Now let $s^0_n = s^1_n = \cdots = s^n_n = 1$, $s_n = 1, k = 1, 2, \ldots$. Then for any $k \geq 1$, we have $\xi_t = (e^s_{2k} + e^{s_{2k}})$ with

$$P_{2k}(x - \sum_{i=1}^{M} \xi_i h_i) = \sum_{n=1}^{\infty} s^n_n e^n$$

so

$$P_{2k}(x - \sum_{i=1}^{M} \xi_i h_i) = A_1 \cdots A_k P_{2k}(x - \sum_{i=1}^{M} \xi_i h_i) = \sum_{n=1}^{\infty} s^n_n e^n.$$ 

Equating coefficients we conclude that

$$\xi_n = 0 \text{ for } n \leq M \quad \text{and} \quad \xi_n = \frac{e^n}{s^n} \text{ for } n > M.$$ 

In particular, applying this for $M = N$ we conclude that

$$\left(\frac{e^n}{s^n}\right)_{n=M}^{\infty} = I.$$ 

And for arbitrary $M \geq N$,

$$\left\|P_{2k}(x - \sum_{i=1}^{M} \xi_i h_i)\right\| = \left(\sum_{n=M+1}^{\infty} \frac{e^n}{s^n}\right) = \left(\sum_{n=M+1}^{\infty} \frac{e^n}{s^n}\right)^{1/2}.$$ 

The last term goes to zero as $M$ goes to infinity and this implies

$$\lim_{M \to \infty} \left\|P_{2k}(x - \sum_{i=1}^{M} \xi_i h_i)\right\| = 0$$ 

so that $x = \sum_{i=1}^{M} \xi_i h_i$.

Remark 5. The case described above is a very primitive example. To go further, it would be very interesting to see what happens if $N$ varies with respect to $S$ and moreover if this approach could be used to approximate an arbitrary lower triangular matrix. Finally one could investigate the connection between upper and lower triangular matrices.

References

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\[ C_p((-\infty, \infty)) = \{ \text{all bounded uniformly continuous functions} \}, \]
\[ C_0((-\infty, \infty)) = \{ \text{all continuous functions with finite limits at } -\infty \text{ and } \infty \}, \]
\[ C_0((-\infty, \infty)) = \{ \text{all continuous functions with limits zero at } -\infty \text{ and } \infty \}, \]
\[ C_0 = \{ \text{all continuous functions having period } 2\pi \}, \]
\[ A^p = \text{the space of almost periodic functions in the sense of Bohr}, \]
\[ L^p = \{ \text{all functions locally integrable with power } p \text{ and having period } 2\pi \}, \]
\[ M((-\infty, \infty)) = \text{the space of all bounded Borel measures}, \]
\[ M_{\text{fin}} = \text{the space of all finite Borel measures having period } 2\pi. \]

We shall regard \( M((-\infty, \infty)), M_{\text{fin}}, L^p(-\infty, \infty) \) as adjoint spaces of \( C_0((-\infty, \infty)), C_0, L^p(-\infty, \infty) \) and \( L^p \) with the \( \ast \)-weak topology. The remaining of the above spaces will be regarded as Banach spaces with the norm topology.

Let \( X \) be any of the spaces listed above, let \( \{ T(t): -\infty < t < \infty \} \subset \mathcal{L}(X) \) be the one-parameter group of left translations and let \( X_{\text{imp}} \) denote the subspace of all impair elements of \( X \). We shall say that \( X \) has property \((E)\) if the \( \mathcal{L}(X_{\text{imp}}) \)-valued cosine function
\[ \gamma(t) = \frac{1}{2} [T(t) + T(-t)] \]
has an exponential representation.

**Theorem.** All spaces \( L^p_1 \) and \( L^p(-\infty, \infty), 1 < p < \infty \), have property \((E)\). The spaces \( C_0((-\infty, \infty)), C_0, L^p(-\infty, \infty) \), \( M_{\text{fin}}, AP, L^p(-\infty, \infty) \), \( L^p, \) \( M((-\infty, \infty)), M_{\text{fin}}, \) \( L^p(-\infty, \infty) \) and \( L^p_1 \) do not have property \((E)\).

The proof will be given in several sections, devoted to various spaces.

The author expresses his warmest thanks to Professor C. Ryll-Nardzewski, who suggested that the problem of the existence of an exponential representation for the cosine function \( \gamma(t) \) may be interesting not only for the space \( C_0, \) considered in [7], but also for other spaces of impair functions. Also to Professor C. Ryll-Nardzewski the author owes many technical hints and some important parts of the proofs.

**1. The spaces \( L^p \) and \( L^p(-\infty, \infty), 1 < p < \infty \), have property \((E)\).** Let \( 1 < p < \infty \) and \( X = L^p \) or \( X = L^p(-\infty, \infty) \). Let
\[ (Re)(s) = \sigma(-s), \quad s \in X, \quad -\infty < s < \infty. \]

Then there is a projector \( P \in \mathcal{L}(X) \) such that
\[ T(t)P = PT(t), \quad -\infty < t < \infty \]
and
\[ T(t)(RP + PR - E) \text{ does not depend of sign}. \]

Indeed, if \( X = L^p \), then, according to a theorem of M. Riesz (cf. [4], Chapter 9), there is a projector \( P \in \mathcal{L}(L^p) \) such that \( P\alpha_n = \alpha_n \) for \( n = 0, 1, 2, \ldots \) and \( P\alpha_n = 0 \) for \( n = -1, -2, \ldots \), where \( \alpha_n = e^{ins}. \)

It is easy to see that this projector satisfies (a) and (b). If \( X = L^p(-\infty, \infty) \) then, according to another theorem of M. Riesz (cf. [1], Chapter XI, § 7, Theorem 8), the Hilbert transformation \( H \) belongs to \( \mathcal{L}(X) \). As known, \( H^2 = -1, HR + RH = 0 \) and \( HT(t) = T(t)H \). Therefore \( P = \frac{1}{2} + \frac{1}{2}H \) is a projector satisfying (a) and (b).

Now we may proceed simultaneously for \( X = L^p \) and \( X = L^p(-\infty, \infty) \). Let \( P \in \mathcal{L}(X) \) be a projector satisfying (a) and (b) and put
\[ G(t) = T(t)P - T(-t)(1-P). \]

By (a) we see that \( \{ G(t): -\infty < t < \infty \} \subset \mathcal{L}(X) \) is a continuous one-parameter group. By (b) we have
\[ G(t) = T(-t)EP + T(t)R(1-P) = T(-t)(1-P)R + T(t)PR = G(t)R, \]
which, in view of the equality \( X_{\text{imp}} = \{ x: x \in X, Rx = -x \} \), implies that
\[ G(t)G(s) = G(t+s). \]

Therefore, \( \mathcal{L}(X_{\text{imp}}) \) is a continuous one-parameter group such that
\[ \gamma(t) = \frac{1}{2} \gamma(t) + \gamma(-t). \]

**2. The spaces \( C_0, AP, L^p_1, L^p_2 \) and \( M_{\text{fin}} \) do not have property \((E)\).** For \( C_0 \) and \( AP \) this was proved in [5]. Following the argumentation used there for \( C_0 \), we shall give the complete proof for \( C_0, L^p_1, L^p_2 \) and \( M_{\text{fin}} \). Let \( X \) denote any of these four spaces. For every \( n = 1, 2, \ldots \) let \( \varphi_n(x) = \sin n x \in X \)

\[ P_n x = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^\pi \varphi_n(s) x(s) ds \varphi_n, \quad x \in X \]
in the case when \( X = C_0 \), \( L^p_1 \) or \( L^p_2 \) and let
\[ P_n x = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^\pi \varphi_n(s) x(s) ds \varphi_n, \quad x \in L^p_2 \]
when \( X = M_{\text{fin}} \). Then \( P_n \in \mathcal{L}(X_{\text{imp}}) \) is a projector and, since
\[ \varphi_n(t) = (\cos nt) \varphi_n, \]
we have

\[ P_n = \frac{2}{\pi} \int_{-\pi}^{\pi} \cos \theta_x(t) \, dt. \]

(b)

Suppose on the contrary to the assertion that there is a continuous one-parameter group \( \{ \theta(t) : -\infty < t < \infty \} \subset \mathcal{L}(X_{\text{rep}}) \) such that \( \theta_\alpha(t) = \frac{1}{2} \theta(t) + \frac{1}{2} \theta(-t) \). Then, by (b), \( P_n \theta(t) \theta_x(t) = \theta(t) P_n \) and consequently, by (a), for every \( n = 1, 2, \ldots \) we have

\[ \theta(t) P_n = e^{it} \theta_x(t) P_n, \quad -\infty < t < \infty, \]

where \( \epsilon_n = 1 \) or \( \epsilon_n = -1 \) do not depend on \( t \). Assume additionally that \( \epsilon_n = 0 \) for \( n = 1, 2, \ldots \) and consider the impair periodic distribution \( S \) with Fourier series

\[ S \sim \sum_{n=-\infty}^{\infty} \epsilon_n e^{itn}. \]

According to a theorem of Helsen [3], if the sequence of Fourier coefficients of a measure on \( [0, 2\pi] \) consists of finitely many distinct values only, then this sequence may be made periodic by a change of a finite number of its elements. The sequence \( \{\epsilon_n\} \) takes only the values \(-1, 0 \) and \( 1 \), but since \( |\epsilon_n| = 1 \) for all \( m \neq 0 \) and \( \epsilon_n \) is impair, it cannot be made periodic by such a change. Therefore \( S \) is not a measure.

