Examples of nuclear systems

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Abstract. The purpose of this paper is to present a detailed study of some concrete examples of nuclear systems, whose general theory has been presented by the author in previous papers. There are three classes of examples. First we consider nuclear systems generated either by a sequence of commuting normal operators or a sequence of permutations. Next, certain matrices which are zero everywhere except on the main diagonal and the diagonal directly above it are considered. Finally a very simple type of lower triangular matrix is discussed. In most cases it is shown that the resulting nuclear Fréchet space has a Schauder basis, but an example is constructed in which all of our methods fail to yield a basis.

The theory of nuclear systems (insofar as it has been developed) was presented in [1], [2]. This theory provides a method of constructing nuclear Fréchet spaces which in principle produces all such spaces whose topology is defined by norms and in practice permits the construction of examples not previously studied. Moreover, several criteria for the existence of Schauder bases have been established. It is the purpose of this paper to study in detail some of the examples of nuclear Fréchet spaces provided by nuclear systems, in most cases proving the existence of a Schauder basis.

We recall now the definitions and results which will be used. Proofs and further explanations are to be found in [1] and [2].

A nuclear system is a sequence $(A_n)$ of injective nuclear operators in $I_n$ with dense range. The associated space, written

$$\tilde{E} = \{ (x_k): x_k \in I_k, x_k = A_k(x_{k-1}), k = 1, 2, \ldots \},$$

is a subset of the countable product of copies of $I_k$ so it may be equipped with the subspace topology whence it becomes a nuclear Fréchet space

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whose topology is defined by a sequence of norms, and all such spaces are obtained in this way ([1], p. 372). The projection operators, \( P_k : E \to E \), \( k = 1, 2, \ldots \) are defined by setting \( P_k(x) = x_k \). Each \( P_k \) is continuous, linear, and has dense range ([1], p. 372). We define \( B_k \) to be the identity map on \( E_k \) and we set \( B_k = A_1 \cdots A_k, \ k = 1, 2, \ldots \).

We recall that a Schauder basis in a topological vector space \( E \) is a sequence \((b_n) \) with the property that for each \( x \in E \) there is a unique sequence \((z_n) \) of scalars such that \( x = \lim_{n \to \infty} \sum_{k=1}^{n} z_k b_k \). We shall call a sequence \((b_n) \) in a topological vector space total if the vector subspace it generates is dense.

For each positive integer \( n \) we denote by \( e^n \in l_2 \) the sequence which is \( 1 \) at the \( n \)th term and \( 0 \) elsewhere and by \( H_n \) the projection of an element of \( l_2 \) onto its \( n \)th coordinate. We denote by \( g \) the subset of \( l_2 \) consisting of those sequences all but finitely many of whose terms are \( 0 \).

A continuous linear map \( D : l_2 \to l_2 \) is a diagonal map with diagonal element \( (l_k) \) if \( D(e^n) = l_n e^n, \ n = 1, 2, \ldots \). The identity map on \( l_2 \) will be denoted by \( I \).

The following results from [1], [2] will be used quite often throughout the paper, so we quote them here for easy reference. Proposition A is essentially proved in [1], p. 372 and appears in the following revised form as Theorem 3 in [2]. Propositions B, C are proved as Propositions 4, 5 respectively in [2].

**Proposition A.** The associated space of a nuclear system \((A_k)\) has a Schauder basis if and only if there exist diagonal nuclear maps \( D_k : l_2 \to l_2, \ k = 1, 2, \ldots \) such that

(i) \( A_k D_{k+1} = f_k D_k, \ k = 1, 2, \ldots \)

(ii) \( f_k \) maps \( \bigcap_{k=1}^{\infty} D_k(l_2) \) into \( A_1 \cdots A_k(l_2) \).

**Proposition B.** If \((A_k)\) is a nuclear system, then \( \hat{E} \) has a Schauder basis if and only if there exists a linear injective map \( S : \varphi \to \bigcap_{k=1}^{\infty} B_k(l_2) \) with \( B_k^{-1} S(\varphi) \) dense in \( l_2 \) for each \( k \geq 0 \) and such that for each \( k \geq 0 \) there exists \( j \geq k \) such that

\[
\sup_{\varphi} \| B_k^{-1} S(\varphi) S^{-1} B_j \|_{B_k^{-1} \mathcal{H}(E)} < \infty.
\]

In this case, if we consider \( \hat{E} \) to be represented by \( \bigcap_{k=1}^{\infty} B_k(l_2) \) via \( P_1(\hat{E}) \), then the basis is the sequence \((S(\varphi))\).

