closed graph theorem there is a \( C > 0 \) such that

\[
\epsilon(T) \leq C \| T \|
\]

for all \( T \in \mathcal{B}(E, Y) \).

Let \( E_n \) be a subspace of \( E \), \( \dim E_n < \infty \). Then there is an \( n \) and a subspace \( F \) of \( l_2 \) such that \( \epsilon(E_n, F) \leq 2 \). Since \( Y \) is a \( \mathcal{S}_\infty \)-space there is a subspace \( G \) of \( Y \) such that \( \epsilon(G, l_2) \leq 1 \) (for each \( n \)). Thus we may choose \( G \subset G \subset Y \) and \( T : E_n \to F \) such that \( T \) is an isomorphism and \( \| T \|, \| T^{-1} \| \leq 2 \).

Since \( d(G, l_2) \leq \lambda \) and \( l_2 \) has the extension property, there is an operator \( T' : E \to G \) such that \( \| T' \| \leq \lambda \| T \| = \lambda \) and \( T' | E_n = T \).

By hypothesis there is a factorization

\[
\begin{array}{c}
E \\
\downarrow \lambda
\end{array}
\begin{array}{c}
F
\end{array}
\]

\[
\begin{array}{c}
\lambda^{-1}
\end{array}
\begin{array}{c}
l_2
\end{array}
\]

such that \( \| A \| = 1 \) and \( \| B \| \leq C \| T' \| \leq C \lambda \).

Let \( T = A(E_n) \). Since \( R A = T ', B | E_n \) is onto \( F \). Let \( D : Z \to E_n \) be defined by \( D = T^{-1} | Z \). Then \( D \) is an isomorphism of \( Z \) onto \( E_n \) and \( D^{-1} = A(E_n) \). Thus \( d(Z, E_n) \leq \| D \| \| D^{-1} \| \leq 2 C \lambda \). Let \( K = \dim E_n \). Since \( Z \subset l_2(n) \) and \( \dim Z = K \), \( K \), we have \( d(E_n, l_2) \leq 2 C \lambda \). Since \( E_n \) was an arbitrary finite dimensional subspace of \( E \), \( E \) is an \( \mathcal{S}_\infty \)-space.

Of course 1.6 is subsumed by 1.7. However, the contrast in the vastly different proofs seems to us, to give a clearer picture of the situation. A further comment is made concerning the constants appearing in 1.6 and 1.7 at the end of the paper.

§ 2. THE GENERALIZATION OF THE LINDENSTRAUSS-PELZTNSKI THEOREM

We first recall Proposition 8.1 of [17] which motivated this paper.

2.1. Theorem. Let \( E \) and \( F \) be infinite dimensional Banach spaces such that \( \langle E, F \rangle \) is \( 1 \)-trivial. Then,

(a) \( \langle E, l_2 \rangle \) is \( 1 \)-trivial;

(b) For any unconditionally convergent series \( \sum n \in E, \sum \| n \|^2 < + \infty \); and

(c) For any \( \mathcal{S}_\infty \)-space \( G, \langle G, E \rangle \) is \( 2 \)-trivial.

Our aim is to obtain the \( p \)-trivial and strongly \( p \)-trivial versions of 2.1. The proof of 2.1 (a) depends on the fact that every infinite dimensional Banach space is a \( \mathcal{S}_\infty \)-space and that the notion of an absolutely summing operator depends only on the domain. Of course this latter statement is valid for \( p \)-absolutely summing operators. Thus the following is true.

2.2. Theorem. If \( \langle E, F \rangle \) is \( p \)-trivial then \( \langle E, l_2 \rangle \) is \( p \)-trivial.

Our next result is the analogue for strongly \( p \)-summing operators. More work is needed since the notion of a strongly \( p \)-summing operator depends on both the domain and range.

In the proof we denote the canonical operators from \( E \to E^* \) and \( E^* \to E^{***} \) by \( J \) and \( J_p \), respectively.
2.3. Theorem. Let \( 1 < p \leq +\infty \). If \( (E, F) \) is strongly \( p \)-trivial and \( F^* \) has the b.a.p. then \( \langle l_q, F \rangle \) is strongly \( p \)-trivial.

Proof. If \( (E, F) \) is strongly \( p \)-trivial then by (0.3) and 1.3 \( (F^*, E^*) \) is \( p \)-trivial and so by 2.3 \( \langle F^*, l_q \rangle \) is \( p \)-trivial. Again by (0.3) and 1.3 \( \langle l_q, F^{**} \rangle \) is strongly \( p \)-trivial. Let \( (f_j) \) be \( p \)-\( (l_q) \) and \( T \) is of \( F^* \).

We want to show that \( \sum \langle T_n f_j, f_j \rangle \) converges.

Since \( \langle T_n f_j, f_j \rangle = \langle J T_n f_j, f_j \rangle \) and \( JT \) is strongly \( p \)-summing, it suffices to show that \( \langle J f_j, f_j \rangle \). But if \( p \) is \( F^{**} \), \( \langle J f_j, f_j \rangle = \langle J^p (f_j) \rangle \) and \( J^p (f_j) = F^p \) and so \( \sum \langle J f_j, f_j \rangle = +\infty \). Thus \( \langle l_q, T \rangle \) is strongly \( p \)-summing.

Since \( \langle l_q, T \rangle \) is \( 2 \)-trivial (indeed \( 1 \)-trivial) \( \langle l_q, l_q \rangle \) is, by 1.4, strongly \( 2 \)-trivial. Since the identity \( \iota : l_q \to l_q \) is not strongly \( 2 \)-summing (since its adjoint is not \( 2 \)-absolutely summing) \( \langle l_q, l_q \rangle \) is not strongly \( 2 \)-trivial. However, \( \iota \) is of \( F_q \). Indeed the following is true.

2.4. Theorem. If \( (E, F) \) is strongly \( p \)-trivial and \( F \) of \( F_q \) then \( \langle l_q, l_q \rangle \) is strongly \( p \)-trivial.

