On Liouville $F$-Algebras

by

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Abstract. This paper investigates the spectra of elements from a Liouville $F$-algebra. A generalization of the notion of a Shilov boundary for a Banach algebra is defined and the principle result concerns the presence of algebraically principal closed maximal ideals on this boundary.

1. Introduction. A commutative $F$-algebra $A$ with an identity element is called a Liouville $F$-algebra if the spectrum of each non-constant element in $A$ is an unbounded subset of the complex plane $C$ [2]. The entire functions $E$ in the topology of uniform convergence on the compact subsets of $C$ is an example of a singly generated Liouville $F$-algebra. Birtel [3] was interested in characterizing $E$ when he defined the Liouville property. The first example of a singly generated Liouville $F$-algebra which properly contains $E$ was constructed in [3].

In Section 2 we investigate conditions which guarantee that the spectrum of an element from a Liouville $F$-algebra is identifiable with $C$. We introduce and investigate a generalization of the Shilov boundary for a Banach algebra. The major result of this study is the existence of algebraically principal closed maximal ideals at non-isolated points on our boundary. The reader is referred to [8] for the basic information on $F$-algebras.

2. Spectra in Liouville $F$-algebras.

Definition 2.1. An $F$-algebra $A$ with identity element $e$ is called a Liouville $F$-algebra provided the following condition is satisfied:

If $a \in A$ and there exists an $M > 0$ such that $|h(a)| \leq M$ for each $h \in M_A$, then $a = le$ for some $l \in C$.

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From this definition it follows immediately that a Liouville F-algebra is a semisimple algebra.

**Proposition 2.2.** If $A$ is a Liouville F-algebra, $x \in A$ and $x \neq ne$ for any $n \in C$, then the spectrum of $x$, $\sigma(x)$, is a dense connected subset of $C$. Moreover, $\sigma(x)^0$ (complement of $\sigma(x)$ in $C$) contains no closed connected subsets other than single points.

**Proof.** Assume that $\sigma(x) \cap N(\epsilon, r) = \emptyset$, where $N(\epsilon, r) = \{z \in C: |z - x| < \epsilon\}$. Now $f(z) = (z - x)^{-1}$ is analytic on $N(\epsilon, r)$, an open subset of $C$ containing $\sigma(x)$. A well known result for F-algebras implies that there exists a unique $y \in C$ such that $h(y) = f(h(z))$ for each $h \in M_d$ [9, Theorem 10.1]. However, $y \in A$ has a bounded spectrum, which contradicts the Liouville hypothesis since $y \neq ne$ for all $n \in C$. Hence, $\sigma(x)$ is a dense subset of $C$.

Now assume that $\sigma(x) \subseteq V_1 \cup V_2$, where $\{V_i\}_{i=1}^{\infty}$ are separating non-empty open subsets of $C$ with $\sigma(x) \cap V_i \neq \emptyset$, $i = 1, 2$. The $\{V_i\}_{i=1}^{\infty}$ may be chosen to be disjoint, for if $V_i \cap V_j \neq \emptyset$ then $\sigma(x) \cap (V_i \cap V_j) \neq \emptyset$ since $\sigma(x)$ is dense in $C$. This contradicts the fact that $V_i$ and $V_j$ separate $\sigma(x)$. For a sufficiently large, $V_i \cap \sigma(u_n) \neq \emptyset$, $i = 1, 2$. Shilov has shown that there exists an idempotent element $u_n \in A_n$ such that $u_n(h) = f(h) = f(u_n(h))$, $h \in M_d$, where $f$ is the analytic function defined on $V_1 \cup V_2$ satisfying $f(V_i \cap V_j) = 1$ if $V_i \cap V_j = \emptyset$. Moreover, if $f > \epsilon$, then $\sigma(u_n) = \emptyset$ and $\sigma(u_n) = \emptyset$. The uniqueness of $u_n \in A_n$ modulo the radical implies $\sigma(u_n) = \emptyset$. Hence, $A$ contains a proper idempotent element [9, Theorem 5.1]. This contradicts the Liouville hypothesis. Therefore, $\sigma(x)$ is a connected subset of $C$.