**Proposition C.** Let \((A_k)\) be a nuclear system and \((b_k)\) a total, linearly independent sequence in \( E \). Then \((b_k)\) is a Schauder basis for \( E \) if and only if for each \( k \geq 0 \), there exists \( j \geq k \) such that

\[
\sup_{\varphi} \| B_k^{-1} S(\varphi) S^{-1} B_j \|_{B_k^{-1} \mathcal{H}(E)} < \infty,
\]

where \( S : \varphi \to l_2 \) is defined by \( S(\varphi^n) = f_k b_k \).

In Section 1, we consider nuclear systems with the property that the eigenvectors of each \( A_k \) can be easily computed and have a relatively transparent dependence on \( k \). In Section 2, we consider nuclear systems generated by a single operator, \( A_1 \), that is, \( A_k = A \) for all \( k \), where \( A \) is a matrix whose terms are \( 0 \) everywhere except the main diagonal and the diagonal just above it. We are able to give sufficient conditions for the existence of a basis and also construct an example in which two methods for obtaining a basis fail. This leads to an example of a Markuschevich basis in a Fréchet nuclear space which is not a Schauder basis.

In Section 3 we consider \( \mathcal{A}_k \) to be a matrix which is \( 0 \) except on the main diagonal and the first \( N \) columns (\( N \) independent of \( k \)). Here \( \hat{E} \) always has a basis.

1. Normal operators and permutations. In the next section we give a generalization of the result in [4]. The idea is that if the operators in a nuclear system all have the same set of eigenvectors, then this set can be used to construct a basis for the associated space.

**Proposition 1.** Let \((A_k)\) be a nuclear system in which each \( A_k \) is normal and \( A_k A_{k+1} = A_{k+1} A_k \) for all \( k \). Then the associated space possesses a Schauder basis.

**Proof.** Let \( \lambda_k \) be an eigenvector of \( A_k \) with eigenspace \( E_k \) which is finite-dimensional since \( A_k \) is compact. If \( \varphi \in E_k \) then \( A_k A_{k+1}(\varphi) = A_{k+1} A_k(\varphi) = \lambda_k A_{k+1}(\varphi) \). Thus \( A_{k+1}(E_k) \subset E_{k+1} \) and \( A_{k+1} E_k \) is a normal operator on \( E_k \) so we can choose a maximal eigenspace \( E_k = E_k \) whose dimension is positive. Repeating the process indefinitely, we obtain a decreasing sequence \((E_k)\) of finite dimensional spaces with positive dimension and hence there exists \( k \) such that \( E_k = E_{k+1} = \cdots = E_{k+s} \) for all \( k \geq k_0 \). It then follows that an orthonormal basis for \( E_k \) is a non-empty set whose elements are eigenvectors for each \( A_k, \ k = 1, 2, \ldots \).

The process can be repeated a number of times at most equal to the dimension of \( E_k \). We obtain an orthonormal basis for \( E_k \) whose elements are eigenvectors for each \( A_k \). Again repeating for each eigenvalue of \( A_k \), we obtain an orthonormal basis \((\varphi_k)\) for each \( k \) such that each \( \varphi_k \) is an eigenvector of each \( A_k \).

Finally, define \( S : l_2 \to l_2 \) by \( S(\varphi^n) = \varphi_k \) and apply Proposition 1. Clearly \( S(\varphi_1) \in B_1(l_2) \) and \( B_k^{-1} S(\varphi) \) is dense in \( l_2 \) for each \( k \). Moreover
we have $A_k = SD_kS^*$ where $D_k$ is a diagonal operator so $B_k = SD_k \ldots D_k S^*$ and so we have,
$$\sup_{k}||B_k|| = \sup_{k}||SD_k D_k \ldots D_k S^*|| = \sup_{k}||SD_k D_k \ldots D_k|| < ||S|| = 1$$
so Proposition B applies to yield the desired result. 

In view of Proposition 1 one might try to construct a nuclear Fréchet space without a basis by making the eigenvectors of the maps $A_k$ different for each $k$. This is perhaps also suggested by the proof of Proposition 2 of (2). However this does not seem to work as is indicated by the next result in which we consider cases in which the eigenvectors are quite different.

**Proposition 2.** Let $(e^i)$ be a sequence of permutations of the natural numbers and let $(e^i)$ be a sequence of elements of $l_1$ each of which has no 0 terms. Define $A_k: l_1 \rightarrow l_1$ by
$$A_k(e^i) = a_k^i e^i, \quad n, k = 1, 2, \ldots$$
Then $(A_k)$ is a nuclear system whose associated space has a Schauder basis.