Proof. Suppose \( F_q \) is of \( F_q \)-space. Then there are operators \( \phi \) of \( F \) such that \( \| \phi \|_q \leq 1 \) and \( P_{F_q} \) is the identity. If \( \phi (x) \in \mathbb{C} \) and \( \mathbb{C} \) is of \( l_q \). Let \( B_0 = \mathbb{C} \) and \( F \) be of the canonical projection. Since \( F_q \) is of \( F_q \)-space there is a \( \phi \) of \( F \) such that \( \| \phi \|_q \leq 1 \). If \( \mathbb{C}_q = \phi (B_0) \) then \( \mathbb{C}_q \) is such that \( \| \phi \|_q \leq 1 \) and \( \mathbb{C}_q \) is \( \mathbb{C} \). By hypothesis \( J_m \mathbb{C}_q \) is \( \langle l_q, l_q \rangle \) and so by (0.3) there is a constant \( M \) (independent of the \( x_i \) such that

\[
\psi (x_i) \leq M \mathbb{C}_q \|
\]

Thus

\[
\psi (T_n) \leq \|
\]

and so

\[
\| \phi (T_n) \| \leq \|
\]

i.e. \( T \) is strongly \( p \)-summing.

As a corollary to 2.4 we obtain a generalization of one of the main results of [34].

2.5. Corollary. Let \( E \) be of \( F_q \) and \( F \) be infinite dimensional. Then \( \langle E, F \rangle \) is not \( p \)-trivial for any \( p \geq 1 \).

Proof. If \( (E, F) \) is \( p \)-trivial then \( \langle E, l_q \rangle \) is \( p \)-trivial and so \( \langle l_q, F^* \rangle \) is strongly \( p \)-trivial. But \( E \) is of \( F_q \) and so by 2.4 \( \langle l_q, l_q \rangle \) is strongly \( p \)-trivial, which is a contradiction.

2.6. Corollary. Let \( E \) and \( F \) be infinite dimensional. Then there is an infinite dimensional subspace \( E_3 \) in \( E \) and \( F \) and a \( T \) such that \( T \) is of \( F_q \) and \( \langle E_3, F \rangle \) is of \( E_q \). (In particular, \( \langle E_3, F \rangle \) is infinite dimensional.

Proof. By 2.5 we need only choose \( E_3 \) and \( F_q \) to be \( F_q \)-spaces. This is possible by (0.11).

Before giving our next result we recall the following fact: If \( Y \) is a finite dimensional subspace of \( l_q \) and \( \epsilon > 0 \) then there is a subspace of \( l_q \) of \( \epsilon \)-dimension \( \sum \| x_i \| < +\infty \). Then \( \sum \| x_i \| < +\infty \). Thus \( \langle l_q, T \rangle \) is strongly \( p \)-summing.

Since \( \langle l_q, T \rangle \) is \( 2 \)-trivial (indeed \( 1 \)-trivial) \( \langle l_q, l_q \rangle \) is, by 1.4, strongly \( 2 \)-trivial. Since the identity \( \iota : l_q \to l_q \) is not strongly \( 2 \)-summing (since its adjoint is not \( 2 \)-absolutely summing) \( \langle l_q, l_q \rangle \) is not strongly \( 2 \)-trivial. However, \( \iota \) is of \( F_q \). Indeed the following is true.

2.7. Theorem. Let \( E \) be a Banach space and \( F \) a \( F_q \)-space for some \( q > 1 \). If \( \langle E, F \rangle \) is \( p \)-trivial \( (1 < \frac{p}{q} \leq \infty ) \) then \( \langle E, F \rangle \) is \( p \)-trivial.

Proof. Since \( E \) is \( p \)-trivial there is a \( C > 0 \) such that \( \| P \| \leq C \| T \| \) for all \( T \) and \( \| T \| \leq \frac{1}{\| T \|} \) when \( \| T \| \leq 1 \). Let \( (x_i) \) be \( F_q \)-space and \( (x_i) \) be \( E \). Let \( W \) be of \( l_q \) such that \( \sum \| x_i \| < +\infty \). Then \( \sum \| x_i \| < +\infty \). Thus \( \sum \| x_i \| < +\infty \). Since \( \langle l_q, l_q \rangle \) is strongly \( p \)-trivial. Of course, 2.7 is a generalization of 2.1 (a).

2.8. Corollary. Let \( E \) be a \( F_q \)-space and \( F \) a Banach space with \( F^* \) having the b.a.p. If \( (E, F) \) is strongly \( p \)-trivial \( (1 < p \leq \infty ) \) then \( \langle E, l_q \rangle \) is strongly \( p \)-trivial.

Proof. If \( (E, F) \) is strongly \( p \)-trivial then by 1.3 \( \langle F^*, E^* \rangle \) is \( p \)-trivial. Clearly \( E^* \) is a \( F_q \)-space. Thus by 2.7 \( \langle F^*, l_q \rangle \) is \( p \)-trivial and so \( \langle E, l_q \rangle \) is strongly \( p \)-trivial.

The results of this section sheds some new light on a conjecture of Grothendieck. Recall that an operator \( T \) is of \( F_q \)-space is nuclear if it can be represented in the form \( T = \sum f_i (x_i) y_i \) with \( (f_i) \in \mathbb{C} \) and \( \sum \| f_i \| < +\infty \) and sup \( \| y_i \| < +\infty \). Grothendieck [10] page 47, has conjectured that if every \( T \) is nuclear then min \( (\dim E, \dim F) \) is finite. It is known ([11] page 319), that to answer this problem of Grothendieck it is enough to show that under the above hypothesis \( F \) is an \( F_q \)-space. With the following observation it follows from 2.4 that to answer this problem it is enough to show that \( F \) is an \( F_q \)-space, a much weaker condition than the above.
2.9. REMARK. Let $T \in \mathcal{L}(E, F)$ be nuclear. Then $T \in \mathcal{L}(E, F)$ and $T \in \mathcal{L}(E, F)$ for every $p$.

Proof. The $p$-absolutely summable assertion is well-known. If $(x_i) \in l_p(E)$, $(y_i) \in l_p(F^*)$ and $T = \sum_{i=1}^{\infty} f_i(x_i)g_i(y_i)$ is a representation of $T$ satisfying the above conditions, then

$$\sum_{i=1}^{\infty} |x_i|_p < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} |f_i(x_i)g_i(y_i)| \leq \left( \sum_{i=1}^{\infty} |f_i(x_i)|^{p'} \right)^{\frac{1}{p'}} \left( \sum_{i=1}^{\infty} |g_i(y_i)|^{p'} \right)^{\frac{1}{p'}} \leq \infty.$$

Again the notion of a nuclear operator depends on both the domain and range. To overcome some of the difficulties in working with such operators, the notion of a fully nuclear operator was introduced. An operator $T \in \mathcal{L}(E, F)$ is fully nuclear [23] if the restricted operator $T_0 \colon E \to T(E)$ is nuclear. Motivated by this we say that $T \in \mathcal{L}(E, F)$ is fully $p$-summing if the restriction $T_0 \colon E \to T(E)$ is strongly $p$-summing. The proof of 2.9 shows that a fully nuclear operator is fully $p$-summing for any $p$. Denote the fully $p$-summing operators from $E$ to $F$ by $\mathcal{B}_p(E, F)$. Clearly if $\mathcal{B}(E, F) = B(E, F)$ then $\mathcal{B}(E, F) = B(E, F)$ for any $F \in F$. Thus by 2.6 (choosing $F_0$ to be an $\mathcal{L}(E, F)$) it follows that $\mathcal{L}(E, F) = B(E, F)$ if and only if $\text{dim}(\text{im}E, \text{dim}F) < \infty$. This together with the above remarks gives a new proof of [23].