Now we shall obtain a contradiction by showing that \( S \) is a measure. We shall use here the argumentation given by professor C. Ryll-Nardzewski. We have

\[ \frac{2}{\pi} \int_{0}^{\pi} e^{it} \sin(n + t) \, dt = -in \epsilon_n e^{-in \epsilon_n} = (x_n - S \ast x_n)(x) \]

and, by (c)

\[ \langle T(t) \theta \rangle_{x_n}(x) = e^{it} \sin(n + t), \]

so that

\[ \langle S \ast x \rangle = -\frac{2}{\pi} \int_{0}^{\pi} T(t) \theta(t) x \, dt \]

for every impair trigonometric polynomial \( x \). Hence, by an application of the Banach–Steinhaus theorem, we infer that there is a constant \( K \) such that

(d)

\[ \|S \ast x\|_X \leq K \|x\|_X \]

for every impair trigonometric polynomial \( x \). If \( X = C_{\text{loc}} \) or \( X = C_{\text{rep}} \) then, \( S \) being impair, we have

\[ |\langle S \rangle, \langle x_{\text{rep}} \rangle| = | - \langle S \ast x_{\text{rep}} \rangle(0) \| \leq K \|x\|_X \]

for every trigonometric polynomial \( x \) with the impair part \( x_{\text{rep}} \) and this implies that \( S \) is a measure.

If \( X = L^1_{\text{loc}} \) or \( X = L^1_{\text{rep}} \), then let \( y_n, n = 1, 2, \ldots \) be an approximative unit in the convolution algebra \( L^1_{\text{rep}} \), such that \( y_n \) are pair trigonometric polynomials and \( \|y_n\| \leq 2 \) for \( n = 1, 2, \ldots \). Then, for every fixed \( t \in (-\infty, \infty) \)

\[ S_{\epsilon_n} = \left( \frac{1}{2} T \right) y_n - T \left( -\frac{1}{2} y_n \right) \rightarrow T(t) S - S \]

as \( n \rightarrow \infty \), in the sense of distributional convergence. On the other hand, applying (d) to the impair trigonometric polynomial \( x = T \left( -\frac{1}{2} y_n \right) x_n \), we see that

\[ \|S_{\epsilon_n}\|_X \leq C \]

for every \( n = 1, 2, \ldots \) and \( t \in (-\infty, \infty) \). It follows that

(e)

\[ T(t) S - S \ast M_{\text{rep}} \quad \text{and} \quad \|T(t) S - S\|_{M_{\text{rep}}} \leq C \]

for every \( t \in (-\infty, \infty) \). Since \( S \) is impair, we have \( \frac{1}{2} T(t) S dt = 0 \) and thus

\[ S = \frac{1}{2} \int_{\infty}^{\infty} (S - T(t) S) dt \]

in the sense of distributional convergence of the Riemann approximating sums. On the other hand, by (e), those approximating sums form a bounded set in \( M_{\text{rep}} \). It follows that \( S \ast M_{\text{rep}} \).

3. Some lemmas on cosine operator functions. In this section \( X \) always denotes a sequentially complete real or complex linear locally convex space and our reasoning will be based on a boundedness principle formulated in Theorem 7.4.4 of the book of Edwards [2]. Throughout this section \( \theta \) denotes a \( \mathcal{L}(X) \)-valued continuous cosine function. According to Sova [6], the infinitesimal generator of \( \theta \) is the linear operator \( A \) defined by the conditions

\[ \mathcal{D}(A) = \left\{ x : x \ast X, \lim_{t \to 0} \frac{\theta(t) x - x}{t} \text{ exists in } X \right\}, \quad A x = \lim_{t \to 0} \frac{\theta(t) x - x}{t} \text{ for } x \in \mathcal{D}(A). \]
Lemma 1. The operator $A$ is sequentially closed and its domain $D(A)$ is sequentially dense in $X$. If $x \in D(A)$, then $\Psi(t)x$ is an $X$-valued function of $t$, twice continuously differentiable on $(-\infty, \infty)$, and such that $\Psi(t)x \in D(A)$, and

$$\frac{d^2}{dt^2} \Psi(t)x = A\Psi(t)x = \Psi(t)Ax$$

for every $t \in (-\infty, \infty)$.

Proof. For every $X$-valued function $f$ continuous on $(-\infty, \infty)$ and every $h > 0$, we have

$$\frac{d}{dt} \int_{-h}^{h} f(t+u+v) du dv = \int_{-h}^{h} \frac{d}{dt} f(t+u+v) du dv \bigg|_{v=0}^{v=h} = \int_{-h}^{h} \left( \frac{d}{du} f(t+u+v) \bigg|_{v=0}^{v=h} \right) du$$

and therefore

$$\int_{-h}^{h} \left( \int_{-h}^{h} f(t+u+v) du \right) dv = \int_{-h}^{h} f(t+u+v) du \bigg|_{v=0}^{v=h} = \int_{-h}^{h} f(t+u+v) du dv$$

and

$$\int_{-h}^{h} f(t+u+v) du dv = \int_{-h}^{h} f(t+u) du \bigg|_{v=0}^{v=h} = \int_{-h}^{h} f(t+u) du \bigg|_{v=0}^{v=h}$$

for every $f \in D(A)$.

We shall apply the former equality to $f(t) = \Psi(t)x$. Since, by the d'Alembert equation, $\Psi(t) = \Psi(-t)$ and

$$\frac{1}{h^2} \int_{-h}^{h} \left( \Psi(t+h) + \Psi(t-h) - 2 \Psi(t) \right) dx = \frac{2}{h^2} \int_{-h}^{h} \Psi(t+h) dx$$

we obtain

$$\frac{1}{h^2} \int_{-h}^{h} \left( \Psi(t+h) + \Psi(t-h) - 2 \Psi(t) \right) dx = \frac{2}{h^2} \int_{-h}^{h} \Psi(t+h) dx$$

and hence

$$\lim_{h \to 0} \int_{-h}^{h} \left( \Psi(t+h) + \Psi(t-h) - 2 \Psi(t) \right) dx = \lim_{h \to 0} \frac{2}{h^2} \int_{-h}^{h} \Psi(t+h) dx$$

This is obvious for $t = 0$ and the argument for $t < 0$ is similar to that for $t > 0$, let us assume that $t > 0$ is fixed. Let $x \in D(A)$ and $t \in (-\infty, \infty)$.

We have

$$\lim_{h \to 0} \int_{-h}^{h} \left( \Psi(t+h) + \Psi(t-h) - 2 \Psi(t) \right) dx = \lim_{h \to 0} \frac{2}{h^2} \int_{-h}^{h} \Psi(t+h) dx$$

and

$$\lim_{h \to 0} \int_{-h}^{h} \left( \Psi(t+h) + \Psi(t-h) - 2 \Psi(t) \right) dx = \lim_{h \to 0} \frac{2}{h^2} \int_{-h}^{h} \Psi(t+h) dx$$

for every $x \in D(A)$ and $t \in (-\infty, \infty)$.

Now we show that

(c) $\Psi(t)x - x = \int_{-h}^{h} \Psi(t)x dx dv du$ for every $x \in D(A)$ and $t \in (-\infty, \infty)$.

This is obvious for $t = 0$ and since the argument for $t < 0$ is similar to that for $t > 0$, let us assume that $t > 0$ is fixed. Let $x \in D(A)$ and $t \in (-\infty, \infty)$.

We have

$$\lim_{h \to 0} \int_{-h}^{h} \left( \Psi(t+h) + \Psi(t-h) - 2 \Psi(t) \right) dx = \lim_{h \to 0} \frac{2}{h^2} \int_{-h}^{h} \Psi(t+h) dx$$

for every $x \in D(A)$ and $t \in (-\infty, \infty)$.
Proof. We have to prove that \( \mathcal{D}(A) \subset \mathcal{D}(K) \). Since \( K \subset A \) and \( \Psi(t) \) is \( \mathcal{D}(K) \), so by Lemma 1, for every \( x \in \mathcal{D}(K) \) and \( t \in (0, \infty) \) we have
\[
\frac{\partial^2}{\partial t^2} \Psi(t)x = \Psi(t)x = \Psi(t)Kx
\]
and consequently
\[
\Psi(t)x = \int_0^t Kx \psi(\tau) d\tau
\]
for every \( x \in \mathcal{D}(K) \). Since the last equality \( \Psi(t)x = \Psi(t)Kx \) are continuous functions of \( t \) and since \( K \) is sequentially closed, we may transport \( K \) before the integrals. We then obtain that
\[
(*) \quad \int_0^t \Psi(\tau)x_\tau d\tau \in \mathcal{D}(K) \quad \text{and} \quad K \int_0^t \Psi(\tau)x_\tau d\tau = \int_0^t \Psi(t)x \tau d\tau
\]
for every \( x \in \mathcal{D}(K) \) and \( t \in (0, \infty) \). Let now \( x \in \mathcal{X} \setminus \mathcal{D}(K) \). Because \( \mathcal{D}(K) \) is sequentially dense in \( \mathcal{X} \), there is a sequence \( (x_n) \), \( n = 1, 2, \ldots \) such that \( x_n \in \mathcal{D}(K) \) and \( \lim_{n \to \infty} x_n = x \). Then
\[
\lim_{n \to \infty} K \int_0^t \Psi(\tau)x_n d\tau = \lim_{n \to \infty} \int_0^t \Psi(t)x_n d\tau = \int_0^t \Psi(t)x d\tau,
\]
by a boundedness principle for locally convex sequentially complete spaces from the book of Edwards and the Lebesque bounded convergence theorem. Hence by the sequential closeness of \( \mathcal{X} \), it follows that \( (*) \) holds true for every \( x \in \mathcal{X} \) and \( t \in (0, \infty) \). Now it is easy to finish the proof.