**Proof.** It is clear that $(A_k)$ is a nuclear system. To see that it has a basis, we apply Proposition A. Let $\Gamma = e^1 \ldots e^n$, $e^n$ is identity and define diagonal operators $D_k: l_1 \rightarrow l_1$ and continuous operators $f_k: l_1 \rightarrow l_1$ by
$$D_k(e^{i_1}) = a_k^{i_1} e^{i_1}, \quad f_k(e^{i_1}, \ldots e^{i_n}) = e^{i_1}, \quad k, n = 1, 2, \ldots$$
Then we have
$$A_k f_{k+1}(e^{i_1}) = A_k(e^i) = a_k^i e^{i_1} = a_k^i f_k(e^{i_1}, \ldots e^{i_n}) = f_k(a_k^i e^{i_1}) = f_k(D_k(e^{i_1})), $$
so that $A_k f_{k+1} = f_k D_k$, $k = 1, 2, \ldots$

Finally, define $\gamma = \epsilon_2 \ldots \epsilon_n \ldots$ by $\gamma^{i_1} = a_k$, so that $D_k(l_1) = \gamma^{i_1} \gamma^{i_2} \gamma^{i_3} \gamma^{i_4} \ldots$.

Then we have,
$$A_1 \ldots A_k(e^i) = a_1^{i_1} a_2^{i_2} \ldots a_k^{i_k} e^{i_1} = \gamma^{i_1} \gamma^{i_2} \gamma^{i_3} \gamma^{i_4} \ldots \gamma^{i_k} e^{i_1}.$$ 

Hence, using the fact that $l_1$ is invariant under permutations,
$$A_1 \ldots A_k(l_1) = \sum_{n=1}^{\infty} \xi_n A_1 \ldots A_k(e^i): \xi = (\xi_i) \in l_1$$
$$= \sum_{n=1}^{\infty} \xi_n \beta_1^{i_1} \beta_2^{i_2} \ldots \beta_k^{i_k} e^{i_1} = \xi e^{i_1}$$
$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \xi_n \beta_k^{i_n} \beta_k^{i_n} e^{i_1} = \xi e^{i_1} = \beta_1^{i_1} \ldots \beta_k^{i_k}.$$ 

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Hence, $f_k(l_1) = D_k(l_1) = \sum_{n=1}^{\infty} A_1 \ldots A_k(l_1) = \sum_{n=1}^{\infty} A_1 \ldots A_k(l_1)$ so the conditions of Proposition A are satisfied and we may conclude that the associated space has a basis.

**Remark.** A slight improvement of Proposition A is possible. If we assume that each $A_k$ is similar to a diagonal matrix via the same similarity transformation, that is, there exists an isomorphism $S: l_1 \rightarrow l_1$ such that $S^{-1} A_k S$ is a diagonal matrix, then the last half of the proof of Proposition 1 will still work showing that $B(A_k)$ is a basis.

2. $\mu - \nu$ matrices. Let $\mu, \nu$ be elements of $l_1$ with $0 < |\mu_n| \leq |\nu_n|$ and define $A: l_1 \rightarrow l_1$ by
$$A(e^i) = \mu e^i, \quad A(e^i) = \mu e^i - \nu e^{i-1}, \quad n > 1.$$ 

Then $A(x) = 0$ if and only if
$$x_{n+1} = \frac{\mu_n}{\nu_n} x_n = \ldots = \frac{\mu_n}{\nu_n} \ldots \frac{\mu_1}{\nu_1} x_1,$$
so that $|\mu_n| > |\nu_n|$ for all $n$ so if $x \in l_1$, then $x = 0$. Thus $A$ is injective. Moreover, it is obvious that $A(\varphi) = \varphi$ so $A$ has dense range. Thus $A$ generates a nuclear system. We now wish to study the existence of a basis with various additional restrictions on $\mu, \nu$.

**Proposition 3.** Let $\mu = \nu$ and suppose that for each $k > 0$, $(|\mu|^2)^{\nu_k}$ is unbounded. Let $(b_k)$ be the sequence in $\hat{E}$ defined by taking $b_k = (b_k)_k$, where $b_k = A^{k+1}(e^i)$, $n, k = 1, 2, \ldots$ Then $(b_k)$ is a total, linearly independent sequence in $\hat{E}$ which is not a Schauder basis for $E$.