§ 3. WEAKLY $p$-SUMMING SERIES AND $p$-TRIALITY

We now consider the implication 2.1(b). Using the same argument as [17] for 2.1(b) one can prove the following result.

3.0. THEOREM. If $\langle X, Y \rangle$ is $p$-trivial and $1 \leq p < 2$ then

$$l_p(X) \subseteq l_{\frac{p^*}{2-p}}(X).$$

Theorem 3.7 below shows that there is no comparable result for $p > 2$ and arbitrary $Y$. Of course the proof of 3.0 relies on the fact that every infinite dimensional Banach space is a $\mathcal{L}(E, F)$-space. We show in 3.11 below that there is a true generalization of 2.1(b). An immediate corollary to 3.0 is the following result.

COROLLARY. If $\langle X, Y \rangle$ is strongly $p$-trivial, $p > 2$ then

$$l_{p'}(X^*) \subseteq l_{\frac{2-p}{2}}(X^*).$$

Theorem 3.8 above shows that the result of Grothendieck [12] is the best possible. We now outline a known technique for constructing basic sequences. This is then used to generalize a result of Grothendieck [12].

Let $P$ and $Q$ be linear subspaces of a Banach space $X$. The indiction of $P$ and $Q$ is defined to be

$$I(P; Q) = \inf \{ |x + y||x \in P, y \in Q| = 1, y \in Q \}.$$

Let $(x_n)$ be a sequence of elements of a Banach space $X$ and let $L(x_n)_{n=1}^{\infty}$ be the finite dimensional subspace spanned by $(x_n, x_{n+1}, \ldots, x_m)$. We define the index of the ordered $n$-tuple $(x_n)_{n=1}^{m}$ to be

$$\theta(x_n)_{n=1}^{m} = \min \{ I(L(x_n)_{n=1}^{m}; L(x_n)_{n=m+1}^{\infty})| 1 \leq p < \infty \}. $$

The index of the sequence $(x_n)$ is defined by

$$\theta(x_n) = \inf \{ \theta(x_n)_{n=1}^{m}| n \geq 1 \}.$$
and

\[ \theta(y_\infty)_{n+1} \geq \frac{1}{1+\varepsilon}. \]

Proof. Let \( n \) be a positive integer and \( \varepsilon > 0 \). Then by (9) there is a subspace \( U \) of \( X \) such that \( d(U, V) < 1+\varepsilon \). Thus there is an isomorphism \( T \) from \( U \) onto \( U \) such that \( \|T\| = 1 \) and \( \|T^{-1}\| \leq 1+\varepsilon \). If \( (e_i)_n \) is the unit vector basis for \( U \), let \( e_i = T(e_i) \) for each \( i \).

Then

\[ \|T^{-1}\| \cdot \|e_i\| = \|T\| \cdot \|T(e_i)\| \geq \|T^{-1}\| \cdot \|e_i\| = \|e_i\|. \]

This implies that \( \|e_i\| \geq \frac{1}{1+\varepsilon} \). Let \( y_i = \frac{a_i}{\|e_i\|} \) for each \( i \), and consider \( (\lambda_i)_n \). Then

\[
\left\| \sum_{i=1}^{n} \lambda_i y_i \right\| = \left\| \sum_{i=1}^{n} \frac{\lambda_i}{\|e_i\|} T(e_i) \right\|
\leq \left\| T \left( \sum_{i=1}^{n} \frac{\lambda_i}{\|e_i\|} e_i \right) \right\| \leq \|T\| \cdot \left\| \sum_{i=1}^{n} \frac{\lambda_i}{\|e_i\|} e_i \right\|_q
\leq \left( \sum_{i=1}^{n} \left| \frac{\lambda_i}{\|e_i\|} \right|^q \right)^{1/q} \left( \sum_{i=1}^{n} \left| \frac{\lambda_i}{\|e_i\|} \right|^q \right)^{1/q} \leq \frac{1}{1+\varepsilon} \left( \sum_{i=1}^{n} \lambda_i^q \right)^{1/q} \leq (1+\varepsilon) \left( \sum_{i=1}^{n} \lambda_i^q \right)^{1/q}.
\]

Thus \( \left( \sum_{i=1}^{n} \lambda_i^q \right)^{1/q} \leq (1+\varepsilon) \left( \sum_{i=1}^{n} \lambda_i^q \right)^{1/q} \), and \( (y_\infty)_{n+1} \) is the desired sequence if we show that \( \theta(y_\infty)_{n+1} \geq \frac{1}{1+\varepsilon} \). We do this by showing that \( \theta(\lambda_i)_{n+1} \geq \frac{1}{1+\varepsilon} \).

Let \( 1 < p < q \), \( \{x_i\}_{i=1}^{n} \subseteq L(\lambda_i)_{n+1} \), and \( \varepsilon \leq L(\lambda_i)_{n+1} \). Then \( w = \sum_{i=1}^{n} a_i T(e_i) \), \( x = \sum_{i=1}^{n} a_i e_i \), and \( \|x + w\| = \|T\left( \sum_{i=1}^{n} a_i e_i \right)\| \); hence

\[
(1+\varepsilon)\|x + w\| \geq \|T^{-1}\| \cdot \|T\left( \sum_{i=1}^{n} a_i e_i \right)\| \geq \left\| \sum_{i=1}^{n} a_i e_i \right\|_q
\geq \left\| \sum_{i=1}^{n} a_i e_i \right\|_q \geq \left\| \sum_{i=1}^{n} a_i e_i \right\|_q \geq \left\| T\left( \sum_{i=1}^{n} a_i e_i \right) \right\| = \|e_i\|.
\]

Thus

\[
\|e_i\| \geq \frac{1}{1+\varepsilon} \|x + w\|.
\]

Consequently, \( \theta(x_\infty)_{n+1} \geq 1+\varepsilon \), and hence \( \theta(y_\infty)_{n+1} \geq 1+\varepsilon \). This completes the proof.