Indeed, if \( x \in \mathcal{D}(A) \) and \( x_1 = \frac{2}{t} \int_0^t \Psi(t)x d\tau \), then \( x_1 = x \) and, by
\[
(*) \quad x_1 \in \mathcal{D}(K) \quad \text{and} \quad \lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{2}{t} \int_0^t \Psi(\tau)x_n d\tau = A x,
\]
which, \( K \) being sequentially closed, implies that \( x \in \mathcal{D}(K) \).

**Lemma 3.** If \( \Psi(t) = \frac{1}{\lambda} \Gamma(t) + \frac{i}{\lambda} \Gamma(-t) \), where \( \Gamma(t) ; -\infty < t < \infty \) is \( \mathcal{D}(X) \) is an one-parameter continuous group with infinitesimal generator \( B \), then \( B = A \).

**Proof.** Let us recall that
\[
\mathcal{D}(B) = \left\{ x; x \in \mathcal{X}, \lim_{t \to 0} \frac{1}{t} (G(t)x - x) \text{ exists in } \mathcal{X} \right\},
\]

\[
Bx = \lim_{t \to 0} \frac{1}{t} (G(t)x - x) \quad \text{for} \quad x \in \mathcal{D}(B).
\]

Then for every \( n = 1, 2, \ldots \) we have
\[
(i) \quad G(t)B^n \subset \mathcal{D}(B) \quad \text{and} \quad \frac{d^n}{dt^n} G(t)x = G(t)B^n x = B^n G(t)x
\]
for every \( t \in (0, \infty) \) and \( x \in \mathcal{D}(B) \). Moreover,
\[
(ii) \quad \text{all the operators } B^n, n = 1, 2, \ldots, \text{ are sequentially closed and}
\]
\[
(iii) \quad \bigcap_{n=0}^{\infty} \mathcal{D}(B^n) \text{ is a sequentially dense subset of } \mathcal{X}.
\]

Having (i), (ii) and (iii), the equality \( B = A \) follows immediately by an application of Lemma 2 to \( K = B^0 \). We may prove (iii) in the same fashion as in the book of Yosida [7], in the case of equicontinuous semi-groups. However, if we want to prove (i) or (ii) without the assumption of equicontinuity, we have to use a new argument.

**Ad (i).** We shall proceed by induction in \( n \). If we assume that \( \mathcal{D}(B^n) \cap \mathcal{X} = \mathcal{D}(B^n) \), then (i) is trivial for \( n = 0 \). Suppose now that (ii) is true for a certain \( n \geq 0 \) and let \( x \in \mathcal{D}(B^{n+2}) \). Then, for every \( h > 0 \) we have
\[
\frac{1}{h} \int_0^h (G(t) \Psi(t)x - \Psi(t)x) dt = \int_0^h \frac{1}{h} G(t)(B^n - 1) B^n x dt
\]
and, passing to the limit as \( h \to 0 \), we obtain that, for every \( t \in (0, \infty) \),
\[
(*) \quad \frac{d^n}{dt^n} G(t)x \biggr|_{t=0} = \int_0^t \frac{1}{h} G(u) B^{n+1} x du.
\]

Indeed, the only non-trivial point in this limit passage is that
\[
\lim_{h \to 0} \int_0^t \frac{1}{h} G(u) B^{n+1} x du = \int_0^t \frac{1}{h} G(u) B^{n+1} x du,
\]
and this may be proved by an application of the boundedness principle from the book of Edwards and the Lebesque bounded convergence theorem, similarly as this was done in the deduction of (c) in the proof of our Lemma 1. From (*) it follows by a differentiation that if \( x \in \mathcal{D}(B^{n+2}) \), then
\[
\frac{d^{n+2}}{dt^{n+2}} G(t)x = G(t)(B^{n+2} x) \quad \text{for every } t \in (-\infty, \infty). \]

On the other hand, if \( x \in \mathcal{D}(B^{n+2}) \) then for \( y = B^n x \in \mathcal{D}(B) \) we have
\[
G(t)B^n y = G(t)y \quad \text{by \ limit } \quad \frac{1}{h} (G(h)y - y) = \lim_{h \to 0} \frac{1}{h} G(h) - 1) G(t)y
\]
so that \( B^n G(t)x = G(t)B^n x = G(t)x \in \mathcal{D}(B) \) and \( G(t)B^{n+2} x = B^n G(t)x \). Thus (i) is proved.

**Ad (ii).** It follows from (i) that, for \( x \in \mathcal{D}(B) \),
\[
G(t)x = x + t Bx + \sum_{k=0}^{n-1} \frac{1}{(n-1)!} B^{n-1} x + \int_0^t \frac{1}{h} G(u) B^n x du.
\]
Applying this formula and proceeding by induction in \( n \), it is easy to prove that for every \( n = 1, 2, \ldots \) the operator \( B^n \) is sequentially closed. Indeed, if \( \mathcal{D}(B^n) = X \) and \( B^n = 1 \), then \( B^n \) is closed. Let now \( n \geq 1 \) and suppose that \( B, B^2, \ldots, B^{n-1} \) all are closed. Let the pair \((x, y)\) lie in the sequential closure of the graph of \( B^n \). Then there is a sequence \((x_k)\), \( k = 1, 2, \ldots \), such that \( x_k \in \mathcal{D}(B^n) \), \( x_k \to x \) and \( B^k x_k \to y \). By the boundedness principle from the book of Edwards and by the Lebesgue bounded convergence theorem, we have

\[
\lim_{k \to \infty} \int \left( \frac{(u-1)}{(n-1)} \right) G(u) B^n x_k du = \int \left( \frac{(u-1)}{(n-1)} \right) G(u) y du
\]

for every \( t \in (-\infty, \infty) \). Since obviously \( \lim_{k \to \infty} G(t)x_k = G(t)x \), it follows that

\[
\lim_{k \to \infty} \left( x_k + B x_k + \ldots + \frac{B^{n-1} x_k}{(n-1)!} \right) G(u) y du
\]

exists for every \( t \). Therefore \( \lim_{k \to \infty} B^n x_k \) exists for every \( n = 1, 2, \ldots, n-1 \), \( B^n \) being closed, \( x \in \mathcal{D}(B^n) \) and \( \lim_{k \to \infty} B^n x_k = B^n x \). Therefore \( x \in \mathcal{D}(B^n) \) and

\[
G(t)x = x + B x + \ldots + \frac{B^{n-1} x}{(n-1)!} + \int \left( \frac{(u-1)}{(n-1)} \right) G(u) y du
\]

for every \( t \in (-\infty, \infty) \). Differentiating this \( n-1 \) times and using (i), we obtain that

\[
\frac{1}{t} (G(t)-1) B^{n-t} x = \frac{1}{t} G(u) y du\text{ for every } t \neq 0.
\]

Finally, passing in the former equality to the limit as \( t \to 0 \), we infer that \( B^n x \in \mathcal{D}(B) \) and \( B^n x \to y \), so that \( B^n \) is sequentially closed.

**Lemma 4.** Let \( X \) denote the field of scalars of the linear structure of \( X \) and define the \( \mathcal{L}(X \times X) \) valued cosine function \( \tilde{\mathcal{F}}_0 \) by the formula

\[
\tilde{\mathcal{F}}_0(x, \lambda) = (\mathcal{F}(0)x, \lambda), \quad x \in X, \lambda \in \mathcal{X}, \quad -\infty < t < \infty.
\]

Suppose that \( \mathcal{F} \) has an exponential representation and that \( A \) is invertible. Then \( \mathcal{F} \) also has an exponential representation.

The invertibility of \( A \) is essential in this lemma. Indeed, let \( X = C \times C \) be the two-dimensional complex space and let \( \mathcal{F}(t) = \begin{pmatrix} \frac{1}{i} t \mathcal{F} \end{pmatrix} \). Then

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

has no square root and therefore \( \mathcal{F} \) has no exponential representation. On the other hand, in this case

\[
\mathcal{F}(t) = \frac{1}{i} \exp \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \right) + \exp \left( -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \right).
\]

**Proof of Lemma 4.** We shall write elements of \( X \times \mathcal{X} \) in the form of columns \( \begin{pmatrix} \frac{1}{i} t \mathcal{F} \end{pmatrix} \), where \( x \in X \) and \( \lambda \in \mathcal{X} \). Let \( \tilde{\mathcal{F}}_0(t) = -\infty < t < \infty \in \mathcal{L}_x(X \times \mathcal{X}) \) be a continuous one-parameter group such that

\[
\frac{1}{t} \tilde{\mathcal{F}}_0(t) + \frac{1}{t} \tilde{\mathcal{F}}_0(-t) = \tilde{\mathcal{F}}_0(t)
\]

and let \( \tilde{\mathcal{F}} \) be its infinitesimal generator. Then, by Lemma 3, \( \tilde{\mathcal{F}} \) is the infinitesimal generator of \( \mathcal{F}_0 \) and so \( \mathcal{D}(\tilde{\mathcal{F}}) = \mathcal{D}(A) \times \mathcal{X} \). It follows that

\[
\mathcal{D}(\tilde{\mathcal{F}}) = L \times \mathcal{X},
\]

where \( L \) is a dense linear subset of \( X \). Therefore we may represent \( \tilde{\mathcal{F}} \) in the form of a matrix

\[
\tilde{\mathcal{F}} = \begin{pmatrix} B & x_1 \\ 0 & \lambda_1 \end{pmatrix}
\]

where \( B \) is a linear operator defined on \( L \) with values in \( X \), \( x_1 \in X \), \( \lambda_1 \in \mathcal{X} \) and \( t \) is a linear form on \( L \). If \( \tilde{z} = \begin{pmatrix} z \end{pmatrix} \in \mathcal{D}(\tilde{\mathcal{F}}) = L \times \mathcal{X} \) then, according to the general rule of multiplication of matrices,

\[
\begin{pmatrix} z \end{pmatrix} \tilde{\mathcal{F}} = \begin{pmatrix} B z + i x_1 \\ \lambda z \end{pmatrix}.
\]