**Proof.** Since $A(\varphi) = \varphi$ it follows that the given expression establishes $(b_k)$ as a sequence in $\hat{E}$. If
$$\sum_{k=1}^{\infty} b_k \xi_k = 0$$
then it follows by taking $k = 1$ and evaluating the sum of sequences at its first coordinate \( 1 \leq j \leq n \),
$$\sum_{k=1}^{\infty} b_k \xi_k = 0$$
which implies that $\xi_1 = \ldots = \xi_n = 0$. Hence $(b_k)$ is linearly independent. Moreover, for each $k$, the vector subspace generated by $(b_k)_{k=1}^{\infty}$ is $A^{k+1}(\mathcal{V}) = \mathcal{V}$ which is dense in $l_1$ so it follows that $(b_k)$ is total in $E$.

To show that $(b_k)$ is not a basis we apply Proposition 4. Clearly we have $B_k = A_k^1$, $S$ is the identity and $B_k^{-1} S(\varphi) = \varphi$ so we must show that for some $k < j$ that by choosing a suitable index $p$ we can make $\|A^{k-j} I_p A^j \| = 0$.
arbitrarily large. We take \( k = 1 \) and we compute the matrix representation,
\[
A^{-1} B A = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & -1 \\
0 & 1 & 0 & \ldots & 0 & -1 \\
0 & 0 & 1 & \ldots & 0 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 1 & -1 \\
0 & \ldots & 0 & 0 & 1 & -1 \\
\end{pmatrix}
\]
or, expressed in more compact form,
\[
A^{-1} B A (e^i) = \begin{cases} 
\varepsilon^i, & n \leq p, \\
- (\varepsilon^1 + \ldots + \varepsilon^n), & n = p + 1, \\
0, & n > p + 1.
\end{cases}
\]
From this it follows that \( \| A^{-1} B A \|_2 \geq \| \varepsilon^1 + \ldots + \varepsilon^n \| = \sqrt{p} \) so that the sequence is unbounded if we choose \( j = 1 \).

Now choose any \( j \geq 1 \). We shall show that \( \sup A^{-1} B A (e^{ij}) = \infty \) which will complete the proof. Let the matrix of \( A' \) be \((a_{i,j}')\), and take \( p > j \). Then we claim
\[
a_{i,j}' = \begin{cases} 
0 & \text{if } s \leq p - j, \\
0 & \text{if } s > p + 1, \\
\mu_\alpha' & \text{if } s = p + 1.
\end{cases}
\]
Indeed, if \( s = 1 \) this is immediate from the definition. Suppose that it holds for some \( s = 1 \). Then to compute the \((p+1)\)-st column of \((a_{i,j}')\) we multiply each row of \((a_{i,j}')\) by the \((p+1)\)-st column of \((a_{i,j}')\). Thus we have,
\[
a_{i,j}' = \sum_{n=1}^{p+1} a_{i,n}' a_{n,j} = \mu_{j+1}' a_{i,j} + a_{i,p+1}' a_{p+1,j} = \mu_{j+1}' a_{i,j} + \mu_{p+1}' a_{i,p+1} a_{p+1,j},
\]
and if \( s > p + 1 \),
\[
a_{i,j}' = \sum_{n=1}^{p+1} a_{i,n}' a_{n,j} = \mu_{j+1}' a_{i,j} + \mu_{p+1}' a_{i,p+1} a_{p+1,j} = 0
\]
and if \( s \leq p - j \),
\[
a_{i,j}' = \sum_{n=1}^{p+1} a_{i,n}' a_{n,j} = \mu_{j+1}' a_{i,j} + \mu_{p+1}' a_{i,p+1} a_{p+1,j} = 0
\]
as \( s < p - j \) implies that \( s < p - j + 1 \) and \( s - 1 < p - j + 1 \) so by the induction hypothesis, both terms vanish and \( a_{i,p+1} = 0 \). Thus the claim is proved.

From this it follows that for some scalars \( \mu_{p-j+1}, \ldots, \mu_p \) we have
\[
A' (e^{ij}) = \sum_{s=p-j+1}^{p} \mu_s' e^i + \mu_{p+1}' e^{ij}
\]
so that
\[
A^{-1} B A A' (e^{ij}) = \sum_{s=p-j+1}^{p} \mu_s' - \mu_{p+1}' \left( \sum_{s=p-j+1}^{p} \mu_s \right)
\]
and
\[
\| A^{-1} B A A' (e^{ij}) \|_2 \geq \left( \sum_{s=p-j+1}^{p} \mu_s' \right)^2 + \left( \sum_{s=p-j+1}^{p} \mu_s \right)^2 \geq \| \mu_{p+1}' \|_2 \| e^{ij} \|_2 \]

Therefore,
\[
\| A^{-1} B A A' (e^{ij}) \|_2 \geq \left( \sum_{s=p-j+1}^{p} \mu_s' \right)^2 + \left( \sum_{s=p-j+1}^{p} \mu_s \right)^2 \geq \| \mu_{p+1}' \|_2 \| e^{ij} \|_2
\]
By our hypotheses, this last term is unbounded in \( p \) so we are finished.