Next let $K$ be a connected subset of $C$, and assume $K \subseteq \sigma(x)$. We assume that $K$ is a subset of $S$, the extended complex plane. If $K$ is an unbounded subset of $C$, we adjoint the point at infinity to $K$. Let $U = S - K$. Now, $\sigma(x) \subseteq U$ and $\sigma(x)$ is a dense subset of $C$. Hence $\sigma(x)$ is dense in $U$. If $U$ is not a connected subset of $C$, then $\sigma(x)$ would not be a connected subset of $C$, which contradicts the previous result. Hence, $U$ is a connected subset of $C$, and by our assumption on $K$, $U$ is also an open subset of $S$. Moreover, $S - U = K$ implies that $U$ is a simply connected region in $S$. If $K$ contains more than one point, then by the Riemann Mapping Theorem, there exists $f \in Hol(U)$ such that $f$ maps $U$ onto the open unit disc [11]. Since $\sigma(x) \subseteq U$ and $A$ is closed under the application of analytic functions, $f : \sigma(x)$ defines a unique element of $A$. The transform $f : \sigma(x)$ is not constant since $\sigma(x)$ is dense in $U$ and $f$ is not constant on $\sigma(x)$. Furthermore, $f : \sigma(x)$ has a bounded spectrum, which contradicts the Liouville property. Hence, $K$ must consist of at most one point.

If $A$ is a commutative F-algebra, then $M_d$ may be topologized in at least two natural ways.

(1) The weak topology on $M_d$ which $M_d$ inherits as a subset of the dual space $A^*$ equipped with the weak topology induced by $A$ [3, p. 6].

(2) The direct limit topology on $M_d$: a set $U \subseteq M_d$ is open and only if $U \cap M_{d+1}$ is open relative to the weak topology on $M_{d+1}$ induced by $A$, for each $n = 1, 2, \ldots$

The weak topology on $M_d$ is the weak* topology determined by the subbasis of sets of the form $V_{m+1} = \{h \in M_d: |\langle h, \delta \rangle| < \epsilon\}$ where $\delta \in A$. The direct limit topology on $M_d$ is stronger than the weak topology on $M_d$.

The technique used in the previous proposition yields the following:

**Corollary 2.3.** If $A$ is a Liouville F-algebra, then $M_d$ is connected in the direct limit topology. Moreover, if $h \in M_d$, then $h$ is not isolated with respect to infinitely many $M_{d+n}$, $n = 1, 2, \ldots$

We note at this point that the known examples of singly generated Liouville F-algebras all have $\sigma(a) = C$ [2, 3]. However, the Liouville hypothesis is not sufficient to guarantee that $\sigma(a)$ is equal to $C$ for all choices of a generator, as Example 2.4 will illustrate. We precede the example with a discussion of a method for forming singly generated Liouville F-algebras which will be referred to later.

If $D$ is a compact subset of $C$, then $Hol(D)$ denotes the algebra of functions which are analytic in $\hat{D}$ (the interior of $D$), and have a continuous extension to $D$. It is well known that $Hol(D)$ is a Banach algebra under the norm, $||f|| = \sup_{z \in D} |f(z)|$ for $f \in Hol(D)$. A theorem of Mergelyan [11], states that $Hol(D)$ is the uniform closure on $D$ of the algebra of polynomials in $z$, if and only if $D$ is a non-separating subset of $C$. Let $\{D(n)\}_{n=1}^{\infty}$ denote an increasing sequence of compact non-separating subsets of $C$, and $\sigma = \bigcup_{n=1}^{\infty} D(n)$. Assume $\sigma^{n+1}: Hol(D(n+1)) \rightarrow Hol(D(n))$ is the natural homomorphism defined by $\sigma^{n+1}(f) = f|D(n)$ for each $f \in Hol(D(n+1))$, $n = 1, 2, \ldots$ Since the sequence $\{D(n)\}_{n=1}^{\infty}$ is an increasing sequence, $\sigma^{n+1}$ is continuous, and $\sigma^{n+1}: Hol(D(n))$ is clearly dense in $Hol(D(n))$. Using Arens' terminology [1], let $Hol(c)$ denote the strong dense in $\lim_{n \to \infty} \{Hol(D(n))\}$, $\sigma^{n+1}$. In Michael's terminology, $Hol(c)$ is the projective limit of the Banach algebras $\{Hol(D(n))\}$, $\sigma^{n+1}$. $Hol(c)$ is clearly a singly generated semisimple F-algebra and $M_{d+1}(c)$ is identifiable pointwise with $\sigma(a) = \bigcup_{n=1}^{\infty} D(n)$.