3.5. Lemma. Let \( (a_i) \) be a sequence in a Banach space \( X \) such that

\[
(*) \quad \left\| \sum_{i=1}^{\infty} \lambda_i a_i \right\| \leq \left( \sum_{i=1}^{\infty} \lambda_i^p \right)^{1/p} (1+\varepsilon)
\]

for every square summable sequence \( (\lambda_i) \). Then

\[
\frac{1}{1+\varepsilon} \geq \sup \left\{ \left( \sum_{i=1}^{\infty} \|a_i\|^{1/p} \right)^{1/p} : a_i \in X, \|a_i\| \leq 1 \right\} \quad \text{for all } p \geq 2.
\]

Proof. Define a linear operator \( S \) from \( l_2 \) into \( X \) by \( S((\lambda_i)) = \sum_{i=1}^{\infty} \lambda_i a_i \) for each \( (\lambda_i)_2 \). Then by (*)

\[
\|S((\lambda_i))\| = \left\| \sum_{i=1}^{\infty} \lambda_i a_i \right\| \leq (1+\varepsilon) \left( \sum_{i=1}^{\infty} \lambda_i^2 \right)^{1/2} \|((\lambda_i))\|_2.
\]

Hence \( S \) is continuous and \( \|S\| \leq 1+\varepsilon \). Now consider \( S \) from \( X^* \) into \( l_2 \). Then

\[
(S^*(a^*))((\lambda_i)) = a^* \left( \sum_{i=1}^{\infty} \lambda_i a_i \right) = a^* \left( \sum_{i=1}^{\infty} \lambda_i \right) \left( \sum_{i=1}^{\infty} a_i \right) = \sum_{i=1}^{\infty} a_i^* (a_i).
\]

Since \( (\lambda_i) \) is arbitrary in \( l_2 \), \( S^*(a^*) = (a^*(a_i))_{n+1} \).

Thus

\[
(1+\varepsilon) \|a^*\| \geq \|S^*(a^*)\| = \sup \{ \left( \sum_{i=1}^{\infty} \|a_i\|^{1/p} \right)^{1/p} : \|a_i\| \leq 1 \} \geq \|S^*(a^*)\|_2 \|a^*\| \leq 1.
\]

Hence \( (1+\varepsilon) \|a^*\| = \sup \{ \left( \sum_{i=1}^{\infty} \|a_i\|^{1/p} \right)^{1/p} : \|a_i\| \leq 1 \} \) for all \( p \geq 2 \) and the proof is complete.

3.6. Corollary. If \( X \) is an infinite dimensional Banach space and \( \varepsilon > 0 \) there is a sequence \( (y_\infty)_{n+1} \) in \( X \) such that \( \|y_\infty\| = 1, \theta(y_\infty)_{n+1} \geq \frac{1}{1+\varepsilon} \), and

\[
(1+\varepsilon) \geq \sup \left\{ \left( \sum_{i=1}^{\infty} \|a_i\|^{1/p} \right)^{1/p} : \|a_i\| \leq 1 \right\}.
\]

3.7. Theorem. If \( X \) is an infinite dimensional Banach space and \( (a_i) \) is an element of \( a_i \) with \( 0 < a_i < 1 \), then there is a basic sequence \( (y_\infty)_{n+1} \) in \( X \) such that \( \|y_\infty\| = a_i \) for all \( i \) and

\[
1 \geq \sup \left\{ \left( \sum_{i=1}^{\infty} \|a_i\|^{1/p} \right)^{1/p} : \|a_i\| \leq 1 \right\}.
\]
Proof. Let $a = \frac{1}{2} \{1 - \sup_i (a_i)\}$. Then
\[
1 - 2a = \sup_i (a_i).
\]
and $1 > a > 0$ since $a_i \to 0$ and $1 > a_i$. For each positive integer $k$ select $i_k$, increasing with $k$, such that
\[
\text{if } i > i_k \text{ then } a_i \leq \frac{a}{2^k + 1}.
\]
Let $\varepsilon = \frac{a}{1 - a}$. Then $\varepsilon > 0$ and $1 + \varepsilon = \frac{1}{(1 - a)}$. Since $X$ is infinite dimensional, we can select $(y_i)_{i=1}^\infty$ such that $\|y_i\| = 1$, and
\[
\theta(y_i)_{i=1}^\infty \geq 1 + \varepsilon = 1 - a,
\]
and
\[
1 - a > 1 + \varepsilon \geq \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\} \quad \text{(this is possible by Corollary 3.6)}.
\]
Let $a_i = a_i y_i$ for $1 \leq i \leq i_k$. Then $\theta(y_i)_{i=1}^{i_k} \geq 1 - a$ and $\|a_i\| = a_i$. For convenience, let $A_1 = (x_i)_{i=1}^{i_k}$, where $x_i = a_i$ for $1 \leq i \leq i_k$ and $x_i = 0$ otherwise. Let $e_i(A_1) = \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}$. Then
\[
e_i(A_1) = \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}
\]
\[
= \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}
\]
\[
= \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}
\]
\[
\leq \sup_i \|a_i\| \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}
\]
\[
\leq (1 - 2a) \left( \frac{1}{1 - a} \right)
\]
(the last inequality is due to (a) and the selection of $(y_i)_{i=1}^\infty$). Thus $e_i(A_1) \leq (1 - 2a) \left( \frac{1}{1 - a} \right)$.