Remark 1. The hypothesis of Proposition 3 is easily obtained, for instance if we choose \( \mu = \varepsilon_1 \) such that
\[
\mu_\varepsilon = \frac{1}{2^n}, \quad n = 1, 2, \ldots
\]
Then if we take \( p = 2^n \), we obtain
\[
|\mu_\varepsilon| p = \frac{2^n}{2^n} \rightarrow \infty \quad \text{as } n \rightarrow \infty.
\]

Remark 2. In a sense one may consider that Proposition 3 shows that the most natural approach to constructing a basis in a nuclear Fréchet space cannot in general succeed. Indeed we may think of \( E \) as a dense subspace of \( F \) with a sequence of (Hilbertable) norms on it, the first being the original \( l_2 \) norm. Then we have taken a complete orthonormal sequence (with respect to the first norm) and shown that it was total and linearly independent in \( E \). Such a procedure is exactly what worked in the cases treated in Section 1, but it fails here.

Actually, we can show more. Recall that a Markushevich basis in a linear topological space \( E \) is a sequence \((b_n, f_n)\), where \( (b_n) \) is total in \( E \), \( (f_n) \) is total in \( E [X(E)] \) and \( f_n(b_m) = \delta_{mn} \). In [3] it is shown that if \( E \) and its strong dual are separable then a Markushevich basis always exists. A Markushevich basis is not necessarily a Schauder basis. For an example in the case of Banach spaces, see [5]. We know of no previous example for Fréchet nuclear spaces, so we show that \( \mu - \varepsilon \) matrices provide such examples.
PROPOSITION 4. With the hypotheses of Proposition 3 there is a sequence 
\((f_n)\) in \(E\) such that \((b_n, f_n)\) is a Markushevich basis but not a Schauder basis. 

Proof. For each \(n = 1, 2, \ldots\) let \(f_n = \Pi_n P_1\). Since \(\Pi_n\) and \(P_1\) are continuous it follows that \(f_n \in E^*\). Also, 
\[ f_n(b_m) = \Pi_n P_1(b_m) = \Pi_n(b_m^{\infty}) = \delta_{mn}. \]
Finally, if \(x \in E\) and \(f_n(x) = 0\) for all \(n\), then \(P_1 x \in I_2\) so we can write 
\[ P_1 x = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \xi \in I_2 \text{ and convergence in } I_2. \]
Hence, for each \(n\), 
\[ 0 = f_n(x) = \Pi_n P_1(x) = \sum_{n=1}^{\infty} \xi_n \eta_n(x^{\infty}) = \xi_n. \]
so it follows that \(P_1(x) = 0\) and since \(P_1\) is injective, \(x = 0\). Thus \((f_n)\) is total in \(E^* (\mathbb{F}_2(E))\) and we are finished.

We cannot try to find a basis for \(E\) using other methods. Let us assume for the rest of this section that \(\mu_n \neq \mu_m\) for \(n \neq m\). Then it is possible to (formally) diagonalize \(A\). Let \(D\) be the diagonal matrix with diagonal elements \((\mu_n)\) and let \(U\) be the following matrix:
\[
\begin{bmatrix}
\frac{v_1}{\mu_1} & \frac{v_2}{\mu_2} & \frac{v_3}{\mu_3} & \cdots \\
\frac{v_3}{\mu_3} & \frac{v_2}{\mu_2} & \frac{v_1}{\mu_1} & \cdots \\
0 & \frac{v_3}{\mu_3} & \frac{v_2}{\mu_2} & \cdots \\
0 & 0 & \frac{v_3}{\mu_3} & \cdots \\
0 & 0 & 0 & \cdots \\
\end{bmatrix}
\]
One can easily verify by direct computation that \(A = UD\). Moreover, \(U\) has a (formal) inverse, \(U^{-1}\) given by the following matrix:
\[
\begin{bmatrix}
\frac{v_1}{\mu_1} & \frac{v_2}{\mu_2} & \frac{v_3}{\mu_3} & \cdots \\
\frac{v_3}{\mu_3} & \frac{v_2}{\mu_2} & \frac{v_1}{\mu_1} & \cdots \\
0 & \frac{v_3}{\mu_3} & \frac{v_2}{\mu_2} & \cdots \\
0 & 0 & \frac{v_3}{\mu_3} & \cdots \\
0 & 0 & 0 & \cdots \\
\end{bmatrix}
\]
and hence we can write \(A = UDU^{-1}\). Indeed all of the above statements are rigorously true if we only apply the operators to \(\varphi\). It would be quite easy (Remark 1) to conclude that \(E\) has a basis if we knew that \(U\) was an isomorphism on \(I_n\). Unfortunately, as we shall see below (Remark 3) this is not necessarily the case. However, we can give a simple condition under which \(U\) is an isomorphism. More detailed computations could lead to sharper results than the following.