**Example 2.4.** Let

$$D(n) = \{x \in C: 0 \leq \Re x \leq 2\pi - 1/n, 1/n \leq |x| \leq n\}$$

$$\cup \{x \in C: \Re x = 2\pi - 1/n, \Im x \geq n + 1, 1/n \leq |x| \leq n\}.$$
LEMMA 2.5. If $a = \bigcup_{n=1}^{\infty} D(n)$ then $A = \text{Hol}(a)$ is a Liouville $F$-algebra. Furthermore, $\bigcup_{n=1}^{\infty} D(n)$ is a proper subset of $C$.

Proof. We note that $D(n) \subseteq D(n+1)$ for each $n = 1, 2, \ldots$ and $a = \bigcup_{n=1}^{\infty} D(n) = C - \{0\}$. Hol$(a)$ is a singly generated semisimple $F$-algebra with an identity element, and $\sigma(a)$ is identifiable with $\sigma(a) = C - \{0\}$. The proof that $A$ is a Liouville $F$-algebra follows from the argument used in the proof of Lemma 3 of [3].

The example may easily be modified to exclude infinitely many points from $\sigma(a)$. We now discuss what conditions must be placed on $A$, which will guarantee that $\sigma(a) = C$ so that $A$ generates $A$.

DEFINITION 2.6. An algebra $A$ is said to admit square roots, if for each $f \in A$ with $f^2 + A = 0$ there is a solution in $A$.

The algebra $A$ in Example 2.4 does not admit square roots, as is demonstrated by the following argument. Since $0 \in \sigma(a)$, $a^{-1} \cdot a$ [9, Theorem 5.2]. If $s \cdot a$ were a solution to $x^2 - a = 0$, then $s^2 = a$, and $\sigma(a) = \pm \sqrt{a}$ for zero elements. Without loss of generality, assume that $s(1) = 1$. Now $s(a)$ is analytic off the positive real axis, and so $s(a)$ defines a single branch of the square root function away from the positive real axis. The choice of $D(n)$ implies that $s(a)$ has a continuous extension to a circle $\Gamma$ about the origin in the relative euclidean topology on $\Gamma$. This clearly is a contradiction. Hence, $x^2 - a = 0$ has no solution in $A$.

Example 2.4 is a counterexample to Theorem 3.1 of Birtle [2]. We now give the correct formulation of his proposition. Let $\sigma(a)$ denote the euclidean interior of $\sigma(a) \subseteq C$.

PROPOSITION 2.7. If $A$ is a singly generated Liouville $F$-algebra with generator $a$, and $A$ admits square roots then $\sigma(a) = C$ provided $\sigma(a)$ is nonempty.

Proof. Without loss of generality we assume the m-base for $A$ is chosen such that $\sigma(a_n) \subseteq \sigma(a_{n+1})$ for each $n = 1, 2, \ldots$ Now $\sigma(a) = \bigcup_{n=1}^{\infty} \sigma(a_n)$.

If $\sigma(a)$ is nonempty, then an elementary application of the Baire Category Theorem implies that $\sigma(a) = C$ for all $n$ sufficiently large.

Let $\sigma(a) \subseteq C$ and without loss of generality we assume that $0 \in \sigma(a)$. Then $a$ is regular in $A$, i.e., $a^{-1} \cdot a$ [9, Theorem 5.2]. Let $s \cdot a$ denote a solution to $x^2 - a = 0$. Now $s : M \rightarrow \sigma(a)$ is a one-one mapping since $s(h^*a) = a(h)$ for each $h \in M$ and $s(h) = \sigma(a)$ is a one-one mapping. Moreover, if $s(h) = s(k)$ for some $h, k \in M$ and $h \neq k$ then $\sigma(a) = \sigma(a)$. Thus $\sigma(a) \subseteq \sigma(a)$. But then $\sigma(a) \subseteq \sigma(a) \subseteq \sigma(a)$. And so $\sigma(a) = \{\}$. This contradicts Proposition 2.2 and thus $\sigma(a) = C$.