Now select a sequence $(\varepsilon_i)$ with $0 < \varepsilon_i < 1$ and $\prod_{i=1}^\infty (1 - \varepsilon_i) = \beta$ for some $\beta > 0$, and let $E_i = [a_i] : i \leq i_k$. Then by Theorem 3.2 there is an infinite dimensional subspace of $X$, say $F_1$, such that $I(E_i, F_1) \geq 1 - \varepsilon_i$. Since $F_2$ is infinite dimensional there is, by Corollary 3.6 a collection $(y_i)_{i=1}^{i_{k+1}}$ in $F_2$ such that
\[
\|y_i\| = 1, \quad \theta(y_i)_{i=1}^{i_{k+1}} \geq 1 - a,
\]
and
\[
\frac{1}{1 - a} \geq \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}.
\]
Again letting $a_i = a_i y_i$, we get $|a_i| = a_i$, $\theta(y_i)_{i=1}^{i_{k+1}} \geq 1 - a_i$. If $A_2 = (x_i)_{i=1}^{i_{k+1}}$, where $a_i = a_i$ for $i_k + 1 \leq i \leq i_{k+1}$, and $a_i = 0$ otherwise, then
\[
e_i(A_2) = \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}
\]
\[
= \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}
\]
\[
= \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}
\]
\[
\leq \sup_i \|a_i\| \sup \left\{ \left( \sum_{i=1}^k |a_i|^2 \|y_i\|^2 \right)^{1/2} : \|a_i\| \leq 1 \right\}
\]
\[
\leq \frac{a}{2^k} \left( \frac{1}{1 - a} \right)
\]
(the latter inequality is due to (b) and the selection of $(y_i)_{i=1}^\infty$). Thus $e_i(A_2) \leq \frac{a}{2^k} \left( \frac{1}{1 - a} \right)$. Let $E_2 = [a_i] : i \leq i_{k+1}$ and choose an infinite dimensional subspace $F_3$ of $X$ such that $I(E_i, F_3) \geq (1 - \varepsilon_i)$. In the same manner as above we get a sequence $(x_i)_{i=1}^{i_{k+1}}$ in $F_3$ such that $|a_i| = a_i$, $\theta(x_i)_{i=1}^{i_{k+1}} \geq 1 - a_i$, and $e_i(A_3) \leq \frac{a}{2^k} \left( \frac{1}{1 - a} \right)$, where $A_3 = (a_i)_{i=1}^{i_{k+1}}$ with $a_i = a_i$ for $i_k + 1 \leq i \leq i_{k+1}$ and $a_i = 0$ otherwise.

Continuing in this manner we get for each $k$ an infinite dimensional subspace $F_k$ and a collection $(x_i)_{i=1}^{i_{k+1}}$ in $F_k$ such that $|a_i| = a_i$, $\theta(x_i)_{i=1}^{i_{k+1}} \geq 1 - a_i$, and $e_i(A_k) \leq \frac{a}{2^k} \left( \frac{1}{1 - a} \right)$, where $A_k = (a_i)_{i=1}^{i_{k+1}}$ with $a_i = a_i$ for $i_{k+1} + 1 \leq i \leq i_{k+1}$ and $a_i = 0$ otherwise and $I(E_{i-1}, F_k) \geq 1 - \varepsilon_{i-1}$, where $E_{i-1} = [a_i] : i \leq i_{k+1}$.
Now,
\[ e_k(x^k) = \sup \left\{ \left( \sum_{i=1}^{n} \|x^i(x^k)\|^p \right)^{\frac{1}{p}} : \|x^k\| \leq 1 \right\} \leq \sum_{i=1}^{n} e_k(A_k) \]
\[ = e_k(A_k) + \sum_{i=1}^{n} e_k(A_k) \leq \left( 1 - 2a \right) \left( \frac{1}{1 - a} \right) + \sum_{i=1}^{n} \frac{a}{2^k} \left( \frac{1}{1 - a} \right) \]
\[ = \left( \frac{1}{1 - a} \right) \left( 1 - 2a \right) + \sum_{i=1}^{n} \frac{a}{2^k} \left( 1 - a \right) = 1. \]
Thus 1 > 1 + \sup \{ \left( \sum_{i=1}^{n} \|x^i(x^k)\|^p \right)^{\frac{1}{p}} : \|x^k\| \leq 1 \} and \( (x_k) \) is the desired sequence.
This completes the proof.

The above proof, combining ideas of Gurarii [13], Grothendieck [12] and Dvoretzky [6] is very useful for constructing operators of specific types.

To illustrate the technique we prove the following theorem.

3.8a. Theorem. Let \( X \) be a Banach space, \( 2 \leq p < \infty \), and \( e > 0 \).
If \( Y \) is an \( L_p \)-space and \( \langle X, Y \rangle \) is \( p \)-trivial then \( X \) is \( p \) finite dimensional.

Proof. Since every infinite dimensional \( L_p \)-space contains a complemented copy of \( l_q \) for \( 1 < q < \infty \) it suffices to prove the theorem for \( Y = l_{p+} \).

Thus choose \( (a_k) \in l_{p+} \cap l_q \) with \( 0 < a_k < 1 \) (e.g. \( \left( \frac{1}{(1 + n)^{1/p}} \right)_{n=1}^{\infty} \)).
Since \( (a_k) \in l_q \), by Theorem 3.7 there is a basic sequence \( (x_k) \) such that
\[ \|x_k\| = a_k \text{ for each } k \text{ and } \sum_{k=1}^{\infty} \|x_k\|^p < \infty \text{ for every } x \in X. \]
Let \( f \) be a sequence of norm-preserving extensions to all of \( X \) of the associated sequence of coefficient functionals. Then by (0.1)
\[ \frac{1}{a_k} \leq \|f\| \leq \frac{2K}{a_k}. \]

Let \( y_k = \frac{a_k}{\|f\|} \) for each \( k \), where \( (a_k) \) is the unit vector basis for \( l_{p+} \).
Define \( T_n(x) = \sum_{k=1}^{n} f_k(x)y_k \). Then for each \( x \in X \),
\[ \|T_n(x)\|_{l_{p+}} = \left\| \sum_{k=1}^{n} f_k(x)y_k \right\|_{l_{p+}} = \left\| \sum_{k=1}^{n} \frac{f_k(x)}{\|f\|} a_k \frac{x_k}{a_k} \right\|_{l_{p+}} \]
\[ \leq \left\| \left( \sum_{k=1}^{n} \|f_k(x)y_k\|^p \right)^{1/p} \right\|_{l_{p+}} \leq \|x\| \left( \sum_{k=1}^{n} \|f_k(x)y_k\|^p \right)^{1/p} \leq \|x\| \left( \sum_{k=1}^{n} \|a_k\|^p \right)^{1/p}. \]
Hence \( \|T_n(x)\|_{l_{p+}} \leq \|x\| \sum_{k=1}^{n} \|a_k\|_{l_{p+}} \), and \( (T_n) \) is pointwise bounded.
If \( n > m \) and \( x \in X \), then
\[ \|T_n(x) - T_m(x)\|_{l_{p+}} = \left\| \sum_{k=m+1}^{n} f_k(x)y_k \right\|_{l_{p+}} \leq \left( \sum_{k=m+1}^{n} \|a_k\|^p \right)^{1/p} \]
Thus \( \langle T(a_k) \rangle \) is Cauchy for every \( a_k \in l_{p+} \). Thus by the Banach–Steinhaus theorem \( T \) is bounded. For \( X \in c_0 \),
\[ \sum_{k=1}^{n} \|T(x_k)\|_{l_{p+}} = \sum_{k=1}^{n} \left\| \sum_{i=1}^{n} f_i(x_k) a_i \frac{x_k}{a_i} \right\|_{l_{p+}} \]
\[ = \sum_{k=1}^{n} \|a_k\|_{l_{p+}} \left( \sum_{i=1}^{n} \left( \frac{2K}{a_i} \right)^p \right)^{1/p} = \sum_{k=1}^{n} \|a_k\|_{l_{p+}} \left( \sum_{i=1}^{n} \left( \frac{2K}{a_i} \right)^p \right)^{1/p}. \]
Now \( \sum_{k=1}^{n} \|T(x_k)\|_{l_{p+}} = +\infty \) since \( (a_k) \in l_{p+} \), and so \( T \) is not \( p \)-absolutely summing. This is a contradiction; hence \( X \) must be finite dimensional. This completes the proof.