PROPOSITION 5. Let \(\mu, \nu\) be such that \(\mu_n \neq \mu_m\) for \(n \neq m\) and 
\[ r = \sup_{l \leq n} \frac{v_l}{l - \mu_n} \leq \frac{1}{2}. \]

Then \(E\) has a basis which is obtained by applying \(P_1^{-1}\) to each of the columns of \(U\) (as an element of \(I_n\)).

Proof. In view of Remark 1, we need only show that \(U\) is an isomorphism. The explicit description of the basis follows from the application of Proposition 3.

To show that \(U\) is an isomorphism, let \(S: I_n \rightarrow I_n\) be the operator defined by \(S(e^n) = e^{n+1}\) and let \(E_n: I_n \rightarrow I_n\) be the diagonal operator whose diagonal is given by the sequence 
\[
\left(\frac{v_1 v_2 \cdots v_{n+1}}{(\mu_n - \mu_{n+1}) \cdots (\mu_{n+1} - \mu_{n+1})}\right)_{n=1}^{m}.
\]
Then clearly we have \(U = I + \sum_{n=1}^{\infty} S^n E_n = I + U_n\). It suffices to show that \(\|U\| < 1\). But \(\|S\| = 1\) and \(\|E_n\|\) is the maximum of the moduli in the given sequence, that is, 
\[ \|E_n\| = \sup_m \left| \frac{v_1 v_2 \cdots v_{n+1}}{(\mu_n - \mu_{n+1}) \cdots (\mu_{n+1} - \mu_{n+1})} \right| \leq \sum_{n=1}^{\infty} \frac{1}{\mu_n - \mu_{n+1}} \leq 1. \]
Hence we have,
\[ \|U\| \leq \sum_{n=1}^{\infty} \|S^n E_n\| \leq \sum_{n=1}^{\infty} \|E_n\| \leq \sum_{n=1}^{\infty} \frac{1}{\mu_n - \mu_{n+1}} = 1. \]

Remark 3. There are some simple cases in which the hypotheses of Proposition 5 are satisfied:

(i) \(\mu_n = \frac{1}{n^2}, \sup_l \frac{v_l}{l - \mu_n} \leq \frac{1}{2}\),

(ii) \(\mu_n = \frac{1}{2n^2}, \sup_l \frac{v_l}{l - \mu_n} \leq \frac{1}{4}\).

Returning now to the situation in Proposition 3, we may well ask if the
method of Proposition 5 will always give a basis. The answer is no, and this will give us an example in which $U$ is not an isomorphism.

**Proposition 6.** Assume that $\mu = \nu$, $0 < \mu_n \neq \mu_m$ for $n \neq m$ and for each integer $j > 0$, the sequence

$$
\left(\frac{\mu_p \mu_p^{j+1}}{\mu_p - \mu_{p+1}}\right)_{j=0}^\infty
$$

is unbounded. Then the sequence $(b_k)$ given by $b_n = a^{-1}_k(b^{j,n})$, where $b^{j,n}$ is the sequence in the $n$th column of $U$, is a total, linearly independent sequence in $\tilde{A}^i$ which is not a Schauder basis.