Since \( \tilde{\mathcal{F}} \) is the infinitesimal generator of \( \mathcal{F} \), we have

\[
\tilde{\mathcal{F}} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}
\]

on \( \mathcal{D}(\tilde{\mathcal{F}}) = \mathcal{D}(A) \times \mathcal{X} \) which implies that

(i) \( \mathcal{D}(A) \subseteq L \)

and

(ii) \( \begin{pmatrix} B^2 + i x_2 & B x_3 + \lambda_3 x_1 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \)

on \( \mathcal{D}(\tilde{\mathcal{F}}) \). Since

\[
\begin{pmatrix} z \end{pmatrix} \mathcal{D}(A) \times \mathcal{X} = \mathcal{D}(\tilde{\mathcal{F}}) \subseteq \mathcal{D}(\tilde{\mathcal{F}}),
\]

we have

\[
\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} = \tilde{\mathcal{F}} \begin{pmatrix} z \end{pmatrix} \mathcal{D}(\tilde{\mathcal{F}}) = \mathcal{D}(A) \times \mathcal{X}
\]

and consequently

(iii) \( x_1 \in \mathcal{D}(A) \).
It follows from (i)–(iii) that

\[ A a_n = B x_n + (l a_n) x_n = B x_n = \text{constant} \]

and so \( a_n = 0 \), since \( A \) is invertible. Now, as we already know that \( a_n = 0 \), we can see from (ii) that also \( \lambda_n = 0 \). Thus

\[ \mathcal{A} \left( \begin{array}{c} B \\ 1 \\ 0 \\ 0 \end{array} \right) = \mathcal{L} \times \mathcal{A}, \]

so that \( \frac{d}{dt} \mathcal{A}(t) \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) = \mathcal{A}(t) \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) = 0 \) and consequently

\[ \mathcal{A}(t) \left( \begin{array}{c} a_n \\ \lambda_n \end{array} \right) = \mathcal{A}(t) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \]

for every \( a \in X \), \( \lambda \in \mathcal{A} \) and \( t \in (-\infty, \infty) \).

Let now \( P \) denote the natural projection of \( X \times \mathcal{A} \) onto \( X \) and let \( \mathcal{J} \) denote the natural imbedding of \( X \) into \( X \times \mathcal{A} \), i.e.

\[ P \left( \begin{array}{c} a \\ \lambda \end{array} \right) = a, \quad \mathcal{J} a = \left( \begin{array}{c} a \\ 0 \end{array} \right) \text{ for } a \in X. \]

Put

\[ \mathcal{A}(t) = P \mathcal{A}(t) \mathcal{J}. \]

Then \( \mathcal{A}(0) = 1 \) in \( \mathcal{J}(X) \) and since, by (iv), \( P \mathcal{A}(t) = P \mathcal{A}(t) \mathcal{J} P \), we have

\[ \mathcal{A}(t) \mathcal{A}(s) = P \mathcal{A}(t) \mathcal{J} P \mathcal{A}(s) \mathcal{J} = P \mathcal{A}(t) \mathcal{J} \mathcal{A}(s) \mathcal{J} = \mathcal{A}(t+s). \]

Therefore \( \mathcal{A}(t) : (-\infty, \infty) \to X \) is a continuous one-parameter group. Moreover, \( \frac{d}{dt} \mathcal{A}(t) + \frac{1}{2} \mathcal{A}(t) = \mathcal{A}(t) \mathcal{A}(t) \mathcal{J} = \mathcal{A}(t) \mathcal{J} = \mathcal{A}(t) \), which gives an exponential representation for \( \mathcal{A} \).

4. The spaces \( L^1(-\infty, \infty) \) and \( M(-\infty, \infty) \) do not have property (B).

Since our reasoning, except of some details, are the same for \( L^1(-\infty, \infty) \) and for \( M(-\infty, \infty) \), we admit in this section the convention that \( X \) denotes \( L^1(-\infty, \infty) \) or \( M(-\infty, \infty) \). Any of our statements concerning \( X \) should be understood as a statement true simultaneously for \( X = L^1(-\infty, \infty) \) and for \( X = M(-\infty, \infty) \). Let us recall that we consider \( L^1(-\infty, \infty) \) under its norm topology and that we consider \( M(-\infty, \infty) \) with the topology of weak convergence of measures, i.e. \( M(-\infty, \infty) \) is regarded as the adjoint space of \( L^1(-\infty, \infty) \) with the \( \lambda \)-weak topology.

Let \( H \) denote the set of all finite linear combinations of Hermite functions \( \varphi_n(t) = \frac{d^n}{dt^n} e^{-t^2}, n = 0, 1, \ldots \). Of course, \( H \subset L^1(-\infty, \infty) \subset M(-\infty, \infty) \) in the obvious sense and it is known that \( H \) is dense in \( L^1(-\infty, \infty) \) in the sense of norm topology (this in particular follows from the statement (1) of our section) and that \( L^1(-\infty, \infty) \) is sequentially dense in \( M(-\infty, \infty) \) in the sense of weak convergence of measures. Therefore

(i) \[ H_{\text{lin}} \text{ is sequentially dense in } X_{\text{lin}}, \]

where \( H_{\text{lin}} \) denotes the set of all finite linear combinations of the functions \( \varphi_n, n = 1, 2, 3, \ldots \).

The infinitesimal generator \( A_\lambda \) of the \( \mathcal{J}(X_{\text{lin}}) \)-valued cosine function

\[ \mathcal{A}(t) = \frac{1}{2} [T(t) + T(-t)] X_{\text{lin}} \]

is defined by the equality

\[ A_\lambda a = \lim_{h \to 0} \frac{1}{h^2} [T(h)a + T(-h)a - 2a], \]

its domain \( \mathcal{D}(A_\lambda) \) being the set of all those elements \( a \in X_{\text{lin}} \) for which this limit exists in the sense of topology admitted in \( X \).

Let \( \mathcal{A} \) denote the space of all space of complex-valued infinitely differentiable functions on \( (-\infty, \infty) \) with compact supports and let \( \mathcal{B} \) be the corresponding space of distributions. In the obvious sense we have \( X \subset \mathcal{B} \). For any \( a \in \mathcal{A} \) let \( a' \) denote its second distributional derivative. We shall show that

(ii) \[ \mathcal{D}(A_\lambda) = \{ a : a \in X_{\text{lin}} \text{ and } a' \in X_{\text{lin}} \}, A_\lambda a = a' \text{ for } a \in \mathcal{D}(A_\lambda). \]

For the proof of (ii), for every real \( h \neq 0 \) put

\[ \delta_h(s) = \max \left( \frac{1}{|h|}, \frac{|s|}{|h|^2}, 1 \right), \quad -\infty < s < \infty. \]

Then it is easy to verify that

\[ \frac{1}{h^2} [T(h)\varphi + T(-h)\varphi - 2\varphi] = \delta_h \varphi' \]

for every \( \varphi \in \mathcal{A} \) and real \( h \neq 0 \). Now suppose that \( a \in X_{\text{lin}} \) and \( a' \in X_{\text{lin}} \).

Then \( T(t)a' \) is an \( X \)-valued function of \( t \), continuous in \( (-\infty, \infty) \).

Therefore, for every real \( h \neq 0 \) the integral \( \int_{-\infty}^{\infty} \delta_h(t) T(t)a' dt \) has a sense and, as is easy to prove, \( a' \in \mathcal{D}(A_\lambda) \).
in the sense of topology animit in $X$. Furthermore, for every $\varphi \in \mathcal{D}$ and \( h \neq 0 \), we have

\[
\int_{-\infty}^{\infty} \frac{\delta_h(t)}{\sqrt{2\pi}} e^{it \varphi'} dt, \varphi \rangle = \langle \varphi', \delta_h \varphi \rangle = \langle \varphi, \delta_h \varphi' \rangle
\]

so that

\[
\int_{-\infty}^{\infty} \frac{\delta_h(t)}{\sqrt{2\pi}} e^{it \varphi'} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\varphi'} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\varphi'} dt
\]

It follows from (a) and (b) that, if $x \in X_{\text{imp}}$ and $\varphi' \in X_{\text{imp}}$, then $x \in \mathcal{D}(A_h)$ and $A_h x = \varphi'$. On the other hand, if $x \in \mathcal{D}(A_h)$, then for every $\varphi \in \mathcal{D}$ we have

\[
\langle A_h x, \varphi \rangle = \lim_{h \to 0} \frac{1}{\sqrt{2\pi}} \langle (T(h) x - T(-h) x - 2h), \varphi \rangle
\]

so that $A_h x$ is equal to the second distribution derivative of $x$. The assertion (ii) is proved.