A stronger theorem is true for \( 1 < p < 2 \). However, the method of proof used in 3.8a is no longer valid.

3.8b. Theorem. Let \( p > 2 \) and \( 1 < p < 2 \). If \( Y \) is an \( L_p \)-space and \( \langle X, Y \rangle \) is \( p \)-trivial then \( X \) is \( p \) finite dimensional.

Proof. Since \( Y \) contains a complemented subspace isomorphic to \( l_q \) we may suppose \( X = l_q \). Since a \( p \)-summing operator is \( r \)-summing for \( r \geq p \) [21] we may suppose \( p = 2 \). Suppose \( X \) is infinite dimensional. By (0.9) we can then find for each integer \( n \) a subspace \( X_n \) of \( X \), dim \( X_n = n \) and \( d(X_n, \mathbb{R}) \leq 1 + e \). Let \( x_1, \ldots, x_n \) correspond under this \( \epsilon \)-isometry to the unit vector basis \( e_1, \ldots, e_n \) of \( l_q \), and let \( e_{n+1}, \ldots, e_{2n} \) denote the unit vector basis of \( l_q \). Also let \( f_1, \ldots, f_n \) denote Hahn–Banach extensions of the coefficient functionals of \( x_1, \ldots, x_n \) to all of \( X \) and let \( (a_k) \) denote the unit vector basis of \( l_q \). Choose \( (b_k) \in l_q \) with \( \sum_{k=1}^{n} \|a_k\|^q = 1 \) and \( \sum_{k=1}^{n} |a_k|^{p} = +\infty \). Let \( (b_k) \in l_q \) with \( \sum_{k=1}^{n} \|b_k\| = 1 \). Consider
\[ R_{n}(x) = \sum_{k=1}^{n} f_k(x) a_k \frac{x_k}{a_k} \]
where \( R_{n}(x) = \sum_{k=1}^{n} \lambda_k b_k u_k \). \( T_n \) is the \( \epsilon \)-isometry of (0.9) and \( S_n(x) = \sum_{k=1}^{n} f_k(x) v_k \). Since, by hypothesis, \( \langle X, l_q \rangle \) is \( 2 \)-trivial and by [17]
\( \langle a_k, b_k \rangle \) is \( 2 \)-trivial there is a constant \( C > 0 \) such that
\[ \|S_n(x)\| \leq C \|S_n\| \quad \text{and} \quad \|R_n(x)\| \leq C \|R_n\|. \]
for all n. Thus by Theorem 4 of [21],
\[ \Pi_1(S_n T_n R_n) \leq \langle T \rangle \Pi_1(S_n) \Pi_1(R_n) \leq (1 + c)O^p = M. \]
Since the domain of \( S_n T_n R_n \) is \( c^2 \), it is easily checked that \( \Pi_1(S_n T_n R_n) = \sum |\beta_i| \). Thus \( \sum |\beta_i| \leq M \) for all n and it follows that \( \langle \lambda \rangle \in L_1 \), contradicting our assumption. Thus \( X \) is finite dimensional.

Actually, by 2.7, \( Y \) need only be a \( BS_{\omega^*} \) space in 3.8a. Since a \( BS_{\omega^*} \) space is a \( BS_\omega \) space for all \( p \geq 1 \) and a \( SF_\omega \) space we obtain the following corollary:

**3.6. COROLLARY.** (a) If \( X \) is a Banach space and \( Y \) is a \( BS_{\omega^*} \) space then \( \langle X, Y \rangle \) is p-trivial \( (1 < p < \infty) \) if and only if \( X \) is finite dimensional.

(b) If \( X^* \) is a \( SF_\omega \) space and \( Y^* \) has the b.a.p. then \( \langle X, Y \rangle \) is strongly p-trivial \( (1 < p < \infty) \) if and only if \( Y \) is finite dimensional.

The proof of (a) is immediate from the above discussion and (b) follows from 2.8 and 1.3.

Using the argument of 3.8a with e.g. \( a_n = (\log(n + 1))^{-1} \) one can constructively prove the following result.

**3.10. THEOREM.** (a) Let \( X \) be a Banach space and \( Y \) an infinite dimensional \( SF_{\omega^*} \) space. If \( 1 < p < \infty \) and \( \langle X, Y \rangle \) is p-trivial then \( X \) is finite dimensional.

(b) Let \( X \) be an infinite dimensional \( SF_\omega \) space, \( Y \) a Banach space with the b.a.p. and \( 1 < p < \infty \). If \( \langle X, Y \rangle \) is strongly p-trivial then \( Y \) is finite dimensional.

We now prove the generalization of 2.1(b).

**3.11. THEOREM.** Let \( X \) be a \( BS_{\omega^*} \) space, \( 1 < p < \infty \), \( q > 1 \) and \( X \) a Banach space. If \( \langle X, Y \rangle \) is p-trivial then \( \ell_p[X] \) is contained in \( \ell_q[Y] \).