**Proof.** Clearly $U(p') = \varphi$ and $U$ is injective on $\varphi$. Hence $U(d') = \varphi$ and $U(p) = UD$ $U^{-1}(p) = \varphi = A^{k}(p) = A^{k}(I,)$ so $b^{j,n} = U(d') = \varphi$ and $b_k = \tilde{E}$. If $\sum a'_{t_i} b_i' = 0$ then $\sum a'_{t_i} U_i' = \sum a'_{t_i} b_i' = 0$ and since $U$ is injective on $\varphi$, for each $k$, the subspace generated by $(b^{j,n})_k$ is $A^{-1}U(p) = \varphi$ so $(b_k)$ is linearly independent. Finally, therefore we conclude that the element in the $p$th row and $(p + 1)$st column of $A^{-1}U(p)U^{-1}A^{k+1}$ is given by

$$
\left\{ \begin{array}{ll}
-\mu_p \sum_{n=0}^{p-1} b_n^{j,n+1} - \sum_{n=0}^{p-1} b_n^{j,n+1}, & m = p, j > 1 \\
0, & m = p, j = 1 \\
\mu_p^{j+1}, & m = p + 1, j > 0 \\
0, & m > p + 1, j > 0
\end{array} \right.
$$

Hence it follows that

$$
\|A^{-1}U(p)U^{-1}A^{k+1}\| \geq \|A^{-1}U(p)U^{-1}A^{k+1}(d^{j,n})\| \geq \frac{\mu_p^{j+1}}{\mu_p - \mu_{p+1}},
$$

and by hypothesis, this last sequence is unbounded with respect to $p$ for each $j > 0$ so the result follows from Proposition C.

**Remark 4.** It is important to note that there exist $\mu - \nu$ matrices which satisfy the hypotheses of both Proposition 3 and 6. Indeed, if we take $\mu$ such that

$$
\mu_n = \begin{cases}
\frac{1}{2^{n-1}}, & n = 2^m - 1 \\
\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}}, & n = 2^m - 1
\end{cases}
$$

then (see Remark 1), the conditions of Proposition 3 are satisfied. Moreover, if we take any $j > 0$ and $p = 2^m - 1$, then

$$
\frac{\mu_p \mu_p^{j+1}}{\mu_p - \mu_{p+1}} = \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} = 2^{m-2}(j+1) + \frac{1}{2^{m-2}}
$$
which is clearly unbounded so Proposition 6 is satisfied. Thus all of
the methods that we know of fail to produce a basis in this case and we
are led to the following.

**Conjecture.** If $A$ is a $\mu - 1$ matrix with $\mu = \lambda$, $0 < \mu_n \neq \mu_m$ for
all $n \neq m$ and such that $\mu$ satisfies (1), then $A$ generates a nuclear system
whose associated space is a nuclear Fréchet space which does not have
a Schauder basis.

3. Lower triangular matrices. In this section we consider a very
simple case of lower triangular matrices which give nuclear systems and
construct a basis by direct computation.

Let $N$ be a fixed positive integer and for each $k = 1, 2, \ldots$ let
$a^{ij}, \ldots, a^{jk}$ be elements of $\ell_1$ such that $a^{ij} = 0$ for $j < i$ and let
$\mu^j \epsilon \ell_1, \mu^j \neq 0$ for all $j, k$. Then we can define $A_k$: $\ell_1 \rightarrow \ell_1$ by

$$A_kx = \sum_{i \geqslant 1} a^{i1}x_i + \sum_{i \geqslant 1} a^{i1}x_i \mu_i^j.$$

As a matrix we can describe $A_k$ by noting that $A_k$ is a diagonal matrix
except for its $i$th column ($1 \leqslant i \leqslant N$) which is the sequence $\mu_i^j + a^{ik}$.

**Proposition 7.** $(A_k)$ is a nuclear system. For each $n$, the map $f_n = P_n P_k$
is in the dual of $\ell_1$.

**Proof.** Each $A_k$ is the sum of a nuclear diagonal map and a map
with finite dimensional range so it is nuclear. If $A_k^*$ is the adjoint of
$A_k$, then we can see by inspection that $A_k^* (\varphi) = \varphi$ so that $A_k^*$ has
dense range so $A_k$ is injective. Next, it is clear that $\varphi' = \Phi A_k (\varphi')$ for $i > N$.

For $1 \leqslant n \leqslant N$, let $\varepsilon > 0$ and define $x = x_n \epsilon A_k (\varphi')$ for $i > N$.

Then

$$\|A_k (x) - \varphi\| = \left\| \sum_{i=0}^{\infty} \left( a^{i1}x_i + \ldots + a^{ik}x_i \right) \right\| = \varepsilon,$$

so $A_k$ has dense range. Thus $(A_k)$ is a nuclear system.

The second statement is obvious since $P_n, P_k$ are continuous.