3. A boundary for $F$-algebras.

DEFINITION 3.1. Let $A$ denote an $F$-algebra. If $P$ is an m-base of zero in $A$, then the boundary $\partial_P$ of $A$ is defined as follows: $\partial_P = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \partial_{kP}$ where $\partial_{kP}$ denotes the shilov boundary of $\hat{A}$. $P$.

PROPOSITION 3.2. If $A$ is an $F$-algebra, then $\partial_P$ is independent of the choice of m-base for $A$.

Proof. Let $P$ and $P'$ denote two choices of m-base for $A$, and $(\hat{A})_{n=1}^{\infty}$ and $(\hat{A}')_{n=1}^{\infty}$ the associated sequence of Banach algebras determined by $P$ and $P'$ respectively. Let $h \in \partial_{P'}$ and $h \in \partial_{P'}$. Then there exists $K$ such that $h \in \hat{A}$. Without loss of generality we assume $m = m(n)$ such that $\hat{A}_n \subseteq \hat{A}$ is a subset of respect to the sequence $(\hat{A})_{n=1}^{\infty}$ implies that given $n$, there exists $m = m(n)$ such that $\hat{A}_n \subseteq \hat{A}$. Thus $h \in \hat{A}_n$. Without loss of generality we assume $m = m(n)$. If $h \in \hat{A}_n$, then suffice to prove that $h \in \hat{A}_n$. For each $f \in \hat{A}_n$, the natural restriction homomorphism of $\hat{A}_n$ into $\hat{A}_n$ defined by $r_n(f) = f | \hat{A}_n$ for each $f \in \hat{A}_n$. Now $\hat{A}_n$ is a compact subset of $\hat{A}_n$ and since for Banach algebras the Shilov boundary depends only on the transform algebras, we may apply Corollary 6.3 [10] to conclude that $\hat{A}_n \cap \hat{A}_n \subseteq \hat{A}_n$. Thus, $h \in \hat{A}_n$ implies $h \in \hat{A}_n$ and it follows that $\partial_P \subseteq \partial_P$. Similarly, $\partial_P = \partial_P$ and the proposition is proven.

If $A$ is now assumed to be a singly generated $F$-algebra and a generator for $A$, then $\sigma(a)$ may be identified with the carrier space of $A$. Moreover, $\hat{A}(x) = P$ implies by Proposition 3.2 that $P$ is independent of the choice of m-base for $A$. Since the topological boundary of $\sigma(a)$, $\sigma(a) - \sigma(a)$ can be identified with $\partial_{kP}$ whenever $\hat{A}_n$ is singly generated [9], we have $\partial_{kP} = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \sigma(a_n)$. Also, $\partial_{kP} = \sigma(a) - \bigcup_{n=1}^{\infty} \sigma(a_n)$ where $\sigma(a_n)$ denotes the euclidean interior of $\sigma(a_n)$.

PROPOSITION 3.3. If $A$ is a singly generated Liouville $F$-algebra with generator $a$, then $\partial_{kP} = C - \bigcup_{n=1}^{\infty} \sigma(a_n)$ (closure in the euclidean topology on $C$).

Proof. Without loss of generality we choose an m-base $P$ for $A$ such that $\sigma(a) \subseteq \sigma(a_n)$ for each $n = 1, 2, \ldots$. If $\sigma(a) = \sigma(a)$ for each $n = 1, 2, \ldots$, then $\partial_{kP} = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \sigma(a_n) = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \sigma(a_n) = \bigcup_{n=1}^{\infty} \sigma(a_n) = \sigma(a)$. Since $\sigma(a)$ is dense in $C$ (Proposition 2.2) we have $\partial_{kP} = \sigma(a) = C$. 