Proof. Since \( \langle X, Y \rangle \) is p-trivial by Theorem 2.7 \( \langle X, \ell_p \rangle \) is p-trivial. Let \( \langle \lambda \rangle \) be a sequence of real numbers such that \( \sum |\lambda_i|^p = 1 \) and let \( \langle a_i \rangle = \ell_p[X] \). Define \( T \in SF_{\omega^*}[X, \ell_q] \) by \( T(x) = \sum |\lambda_i|^{1/p} |x_i^{*}(x_i)| e_i \), where each \( x_i^* \) is chosen in \( X^* \) in such a way that \( |x_i^*(x_i)| = 1 \) and \( a_i^* (a_i) = |x_i|^p \) and \( (e_i) \) is the unit vector basis in \( \ell_q \). Then

\[ ||T||_{\ell_q} = \left( \sum_{i=1}^\infty |\lambda_i|^{2/p} |x_i^{*}(x_i)|^2 \right)^{1/2} \leq \left( \sum_{i=1}^\infty |\lambda_i|^2 \right)^{1/2} ||x|| = ||x||. \]

Hence \( ||T|| \leq 1 \) and

\[ ||T(a_i)||_{\ell_q} = \left( \sum_{i=1}^\infty |\lambda_i|^{2/p} a_i^* (a_i) e_i \right)_{\ell_q} \geq |\lambda_i| ||x_i^*|| \Rightarrow ||T||_{\ell_q} \leq ||x||. \]

for each \( i \). Hence

\[ \left( \sum_{i=1}^\infty |\lambda_i|^2 |x_i^*(x_i)|^2 \right)^{1/2} \leq \left( \sum_{i=1}^\infty |T(x_i)|^{2/p} \right)^{1/2} \leq \epsilon_{q,T} \sup \left\{ \left( \sum_{i=1}^\infty |x_i^*(x_i)|^2 \right)^{1/2} : ||x_i|| < 1 \right\} \]

\[ \leq \epsilon_{p,q}(\epsilon_{q,T}) \leq \epsilon_{p,q}(\epsilon_{q,T}). \]

Thus \( \sum |\lambda_i|^2 |x_i^*(x_i)|^2 = \infty \) for every sequence of real numbers \( \langle \lambda \rangle \) in \( \ell_p \) with \( ||\lambda|| = 1 \). Thus \( \left( \sum |x_i^*(x_i)|^2 \right)^{1/2} = \ell_q \). Thus \( \sum |\lambda_i|^2 \leq \infty \).

Letting \( p = 1 \), \( q = 2 \). We see that 2.1(b) is an immediate consequence of 3.11.

We now give the dual result.

**3.12. COROLLARY.** Let \( 1 < p < \infty \), \( q > 1 \) and let \( X \) be a \( BS_{\omega^*} \) space and \( Y^* \) have the b.a.p. Then, if \( \langle X, Y \rangle \) is strongly p'-trivial then \( \ell_p[X^*] \) is contained in \( \ell_q[Y^*] \).

Unfortunately the hypotheses of 3.11 and 3.12 are nearly met.

**3.13. COROLLARY.** If \( X \) is a \( BS_{\omega^*} \) space \( p \geq 2 \), \( q > 1 \), \( X \) has the b.a.p. and \( \langle X, Y \rangle \) is p-trivial then \( X \) is finite dimensional.

Proof. By 3.7, if \( \dim X = +\infty \) there is an \( \langle a_i \rangle \in \ell_p[X] \) with \( ||a_i|| \leq \frac{1}{\ln(i + 1)} \), i.e. \( \langle a_i \rangle \in \ell_p[X] \) for any \( g > 1 \). This contradicts 3.11.

### § 4. REMARKS AND UNSOLVED PROBLEMS

In this section we make some remarks concerning the preceding section and raise some unsolved problems.

**Remark 1.** For \( p \geq 2 \) there is no analogue of 2.1(c).

Indeed \( \langle a, \ell_p \rangle \) is p-trivial for \( 1 < q < 2 \) and any \( p > 2 \). However by 3.10 if \( \langle X, \ell_p \rangle \) is p-trivial for any \( p \) then \( \dim X = +\infty \). We do not know if there is any analogue for \( p = 2 \). This question is closely related to the following problem.

**Problem 1.** ([17] p. 319) If \( \langle X, Y \rangle \) is p-trivial for some fixed \( p, 1 < p < 2 \), \( \langle X, Y \rangle \) 1-trivial?

A somewhat weaker question is the following.

**Problem 2.** If \( \langle X, Y \rangle \) is p-trivial for some fixed \( p, 1 < p < 2 \), \( \langle X, Y \rangle \) 2-trivial?

Of course there is a dual to 2.1(c) for strongly \( \alpha \)-summing operators.

**Remark 2.** If \( \langle X, Y \rangle \) is strongly \( \alpha \)-trivial then for any \( SF_\omega \) space \( Z \), \( \langle Y, Z \rangle \) is strongly \( \alpha \)-trivial, whenever \( Y \) has the b.a.p.
Indeed if \( \langle X, Y \rangle \) is strongly \( \infty \)-summing then \( \langle X', X' \rangle \) is \( 1 \)-trivial and so \( \langle Z', Y' \rangle \) is \( 2 \)-trivial. This is true since \( Z' \) is an \( \mathcal{L}_\infty \)-space [18]. But then \( \langle Y, Z \rangle \) is strongly \( 2 \)-trivial.

There are obvious problems analogous to Problems 1 and 2 for strongly \( p \)-summing operators.

An affirmative answer to the following problem would settle several of the outstanding problems concerning \( p \)-trivial spaces.

**Problem 3.** Is every infinite dimensional Banach space either a \( \mathcal{S}_1 \), \( \mathcal{S}_2 \) or \( \mathcal{S}_\infty \)-space?

In our next remark we summarize what is known about \( p \)-triviality and \( \mathcal{S}_\infty \)-spaces.

**Remark 3.** First recall that if \( X \) is an \( \mathcal{L}_\infty \)-space and \( Y \) and \( \mathcal{S}_\infty \)-space \( 1 \leq p < 2 \) then \( \langle X, Y \rangle \) is \( p \)-trivial for all \( p \geq 2 \). We show that this is false for \( 1 < p < 2 \). Indeed if \( X \) is an \( \mathcal{L}_\infty \)-space and \( 1 < p < 2 \) then if \( \langle X, Y \rangle \) is \( p \)-trivial, \( \dim Y < +\infty \).

Indeed if \( \langle X, Y \rangle \) is \( p \)-trivial, \( \langle X, \lambda \rangle \) is \( p \)-trivial. Since \( X \) is an \( \mathcal{L}_\infty \)-space, \( X^* \) is an \( \mathcal{L}_\infty \)-space [18] and so ([17] Prop. 7.5, p. 311) \( X^* \) has a complemented subspace isomorphic to \( l_2 \). Then \( X^\ast \) contains an isomorphic copy of \( l_2 \).