It follows immediately from (ii) that

\[
\varphi_k \in \bigcap_{n=1}^{\infty} \mathcal{D}(A_n^k) \quad \text{and} \quad \varphi_{n+1} = A_n^k \varphi_k \quad \text{for} \quad n = 1, 2, \ldots
\]

After this preparation we shall prove the assertion stated in the title of this section. The proof will be by proceeding ad absurdum. We assume that there is a one-parameter continuous group \( \varphi(t) : -\infty < t < \infty \) \( \in \mathcal{D}(X_{\text{imp}}) \) such that

\[
\frac{1}{2} \varphi(t) + \frac{1}{2} \varphi(-t) = \varphi(t).
\]

Under this assumption we shall prove some lemmas, which will lead us to a contradiction.

For any $x \in X$ let $\mathcal{F}x$ be its Fourier transform, i.e.

\[
(\mathcal{F}x)(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iu \varphi}(x) dx, \quad -\infty < u < \infty, \quad \text{if} \quad x \in L^1(-\infty, \infty)
\]

and

\[
(\mathcal{F}x)(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iu \varphi}(x) dx, \quad -\infty < u < \infty, \quad \text{if} \quad x \in L^\prime(-\infty, \infty)
\]

Then $\mathcal{F} x \in C_b(-\infty, \infty)$. In particular,

\[
(\mathcal{F}x)(u) = \frac{iu}{\sqrt{2\pi}} e^{iu^2}.
\]

If $x \in \mathcal{D}(A_n^k)$, then, by (ii),

\[
(\mathcal{F}A_n^k x)(u) = (-u^2)^n(\mathcal{F}x)(u).
\]

For any real $t$ and $u$ put

\[
g(u) = \begin{cases} \frac{\sqrt{2}}{1+iu} \mathcal{F}(\mathcal{F}(x)) & \text{if} \quad u \neq 0, \\ 1 & \text{if} \quad u = 0. \end{cases}
\]

Then, by fixed $t$, $g(u)$ is a pair function of $u$, continuous in $(-\infty, \infty)$ except, perhaps, of the point $u = 0$.

**Lemma A.** For every $t \in (-\infty, \infty)$ and $x \in X_{\text{imp}}$ we have

\[
\mathcal{F}(\mathcal{F}(x))(u) = g(u).
\]

**Proof.** From Lemma 3 of Section 3 and from the property (i) of one-parameter groups stated in the proof of this lemma, it follows that

\[
\mathcal{F}(t) \mathcal{D}(A_n^k) = \mathcal{D}(A_n^k) \quad \text{and} \quad \mathcal{F}(t) A_n^k x = A_n^k \mathcal{F}(t)x
\]

for every $t \in (-\infty, \infty)$, $x \in \mathcal{D}(A_n^k)$ and $n = 1, 2, \ldots$ Therefore, by (iii), for every $t \in (-\infty, \infty)$ we have $\mathcal{F}(t) \varphi_{2n+1} = A_n^k \varphi_n$ and $\mathcal{F}(t) \varphi_{2n+1}(u) = (-u^2)^n(\mathcal{F}(t) \varphi_n)(u) = g(u)(-u^2)^n(\mathcal{F}(t) \varphi_n)(u)$, so that our lemma is true for every $x \in X_{\text{imp}}$. Let now $x \in X_{\text{imp}} \setminus H_{\text{imp}}$. Then, by (i), there is a sequence $x_1, x_2, \ldots$ of elements of $H_{\text{imp}}$ converging to $x$. For any $t \in (-\infty, \infty)$ we have $\lim \mathcal{F}(t) x_n = \mathcal{F}(t)x$.

If $x \in L^1(-\infty, \infty)$; then it follows that, for every $t \in (-\infty, \infty)$,

\[
\lim_{n \to \infty} \mathcal{F}(t)x_n = \mathcal{F}(t)x \quad \text{and} \quad \lim_{n \to \infty} \mathcal{F}(t)x_n = \mathcal{F}(t)x
\]

in the sense of uniform convergence on $(-\infty, \infty)$ and so, for every $u \in (-\infty, \infty)$,

\[
(\mathcal{F}(t)(x))(u) = \lim_{n \to \infty} (\mathcal{F}(t)x_n)(u) = \lim_{n \to \infty} g_n(u)(\mathcal{F}(t)x_n)(u) = g_n(u)(\mathcal{F}(t)x)(u).
\]

Therefore, in the case of $X = L^1(-\infty, \infty)$ our lemma is proved. If $x \in X_{\text{imp}} (-\infty, \infty) \setminus H_{\text{imp}}$ then we cannot assert that the convergence in
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It follows from (a), (b) and (c) that for every \( u \in (-\infty, \infty) \) there is a complex number \( k(u) \) such that

\[
g_t(u) = e^{k(u)t}
\]

for every \( t \in (-\infty, \infty) \). Since for every \( x \in \mathcal{D}_\text{map} \) we have

\[
\frac{1}{2}(e^{k_0u} + e^{-k_0u})(\mathcal{F}x) = \frac{1}{2}\mathcal{F}(\mathcal{T}(t)+\mathcal{T}(-t))x(x)
\]

and since \( e^{k_0u} = g_t(0) = 1 \), it follows that \( \frac{1}{2} e^{k_0u} + \frac{1}{2} e^{-k_0u} = \text{cost}_x \) for every real \( u \) and \( t \) and consequently for every real \( u \) we have \( k(u) = iu \) or \( k(u) = -iu \). But we already know that, for every fixed \( t \), \( g_t(u) = e^{k_0t} \) is a pair function of \( u \), continuous everywhere except, perhaps, of the point \( u = 0 \). It follows that \( k(u) = i|u| \) for every \( u \) or \( k(u) = -i|u| \) for every \( u \), which completes the proof.

**Lemma C.** For every \( t \in (-\infty, \infty) \) and every \( x \in \mathcal{D}_\text{map} \) we have

\[
\mathcal{F}(t)x = \mathcal{F}_t(0)x \pm \frac{1}{2}i\mathcal{H}(\mathcal{T}(t) - \mathcal{T}(-t))x,
\]

where \( \mathcal{H} \) is the Hilbert transformation, i.e.

\[
(Hx)(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(u)}{u-t} du,
\]

and

\[
\text{cost}_t \Leftarrow \frac{1}{2}(e^{k_0t} + e^{-k_0t}) \text{sign}_x(\mathcal{F}(t)x) \text{ and therefore}
\]

\[
\mathcal{F}(\mathcal{T}(t) - \mathcal{T}(-t))x](u) = \text{cost}_t \Leftarrow \frac{1}{2}(e^{k_0u} + e^{-k_0u}) \text{sign}_x(\mathcal{F}(u)x) = e^{k_0u}(\mathcal{F}(u)x).
\]

Now consider the following statement:

(S) For every \( t \in (-\infty, \infty) \) there is \( c_t \in (0, \infty) \) such that

\[
||\mathcal{H}(\mathcal{T}(t) - \mathcal{T}(-t))x||_{L^\infty(-\infty,\infty)} \leq c_t||x||_{L^\infty(-\infty,\infty)}
\]

for every \( x \in \mathcal{D}_\text{map} \).

**Lemma C** implies **S**. This is obvious if \( x \in \mathcal{D}_\text{map} \). If \( x = \mathcal{M}(-\infty, \infty) \) then the operator \( \mathcal{F}_t(0) - \mathcal{F}(t) \mathcal{M}_\text{map}(-\infty, \infty) \), continuous in the \( \epsilon \)-weak topology, is, by the closed graph theorem, also continuous with respect to the norm topology in \( \mathcal{D}_\text{map} \). On the other hand if \( x \in \mathcal{D}_\text{map} \) then \( \mathcal{H}(\mathcal{T}(t) - \mathcal{T}(-t))x \in \mathcal{D}_\text{map} \), so that also in this case **S** follows from **Lemma 3**.

Now we shall show that **S** is not true. Let \( 0 < c < b \). For any function \( x \) defined on \((a, b)\) let

\[
(Kx)(s) = x(a+b-s).
\]

Then \( \|y_k\| = \|y_k\| \), in the sense of norm in \( L^1(-\infty, \infty) \). By the closed graph theorem, the operator \( \mathcal{F}(t) \in \mathcal{L}(\mathcal{M}_\text{map}(-\infty, \infty)) \) is continuous also with respect to the norm topology in \( \mathcal{M}_\text{map}(-\infty, \infty) \). Consequently \( \mathcal{F}(\mathcal{T}(t)) \) is a bounded subset of \( \mathcal{M}_\text{map}(-\infty, \infty) \) and therefore \( \mathcal{F}(\mathcal{T}(t)) \) is a bounded subset of \( \mathcal{O}_2(-\infty, \infty) \). But \( y_k \in L^2_{\text{map}}(-\infty, \infty) \) and so, as we have stated above, \( \mathcal{F}(\mathcal{T}(t)) y_k = g_t y_k \). Since \( \mathcal{F}(\mathcal{T}(t)) y_k = g_t y_k \), we see that \( g_t \) is bounded.

**Lemma B.** We have \( g_t(u) = e^{k_0t} \) for every real \( u \) and \( t \neq 0 \) for every real \( u \) and \( t \).