**Proposition 8.** The associated space of $(A_k)$ has a Schauder basis.

**Proof.** For $i > N$ it follows that $\varphi' \epsilon \cap A_k \ldots A_k (\varphi')$ so we can define
$b_i = P_{i-1} (\varphi') \epsilon \ell_1$. Now $P_i (\ell_1)$ is a dense subspace of $\ell_1$ so if we define $\Pi_i: \ell_1 \rightarrow \ell_1$ by $\Pi_i(x) = (x_1, \ldots, x_N)$ then $\Pi_i (\ell_1) = \ell_1$. Hence there exists $b_1, \ldots, b_N$ in $\ell_1$ such that $\Pi_i (b_i) = \varphi' \epsilon \ell_1, i = 1, \ldots , N$.

We claim that $(b_i)$ is a Schauder basis for $\ell_1$. First suppose that $x \epsilon \ell_1$ and $x = \sum \xi_i b_i$. Then by Proposition 7, for each $n$, $f_n (x) = \sum \xi_i f_n (b_i)$.

Thus we obtain,

$$\xi_n = \begin{cases} f_n (x) & n \leqslant N, \\ f_n (x) - \sum_{i=0}^{n} f_i (x) f_n (b_i) & n > N. \end{cases}$$

This shows that the representation is unique and we need only show that for $x \epsilon \ell_1$, the series $\sum \xi_i b_i$ converges to $x$, where $(\xi_i)$ is given by the above relations.

We consider for $n = 1, 2, \ldots$

$$\Pi_n P_k (x - \sum_{i=0}^{N} \xi_i b_i) = f_n (x) - \sum_{i=0}^{N} f_i (x) f_n (b_i)$$

$$= 0, \quad n \leqslant N, \quad \xi_n, \quad n > N,$$

and since $P_i (x - \sum_{i=0}^{N} \xi_i b_i) \epsilon \ell_1$ we have $\xi \epsilon \ell_1$ and

$$\Pi_i (x - \sum_{i=0}^{N} \xi_i b_i) = \sum_{n=N+1}^{\infty} \xi_n \varphi''.$$

Hence for $M > N$,

$$P_i (x - \sum_{i=0}^{M} \xi_i b_i) = P_i (x - \sum_{i=0}^{N} \xi_i b_i) - \sum_{n=N+1}^{\infty} \xi_n P_i (b_i)$$

$$= \sum_{n=N+1}^{\infty} \xi_n \varphi'' - \sum_{n=N+1}^{\infty} \xi_n \varphi'' = \sum_{n=N+1}^{\infty} \xi_n \varphi''.$$
Now let $x^a_0 = \mu_1 \ldots \mu_a$, $x^a_{1} = 1$, $n, k = 1, 2, \ldots$. Then for any $k \geqslant 1$, we have $\zeta^a = \langle e^a_k, e^a_k \rangle$ with

$$P_{k+1}(x - \sum_{i=1}^{k} \xi_i h_i) = \sum_{n=1}^{\infty} \xi_n e_n a^n$$

so

$$P_{k+1}(x - \sum_{i=1}^{k} \xi_i h_i) = A_1 \ldots A_k P_{k+1}(x - \sum_{i=1}^{k} \xi_i h_i) = \sum_{n=1}^{\infty} \xi_n^2 e_n a^n.$$

Equating coefficients we conclude that

$$\xi_n^a = 0 \text{ for } n \leqslant k \text{ and } \xi_n^a = \frac{\xi_n^a}{\xi_n^a} \text{ for } n > k.$$

In particular, applying this for $M = N$ we conclude that

\[
\left( \sum_{n=1}^{\infty} \xi_n^a \right) \left( \sum_{n=1}^{\infty} \xi_n^a \right) = \left( \sum_{n=1}^{\infty} \xi_n^a \right) = 1.
\]

And for arbitrary $M \geqslant N$,

\[
\left\| P_{M}(x - \sum_{i=1}^{M} \xi_i h_i) \right\| = \left\| \sum_{n=M+1}^{\infty} \xi_n e_n \right\| = \left( \sum_{n=M+1}^{\infty} \xi_n e_n \right)^{\frac{1}{2}}.
\]

The last term goes to 0 as $M$ goes to $\infty$ and this implies

$$\lim_{M \to \infty} \left\| P_{M}(x - \sum_{i=1}^{M} \xi_i h_i) \right\| = 0 \text{ so that } \left\| P_{M}(x - \sum_{i=1}^{M} \xi_i h_i) \right\| \to 0.$$

Remark 5. The case described above is a very primitive example. To go further, it would be very interesting to see what happens if $M$ varies with respect to $k$ and moreover if this approach could be used to approximate an arbitrary lower triangular matrix. Finally one could investigate the connection between upper and lower triangular matrices.

References