Choose \( (a_i) \), \( (\lambda_i) \), \( \lambda_i > 0 \) and define \( T \) by \( T((x_i)') = \sum \lambda_i a_i \beta_i(x_i') \), where \( (\beta_i) \) corresponds to the unit vector basis of \( l_1 \) and \( (a_i) \) is the unit vector basis of \( l_1 \). Then \( \| T(x^*) \| \leq \| x^* \| \sup \| \beta_i \| \sum \lambda_i a_i \| \) and \( T \) is continuous. Define \( S \in \mathcal{S}(X, l_1) \) by \( Sx = \sum f_i(x) a_i x_i \) then clearly \( S^* = T \). If \( (\omega_i) \) corresponds to the unit vector basis of \( c_0 \) in \( X^\ast \) then \( (\omega_i) \) is weakly \( p \)-summing. But \( \| Tu \| = \| u \| \) and so \( \sum \| Tu \| = +\infty \), i.e. \( T \) is not absolutely \( p \)-summing. By [21] \( S \) is not \( p \)-absolutely summing, contradicting our hypothesis.

If \( X \) is an \( \mathcal{S}_\infty \)-space \( 1 < p < \infty \) and \( \langle X, Y \rangle \) is \( q \)-trivial for some \( q > 1 \) then \( \dim Y < +\infty \).

Indeed and \( \mathcal{S}_\infty \)-space for \( 1 < p < \infty \) is \( \mathcal{S}_1 \)-space [24]. Thus by 2.5, \( \dim Y < +\infty \).

The analogous results for \( \mathcal{S}_p \)-spaces and for the strongly \( p \)-trivial case is covered in the above or the unsolved problems.

Finally we mention the following striking result of S. Kwapien [26]:

Let \( E \) be an \( \mathcal{L}_\infty \)-space and \( F \) be an \( \mathcal{L}_\infty \)-space, \( p > 2 \). Then

(i) \( \langle E, F \rangle \) is \( q \)-trivial for any \( q > p \); and

(ii) \( \langle E, F \rangle \) is not \( p \)-trivial.

We end this paper with some remarks on Hilbertian operators.

Lindenstrauss and Pełczyński ([17] Prop. 5.3, p. 294) have shown that the property of being a Hilbertian operator is a local property of the domain.

More precisely, an operator \( T \) from \( X \) to \( Y \) is Hilbertian if and only if there is a constant \( C \) such that for every finite dimensional subspace \( E \) of \( X \) there are operators \( E \to l_1 \) and \( E \to l_2 \) such that \( V_{E, N} \cdot U_{E, N} \cdot \| V_{E, N} \| \cdot \| U_{E, N} \| \leq C \).

Motivated by this result we define the Hilbertian Constant \( h = h(X, Y) \) of the Banach spaces \( X \) and \( Y \) by

\[
h(X, Y) = \inf \{ c \geq 0 : T \text{ is weakly } c \text{-summing} \}
\]

where \( \epsilon(T) \) is the infimum of the constants \( C \) satisfying the above.

It follows from [17] that if \( X \) is an \( \mathcal{L}_\infty \)-space and \( Y \) an \( \mathcal{S}_p \)-space \( 1 \leq p \leq 2 \) then \( h(X, Y) \leq \epsilon(C) \), where \( C \) is the Grothendieck constant [17].

From the definition it is clear that if \( X \) or \( Y \) is isomorphic to a Hilbert space then \( h(X, Y) \leq \epsilon(C) \), where \( C \) is the isomorphism of Hilbert space \( H \). We now prove that if every \( T \in \mathcal{L}(X, Y) \) is Hilbertian then \( h(X, Y) < +\infty \).

First observe that if every \( T \in \mathcal{L}(X, Y) \) is Hilbertian the same is true for any complemented subspace \( X_0 \) of \( X \), in particular for the subspaces of finite co-dimension in \( X \). By this remark and the Lindenstrauss–Pełczyński theorem above if \( h(X, Y) = +\infty \), we can construct a sequence of disjoint finite dimensional spaces \( (F_n) \) in \( X \) and operators \( T_n : X \to Y \) \( \| T_n \| = 1 \) such that for any \( U_n : F_n \to \mathbb{R} \), \( V_n \), the restriction of \( T_n \) to \( F_n \), \( \| T_n \| \cdot \| U_n \| \cdot \| V_n \| > \epsilon(C) \| I - P_{n-1} \|^{-1} \) where \( P_n \) is the identity on \( X \) and \( P_n \) the projection onto \( X_n \) determined by \( X = F_n \oplus X_n \). Define \( T : X \to Y \) by \( T = \sum \frac{1}{2^n \| P_{n-1} \|} T_n | I - P_{n-1} \| \). Then \( \| T \| \leq 1 \) and since \( (F_n) \) are disjoint

\[
\| T \| = \frac{1}{2^n \| P_{n-1} \|} \| I - P_{n-1} \| = \frac{1}{2^n \| P_{n-1} \|} \| T_n \| \| I - P_{n-1} \| \leq \frac{1}{2^n \| P_{n-1} \|} \| U_n \| \| I - P_{n-1} \| \| I - P_{n-1} \| \geq 2^{-n}
\]

and this contradicts [17].

With this observation we can sharpen 1.6. If \( X \) is a Banach space, \( Y \) an \( \mathcal{L}_\infty \)-space and every \( T \in \mathcal{L}(X, Y) \) is Hilbertian then \( X \) is an \( \mathcal{S}_p \)-space, where \( \beta \leq h(X, Y) \lambda(1+\epsilon) \), where \( \epsilon > 0 \) is arbitrary.

Indeed, if \( F \subset X \), \( \dim F < +\infty \) then there is an operator \( S : F \to Y \) with \( \delta(S^\ast, S(F)) < \lambda(1+\epsilon) \). Since \( Y \) is an \( \mathcal{L}_\infty \)-space and \( S \) has finite
rank, there is an extension \( \hat{S} \) of \( S \) to all of \( X \) with \( \| \hat{S} \| \leq \lambda \| S \| \). We have, by hypothesis

\[
\begin{align*}
F & \xrightarrow{R_X} Y \\
\gamma_B & \downarrow \\
\ell_2 & \\
\|V_E\|/\|V_S\| & \leq h(X, Y)(1+\varepsilon)\lambda^{-1}
\end{align*}
\]

Now \( V_E \) must be onto \( S(X) \) and so \( SE \) is isomorphic to a factor space of \( \ell_2 \) with a bound on the isomorphism no larger than \( h(X, Y)(1+\varepsilon) \). Since this constant is independent of \( F, X \) is an \( L_p \)-space with \( \beta \leq h(X, Y)(1+\varepsilon) \).

References


Received February 26, 1971