**Proof.** For every \( x \in \mathcal{D}_\text{map} \) we have \( \mathcal{F}(\mathcal{T}(t)x) = \mathcal{F}(0)x \) and

\[
g_t \mathcal{F}(\mathcal{T}(t)x) = \mathcal{F}(\mathcal{T}(t) + t)x = \mathcal{F}(\mathcal{T}(t)) \mathcal{F}(t)x = g_t \mathcal{F}(t)x = g_t \mathcal{F}(\mathcal{T}(t)x),
\]

so that, since \( g_t(0) = 1 \), we have

\[
g_t(u) = 1,
\]

\(-\infty < u < \infty,
\)

and

\[
g_{t+t}(u) = g_t(u)g_t(u),
\]

\(-\infty < u < \infty,
\)

Moreover,

\[
edd (\text{for every fixed } u \in (-\infty, \infty)) g_t(u) \text{ is a function of } t \text{ measurable on } (-\infty, \infty).
\]

Indeed, let \( u \neq 0 \) be fixed and let \( x \in \mathcal{D}_\text{map} \) be such that \( \mathcal{F}(x)(u) = 1 \). Then \( g_t(u) = g_t(u) \mathcal{F}(x)(u) = \mathcal{F}(\mathcal{T}(t)) \mathcal{F}(x)(u) \). In the case of \( \mathcal{X} \in \mathcal{L}(-\infty, \infty) \), \( \mathcal{F}(\mathcal{T}(t)x) = \mathcal{O}_2(-\infty, \infty) \) is a continuous \( \mathcal{O}_2(-\infty, \infty) \)-valued function of \( t \) and therefore \( g_t(u) \) is continuous in \( t \). If \( \mathcal{X} = \mathcal{M}(-\infty, \infty) \) then \( \mathcal{F}(\mathcal{T}(t)x) = \mathcal{C}_0(-\infty, \infty) \) depends on \( t \) continuously in the sense of the \( \epsilon \)-weak topology in \( L^\infty(-\infty, \infty) \), and so \( f_n(t) = n \int \mathcal{F}(\mathcal{T}(t)x)(n,t) \), \( n = 1, 2, \ldots \), is a sequence of continuous functions of \( t \), converging pointwise to \( g_t(u) \).
Let $E_{a,b}$ be the operator of restriction to $(a, b)$ of functions defined on $(-\infty, \infty)$. Let $E$ and $F$ be operators of extension of functions from $(a, b)$ onto $(-\infty, \infty)$ defined by the formulae

$$(Ec)(s) = \begin{cases} 
\frac{1}{\pi} \int_a^b \frac{c(t)}{s-t} dt, & s \in (a, b), \\
0, & \text{otherwise}
\end{cases}$$

and

$$(Fu)(s) = \frac{1}{\pi} \int_a^b \frac{u(t)}{s-t} dt.$$ 

Consider the operators $A: \mathcal{D}(a, b) \to \mathcal{C}^m(a, b)$ and $B: \mathcal{L}^1(a, b)$ defined as follows:

$$A = R_{a,b}HE, \quad \text{i.e.} \quad (A\phi)(s) = \frac{1}{\pi} \int_a^b \frac{\phi(u)}{s-u} du, \quad \phi \in \mathcal{D}(a, b), \quad s \in (a, b),$$

$$B = R_{a,b}HE, \quad \text{i.e.} \quad (B\phi)(s) = \frac{1}{\pi} \int_a^b \frac{\phi(u)}{s-u} du, \quad \phi \in \mathcal{L}^1(a, b), \quad s \in (a, b).$$

Then, as is easy to verify, $AK\phi + B\phi = R_{a,b}HE[T(a+b) - T(-a-b)]\phi$ or, which is the same, in view of $K^2 = 1$,

$$A\phi = R_{a,b}HE[T(a+b) - T(-a-b)]B\phi,$$

for every $\phi \in \mathcal{D}(a, b)$. From this formula we see that (S) implies the following:

(S) \quad \|A\phi\|_{\mathcal{L}^2(a,b)} \leq c\|\phi\|_{\mathcal{L}^2(a,b)}, \quad c = \text{const}, \quad \text{for every} \quad \phi \in \mathcal{D}(a, b).

However, (S) is not true. Indeed, for every $\epsilon > 0$, let $\phi \in \mathcal{D}(a, b)$ have the following properties:

$$\sup_{u \in (a, a+3\epsilon)} \phi(u) \leq \frac{1}{\epsilon}, \quad \phi(u) = \frac{1}{\epsilon} \text{ for } u \in (a+\epsilon, a+2\epsilon).$$

Then

$$\|\phi\|_{\mathcal{L}^2(a,b)} \leq 3 \leq \|A\phi\|_{\mathcal{L}^2(a,b)}$$

and, on the other hand, if $s \in (a+3\epsilon, b)$ then

$$(A\phi)(s) = \frac{1}{\pi} \int_a^b \frac{\phi(u)}{s-u} du \geq \frac{1}{\pi} \int_{a+2\epsilon}^b \frac{\phi(u)}{s-u} du = \frac{1}{\pi \epsilon} \log(|s-a-\epsilon|) \geq \frac{1}{\pi \epsilon} \log(s-a-2\epsilon) \geq \frac{1}{\pi \epsilon} \log(\frac{s-a}{s-a-\epsilon})$$

so that

$$\|A\phi\|_{\mathcal{L}^2(a,b)} \geq \frac{1}{\pi \epsilon} \int_{a+3\epsilon}^b \frac{1}{s-a-\epsilon} ds - \frac{1}{\pi \epsilon} \log(s-a) \geq \frac{1}{\pi \epsilon} \int_{a+3\epsilon}^b \frac{1}{s-a-\epsilon} ds - \frac{1}{\pi \epsilon} \log(s-a).$$

5. The spaces $C_{m,0}(-\infty, \infty)$, $C_{a}(-\infty, \infty)$, $C_{b}(-\infty, \infty)$, and $L^m(-\infty, \infty)$ do not have property (E). For $C_{a}(-\infty, \infty)$ and $C_{b}(-\infty, \infty)$ this follows easily from the results of the preceding section and from Lemma 4 of Section 3. Indeed, the adjoint space of $C_{b}(-\infty, \infty)$ with the $m$-weak topology may be represented as $M_{b}(-\infty, \infty)$ with the topology of weak convergence of measures, in the sense that any $m \in M_{b}(-\infty, \infty)$ defines a continuous linear form on $C_{b}(-\infty, \infty)$ according to the formula

$$m(x) = \int_{-\infty}^{0} x(s)m(ds), \quad x \in C_{b}(-\infty, \infty).$$

In this representation, for any $t \in (-\infty, \infty)$ the operator adjoint to $\varphi_{t}(x) \in \mathcal{L}^m(M_{b}(-\infty, \infty))$ equals again $\varphi_{t}(x)$, but now viewed upon as an element of $\mathcal{L}^m(M_{b}(-\infty, \infty))$. Therefore, if a one-parameter group should have an exponential representation for $\varphi_{t} \in \mathcal{L}^m(M_{b}(-\infty, \infty))$, then the corresponding group of adjoint operators would give an exponential representation for $\varphi_{t} \in \mathcal{L}^m(M_{b}(-\infty, \infty))$, contrary to the result of the preceding section.

The adjoint space of $C_{a}(-\infty, \infty)$ with the $m$-weak topology may be represented as the direct sum $M_{a}(-\infty, \infty) + C$, where $C$ is the field of complex numbers and $M_{a}(-\infty, \infty)$ is equipped with the topology of weak convergence of measures, in the sense that any element $m + \lambda \cdot M_{b}(-\infty, \infty) + C$ defines a continuous linear form on $C_{a}(-\infty, \infty)$ according to the formula

$$\langle m + \lambda \rangle(x) = \int_{-\infty}^{\infty} x(s)(m(ds) + \lambda \lim_{s \to \infty} \varphi_{t}(x)(ds), \quad x \in C_{a}(-\infty, \infty).$$

In this representation, we have for operators $\varphi_{t}(x) \in \mathcal{L}^m(M_{b}(-\infty, \infty))$,

$$\langle \varphi_{t}(x)(m + \lambda) \rangle = \int_{-\infty}^{\infty} \varphi_{t}(x)(m(ds) + \lambda \lim_{s \to \infty} \varphi_{t}(x)(ds), \quad x \in C_{a}(-\infty, \infty).$$

so that

$$\langle \varphi_{t}(x)(m + \lambda) \rangle = \varphi_{t}(x)(m + \lambda)$$

for every $m + \lambda \in M_{b}(-\infty, \infty) + C$ and $t \in (-\infty, \infty)$, where $\varphi_{t}(x)$ is treated as an element of $\mathcal{L}^m(M_{b}(-\infty, \infty))$. Moreover, according to the statement (ii) of Section 4, the infinitesimal generator $A_{m}$ of $\varphi_{t}(x)$ in $\mathcal{L}^m(M_{b}(-\infty, \infty))$ is the operator of the second derivative in the sense of distributions, defined on the set $A_{m} = \{m; \text{m is bounded}\}$. If $m \in A_{m}$ and $m'' = A_{m}m = 0$ then the density of $m$ with respect to the Lebesgue measure is a linear function. But since $m$ is a bounded
measure, this is possible only if \( m = 0 \). Therefore \( A_a \) is an invertible operator.

Now suppose that a one-parameter group gives an exponential representation for \( \mathcal{G}_t(a) \) in \( \mathcal{L}(C_{\text{loc}}(\mathbb{R})) \). Then the corresponding group of adjoint operators would give an exponential representation for \( \mathcal{F}_t(a) \) in \( \mathcal{L}(\mathcal{R}(\mathbb{R}) \to C) \). But then, since the operators \( \mathcal{F}_t(a) \) have the form (4) and since \( A_a \) is invertible, by Lemma 4 of Section 3, \( \mathcal{F}_t(a) \) would have an exponential representation in \( \mathcal{L}(\mathcal{R}(\mathbb{R}), \infty, \infty) \), which is impossible, as we already know from Section 4.

The above method of proof was suggested to the author by professor C. Byll-Narkiewicz, whose suggestion was also that \( \mathcal{G}(\mathbb{R}, \infty, \infty) \) with the *-weak topology may be treated similarly to \( \mathcal{L}(\mathbb{R}, \infty, \infty) \).

A direct approach to \( C_0(-\infty, \infty) \) (without a use of adjoint operators), which was a former idea of the author, is more complicated. However, this direct proof works without any further additional complications also for \( C(-\infty, \infty), C_b(-\infty, \infty) \) and \( \mathcal{L}^p(-\infty, \infty) \). We shall present it for the last two spaces.

Proof of the Fact that \( C_0(-\infty, \infty) \) and \( \mathcal{L}^p(-\infty, \infty) \) do not have property (E). We admit the convention that \( X \) always denotes \( C_0(-\infty, \infty) \) or \( \mathcal{L}^p(-\infty, \infty) \) and that any statement concerning \( X \) should be considered as a statement true for \( X = C_0(-\infty, \infty) \) and for \( X = \mathcal{L}^p(-\infty, \infty) \) simultaneously. Let us recall that \( C_0(-\infty, \infty) \) is considered with the norm topology, while \( \mathcal{L}^p(-\infty, \infty) \) is regarded as the adjoint space of \( \mathcal{L}(\mathbb{R}, \infty, \infty) \) with the *-weak topology. The Fourier transforms of elements of \( X \) are temperated distributions, i.e., they are elements of the space \( \mathcal{S} \) of L. Schwartz.

Let \( H \) denote the set of all finite linear combinations of Hermite functions \( \varphi_n(x) = \frac{1}{\sqrt{n!}} \exp(-x^2) \), \( n = 0, 1, \ldots \). We shall need the fact that

(I) \( H \) is a dense subset of the space \( \mathcal{S} \).

Here \( \mathcal{S} \) is the space of L. Schwartz of indefinitely differentiable rapidly decreasing functions.

Let \( S \in \mathcal{S} \) and \( \langle S, \varphi \rangle = 0 \) for every \( \varphi \in H \). Then (I) will be proved if we show that \( S = 0 \). Let \( \varphi_n(u) = u^n \exp(-u^2) \), and for every real \( s \) the series \( \sum_{n=0}^{\infty} \frac{(-is)^n}{n!} \varphi_n \) converges in the sense of the topology of \( \mathcal{S} \), so that

(a) \[ \langle S, e^{-is^2} \rangle = \sum_{n=0}^{\infty} \frac{(-is)^n}{n!} \langle S, \varphi_n \rangle = 0, \]

where \( S \) acts onto \( e^{-is^2} \) as a function of \( u \).

If \( s = s_{1,n} < s_{2,n} < \ldots < s_{m,n} = b, m = 1, 2, \ldots \), is a nor-

mal sequence of partitions of the interval \([a, b]\), then the Riemann sums

(b) \[ \sigma_n(u) = \sum_{k=1}^{n} \varphi(\epsilon_{k,n} e^{-is_{k,n}^2} (s_{k,n} - s_{k-1,n}), n = 1, 2, \ldots, \]

form a sequence of functions of \( u \), such that, for every fixed \( l = 0, 1, \ldots, \)

the sequence \( \frac{d\sigma_n(u)}{du}, n = 1, 2, \ldots, \) is bounded and converges to

\[ \frac{d(\mathcal{F}(\varphi))(u)}{du}, \]

almost uniformly in \( u \) on \( (-\infty, \infty) \). Therefore \( \tau_\varphi(u) = e^{-is} \varphi(u), n = 1, 2, \ldots, \)

is a sequence of functions of \( u \), converging to \( e^{-it}\mathcal{F}(\varphi)(u) \) in the sense of the topology of \( \mathcal{S} \). Moreover, by (a) and (b), \( \langle S, \tau_\varphi \rangle = 0 \) and \( \langle S, e^{-is}\mathcal{F}(\varphi) \rangle = \lim_{n \to \infty} \langle S, \tau_\varphi \rangle = 0 \). Since \( e^{-is} S \in \mathcal{S} \) for all \( s \) and \( \mathcal{S}(-\infty, \infty) \) is dense in \( \mathcal{S} \), it follows that \( e^{-is} S = 0 \) as an element of \( \mathcal{S} \) and therefore also as an element of \( \mathcal{S}(-\infty, \infty) \). It follows that \( S = 0 \) as an element of \( \mathcal{S}(-\infty, \infty) \). But \( S \in \mathcal{S} \) and \( \mathcal{S}(-\infty, \infty) \) is dense in \( \mathcal{S} \), and therefore \( S = 0 \) as an element of \( \mathcal{S} \).

The statement (I) is proved.

Similarly as in Section 4, it may be proved that if \( A_a \) is the infinitesimal generator of \( \mathcal{L}(X_{\text{mbox} b}) \)-valued cosine function

\[ \mathcal{G}_t(a) = \frac{1}{2} \left( [\mathcal{T}(t) + \mathcal{T}(-t)] \right) X_{\text{mbox} b} \]

then

\[ \mathcal{D}(A_a) = \{ x \in X_{\text{mbox} b} : e^{t \cdot x_{\text{mbox} b}} \}

\]

where \( x' \) denotes the second distributional derivative of \( x \).

After this preparation suppose that \( \mathcal{S}(t); -\infty < t < \infty \subset \mathcal{S}(X_{\text{mbox} b}) \) is a one-parameter continuous group such that

\[ \langle \mathcal{F}(\varphi), \varphi \rangle = \frac{1}{2} \left( \mathcal{T}(t) + \mathcal{T}(-t) \right) \varphi, \]

for every \( t \in (-\infty, \infty) \). An investigation of the structure of this group will lead us to a contradiction.

For any real \( t \) let the distribution \( \xi_t \in \mathcal{D}'(0, \infty) \) be defined by the equality

\[ \langle \mathcal{F}(\xi_t), \varphi \rangle = \langle \mathcal{F}(\varphi)(t), \varphi \rangle_{(0, \infty)}, \]

This is a correct definition because \( \langle \mathcal{F}(\varphi)(t), \varphi \rangle_{(0, \infty)} = \frac{1}{\sqrt{2}} e^{-\frac{1}{4}t^2} \) is positive and infinitely differentiable in \((0, \infty)\). Both terms in this equality are elements of \( \mathcal{S}'(0, \infty) \).

**Lemma A1.** For every \( x \in \mathcal{L}(X_{\text{mbox} b}) \) and every \( t \in (-\infty, \infty) \) we have

\[ \langle \mathcal{F}(\xi), \varphi \rangle_{(0, \infty)} = \langle \mathcal{F}(\varphi)(t), \varphi \rangle_{(0, \infty)}. \]
Proof. By an argument to that similar used at the beginning of the proof of Lemma A in Section 4, it follows that Lemma A1 is true for every \( x \in \mathcal{H}_{\text{imp}} \). Let now \( x \in \mathcal{H}_{\text{imp}} \). Then, by (I), there is a sequence \( \{x_n\}_n \) and elements of \( \mathcal{H}_{\text{imp}} \) converging to \( x \) in the sense of topology of the space \( \mathcal{S} \). Now, as it is easy to see, \( \langle F \xi \rangle_{n \to 0} = \lim_{n \to \infty} \langle F \xi \rangle_{x_n \to x} = \lim_{n \to \infty} \langle F \xi \rangle_{x_n \to x} = \lim_{n \to \infty} \langle F \xi \rangle_{x_n \to x} \) in the sense of convergence in \( \mathcal{S} \).

**Lemma A2.** For every \( t \in (-\infty, \infty) \) the distribution \( g_t \in \mathcal{D}'(0, \infty) \) is a function continuous in \((0, \infty)\).

**Proof.** It is sufficient to show that for every \( \phi \in \mathcal{D}(0, \infty) \) the distribution \( g_t \) is equal to \( \phi \) on \((0, \infty)\) and zero on \((-\infty, 0)\). For a fixed \( \phi \in \mathcal{D}(0, \infty) \) let \( h \in \mathcal{S}_{\text{imp}} \) be such that \( \langle \mathcal{F}h \rangle_{(a, \infty)} = 1 \) for every \( a \in \mathcal{S}_{\text{imp}} \). Let \( \phi \in \mathcal{D}(-\infty, 0) \) be equal \( \phi \) on \((0, \infty)\) and zero on \((-\infty, 0)\). For any \( \psi \in \mathcal{C} \) let \( x_\psi \) and \( x_\psi \) denote respectively the pair and the imp part of \( x \).

Then
\[
\langle F^{-1}g_t, \phi \rangle = \langle \langle F^{-1}g_t, Fh \phi \rangle \rangle = \langle \langle F^{-1}g_t, Fh \phi \rangle \rangle
\]
\[
= \langle \langle F^{-1}g_t, Fh \phi \rangle \rangle = \langle \langle F^{-1}g_t, Fh \phi \rangle \rangle
\]
\[
= \langle \langle F^{-1}g_t, Fh \phi \rangle \rangle = \langle \langle F^{-1}g_t, Fh \phi \rangle \rangle
\]
\[
\leq \|Fh\|_{L^1(-\infty, 0)} \sup_{s \in \mathbb{R}} |\phi(s)|
\]

which proves that \( g_{F^{-1}} \) is a bounded measure on \((-\infty, \infty)\). We must only make clear that, in the case of \( X = \mathcal{K}(\mathcal{F})(-\infty, \infty) \), \( \|\cdot\|_{L^1} \) should be understood as ess sup and \( \|\cdot\|_{L^1(X_{\text{imp}})} \) as the corresponding norm for operators. Since \( \mathcal{F} \) is continuous in the s-weak topology, it is also continuous in the norm topology, so that \( \|\mathcal{F}h\|_{L^1(X_{\text{imp}})} < \infty \).

On account of Lemma A2, it is convenient to extend \( g_t \) to a pair function on \((-\infty, 0)\), whose value at 0 is 1. Henceforth by \( g_t \) we shall mean such an extended function. Let \( Z_{\text{imp}} \) denote the set of all imp functions continuous in \((-\infty, \infty)\) with compact supports not containing zero. Let \( Y_{\text{imp}} = \{x \in \mathcal{S}_{\text{imp}} : x \in Z_{\text{imp}}\} \). Then clearly \( Y_{\text{imp}} \subseteq \mathcal{S}_{\text{imp}} \).

**Lemma A3.** \( \mathcal{F} \mathcal{Y}_{\text{imp}} \subseteq \mathcal{Y} \) and \( \mathcal{F} \mathcal{Y}(t) = g_t \mathcal{F} \mathcal{Y} \) for every \( t \in (-\infty, \infty) \) and \( x \in \mathcal{Y}_{\text{imp}} \).

**Proof.** If \( x \in \mathcal{Y}_{\text{imp}} \), then also \( \mathcal{F}^{-1}(g_t \mathcal{F} x) = \mathcal{Y}_{\text{imp}} \), and therefore in Lemma A3 the inclusion follows from the equality. Let \( x \in \mathcal{Y}_{\text{imp}} \). Then there are positive numbers \( a, b \) with \( b > a \), and a sequence \( \{a_n\}_n \) for every \( \mathcal{H}_{\text{imp}} \), such that \( a_n \to a \mathcal{F} \) uniformly on \((-\infty, \infty)\) as \( n \to \infty \) and \( \sup_{x \mathcal{S}} \|x\|_{\mathcal{S}_{\text{imp}}} \mathcal{S} \subseteq [b, a] \) \( \{F(t)\} \). Put \( a_n = \mathcal{F} x_n \). Then \( x_n \in \mathcal{H}_{\text{imp}} \) and \( x_n \to x \) uniformly on \((-\infty, \infty)\) as \( n \to \infty \), which implies that \( \mathcal{F} \mathcal{Y}(t) = \mathcal{Y} \) in the sense of convergence in \( \mathcal{Y} \). Consequently, \( \mathcal{F} \mathcal{Y}(t) = \mathcal{Y} \) in the sense of convergence in \( \mathcal{Y} \). On the other hand, the sequence \( g_t \mathcal{F} x_n \to g_t \mathcal{F} x \) in the sense of topology in \( X \), and consequently \( \mathcal{F} \mathcal{Y}(t) = \mathcal{Y} \), in the sense of convergence in \( \mathcal{Y} \).

By Lemma A1, for every \( n = 1, 2, \ldots \) we have \( \mathcal{F} \mathcal{Y}(t) = g_t \mathcal{F} x, \) and \( \mathcal{Y} \mathcal{Z}_{\text{imp}} \) is an imp expansion with the one-point support at zero. Therefore \( \mathcal{F} \mathcal{Y}(t) = g_t \mathcal{F} x = \sum_{2}^{\infty} C_n \mathcal{F} \mathcal{Z}_{\text{imp}} \), so that the function \( \mathcal{F} \mathcal{Y}(t) = g_t \mathcal{F} x \) is measurable, bounded on \((-\infty, \infty)\), equals to an imp function polynomial. This is possible only if this polynomial vanishes identically, i.e., only if \( \mathcal{F} \mathcal{Y}(t) = g_t \mathcal{F} x \).

Our further considerations follow very closely the reasoning of Section 4.

Lemma B may be transported without any change to the present section. The only difference in the proof is that instead of Lemma A now we use Lemma A1 and that the proof of the fact that (c) for every fixed \( u \in (-\infty, \infty) \), \( g_t(u) \) is a function of \( t \) measurable on \((-\infty, \infty)\) must be a modified. Now we prove (c) as follows. Given a fixed \( u \neq 0 \), we take a function \( x \in \mathcal{H}_{\text{imp}} \) such that \( \mathcal{F} \mathcal{Y}(t) = u \) and we take a \( \delta \)-like sequence \( \{y_n\}_n \), \( n = 1, 2, \ldots \), of non-negative functions in \( \mathcal{S} \).

Then
\[
\mathcal{F} \mathcal{Y}(t) = \{y_n \mathcal{F} \mathcal{Y}(t) \mathcal{Y}(x) \}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F} \mathcal{Y}(t) \mathcal{Y}(x) \mathcal{Y}(t)(u) \, dt
\]

\[
\sup_{x \mathcal{S}_{\text{imp}}} |\mathcal{F} \mathcal{Y}(t) \mathcal{Y}(x) \mathcal{Y}(t)(u)|
\]

\[
0 \leq x \mathcal{S}_{\text{imp}} \subseteq [b, a] \mathcal{S}_{\text{imp}}
\]

\[
\lim_{t \to \infty} \mathcal{F} \mathcal{Y}(t) \mathcal{Y}(x) \mathcal{Y}(t)(u) = g_t(u) \mathcal{F} x
\]

Lemma C together with its proof may be transported to the present section without any change. By means of an argument similar to that used in Section 4, this lemma implies the following statement:

(S1) For every \( t \in (-\infty, \infty) \) there is \( g_t \) such that

\[
\sup_{x \mathcal{S}_{\text{imp}}} \mathcal{H} \mathcal{Y}(t) \mathcal{Y}(x) \mathcal{Y}(t)(u) \leq g_t \sup_{x \mathcal{S}_{\text{imp}}} |\mathcal{Y}(x)|
\]

for every \( x \in \mathcal{Y}_{\text{imp}} \).
Further, for arbitrarily fixed positive a and b, b > a, by a similar reasoning as in Section 4, (S1) implies the following statement

\[ \sup_{a \in (0, 1)} \left| \int_V \frac{x(u)}{u - \omega} \, du \right| \leq \text{const} \cdot \sup |x(u)| \text{ for every } x \in D(a, b). \]

Now the whole indirect proof is completed by showing that (S1) is not true. Indeed, if \( \tilde{a} \in \left(0, \frac{b-a}{2}\right) \) and \( \tilde{a} \in D(a, b) \) is such that \( 0 < |\tilde{a}(s)| \leq 1 \) for \( s \in (a, b) \), and that \( \tilde{a}(s) = 1 \) for \( s \in (a+\epsilon, b-\epsilon) \), then

\[ \int_a^b \frac{x(u)}{u - \omega} \, du \geq \int_{a+\epsilon}^{b-\epsilon} \frac{du}{u - \omega} = \log \left( \frac{b-a}{\epsilon} \right) - 1. \]

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On uniform symmetrization of analytic matrix functions

by

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Abstract. Let \( A \) be a real-analytic function of \( \xi \) on open set \( M = \mathbb{R}^n \), which values \( A(\xi) \) are \( m \times m \) matrices with purely diagonal and real canonical Jordan form. If the characteristic roots of \( A(\xi) \) are restricted to change their multiplicities only in a suitable, very simple manner, then for every \( \xi \in M \) we construct a hermitean positive \( m \times m \) matrix \( H(\xi) \), such that \( H(\xi):A(\xi) \) is hermitean and that \( |H(\xi)| \) and \( |H^{-1}(\xi)| \) are locally bounded functions of \( \xi \).

1. The result. Let \( A \) be a function defined on a set \( M \), which values are \( m \times m \) complex matrices. We shall say that \( A \) is uniformly symmetrizable on \( M \) if the following condition is satisfied:

there is a constant \( c \geq 1 \), such that for every \( \xi \in M \) there is a hermitean \( m \times m \) matrix \( H(\xi) \), such that \( c^{-1} \leq H(\xi) \leq c \) and that \( H(\xi)A(\xi) \) is hermitean.

According to Kreiss [2], [3], the uniform symmetrizability of \( A \) on \( M \) is equivalent to either of the following conditions:

there is a constant \( d \geq 1 \), such that for every \( \xi \in M \) there is an singular \( m \times m \) matrix \( T(\xi) \), such that \( \|T(\xi)\| \leq d \), \( \|T^{-1}(\xi)\| \leq d \) and that \( T^{-1}(\xi)A(\xi)T(\xi) \) is purely diagonal and real;

\( \mathrm{sup} \left( \left| \exp(itA(\xi)) \right| ; \xi \in (-\infty, \infty), \xi \in M \right) < \infty; \)

\( \mathrm{sup} \left( \left| (E-itA(\xi))^{-1} \right| ; \xi \in (-\infty, \infty), \xi \in M \right) < \infty \), where \( E \) denotes the unit \( m \times m \) matrix.

The theorem, which we state below may be treated as a contribution to the following problem. Let \( A \) be a matrix-valued function on a set \( M \) and suppose that \( A(\xi) \) is symmetrizable for every fixed \( \xi \in M \). Under which additional conditions \( A \) is uniformly symmetrizable on \( M \)? Our additional conditions have the form of restrictions on the behaviour of characteristic roots of \( A(\xi) \) near the points of branching. We consider only the simplest case, when two roots come together.

Theorem. Let \( M \subset \mathbb{R}^n \) be open and let \( A \) be an analytic function on \( M \), which values are \( m \times m \) complex matrices. Suppose that for every \( \xi \in M \) the matrix \( A(\xi) \) has purely diagonal and real canonical Jordan form. Moreover,