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Without loss of generality we may assume \( \sigma(a_n) \neq \emptyset \). Clearly \( \Gamma_n \subseteq C - \bigcup_{n=1}^{\infty} \sigma(a_n) \). Let \( \lambda : C - \bigcup_{n=1}^{\infty} \sigma(a_n) \) \( \rightarrow [z : -1 < r] \) for any \( r > 0 \), then \( \overline{N}(\lambda) \cap \overline{\Gamma_n} \neq \emptyset \). Assume \( \overline{N}(\lambda) \cap \overline{\Gamma_n} = \emptyset \) for some \( r > 0 \). Now \( \{\sigma(a_n) \cap \overline{N}(\lambda)\} \) where the complement is taken with respect to \( \overline{N}(\lambda) \) is a compact subset of \( \overline{N}(\lambda) \). Let \( D_\lambda \) denote the component of \( \overline{N}(\lambda) \) \( \cap \overline{\Gamma_n} \) containing \( \lambda \) and let \( D = \bigcap_{n=1}^{\infty} D_\lambda \).

We prove that \( D \) must contain a point \( x_\lambda \in \sigma(a_n) \) where \( x_\lambda \neq \lambda \). Let \( y_\lambda \in \sigma(a_n) \) denote the circle of radius \( r \) about \( \lambda \). Note \( y_\lambda \in \sigma(a_n) \) for any \( n = 1, 2, \ldots \) since \( \sigma(a_n) \) is a non-separating subset of \( C \). Moreover, since \( \sigma(a_n) \) is a dense connected subset of \( C \) we know that \( \sigma(a_n) \cap y_\lambda \neq \emptyset \). Without loss of generality we may assume that \( \sigma(a_n) \cap y_\lambda \neq \emptyset \). But \( \sigma(a_n) \) being a non-separating subset of \( C \) implies that there exists an arc joining \( y_\lambda \) to \( \infty \) which misses \( \sigma(a_n) \) and intersects \( \gamma_\lambda \). Thus, there exists \( x_\lambda \in D_\lambda \cap y_\lambda \), for each \( n = 1, 2, \ldots \). Because \( y_\lambda \) is compact, \( \{x_\lambda \}_{n=1}^{\infty} \) has a limit point \( x_\lambda \in D_\lambda \) since \( x_\lambda \in D_\lambda \) for each \( j \geq k \). Moreover, \( x_\lambda \neq \lambda \) since \( x_\lambda \neq \gamma_\lambda \).

Now \( D = \bigcap_{n=1}^{\infty} D_\lambda \). Since \( \{D_\lambda \}_{n=1}^{\infty} \) is a decreasing sequence of compact connected subsets of \( C \), \( D \) is a compact, connected subset of \( C \). Also, \( D \cap \sigma(a_n) \subseteq D \cap \bigcup_{n=1}^{\infty} \sigma(a_n) \) \( \cup \bigcap_{n=1}^{\infty} \sigma(a_n) \cap \sigma(a_n) = \emptyset \) since \( D \cap \sigma(a_n) = \emptyset \) for each \( m \geq n \) and \( D \cap \overline{\Gamma_n} = \emptyset \). Thus, \( D \) is a closed connected subset of \( \sigma(a_n) \) which contains more than one point. This contradicts Proposition 2.2. Thus, \( \overline{N}(\lambda) \cap \overline{\Gamma_n} \neq \emptyset \) for any \( r > 0 \) and the proposition is proven.

Even if \( \mathcal{A} \) is a Liouville \( \mathcal{F} \)-algebra with generator \( a \), \( \Gamma_a \) is in general not a connected subset of \( C \). However, we do have the following proposition.

**Proposition 3.4.** If \( \mathcal{A} \) is a singly generated Liouville \( \mathcal{F} \)-algebra with generator \( a \), then \( \Gamma_a \) (closure now taken in the extended complex plane \( S \)) is a compactified subset of \( S \).

**Proof.** We again assume without loss of generality that \( \sigma(a_n) \) is a non-empty connected subset of \( S \) and hence \( \sigma(a_n) \) (closure in \( S \)) is a compact, connected subset of \( S \). Moreover, \( \sigma(a_n) \) \( \subseteq \sigma(a_{n+1}) \) implies that \( D = \bigcap_{n=1}^{\infty} \sigma(a_{n+1}) \) is a compact, connected subset of \( S \). If \( \lambda \in \overline{\Gamma_n} \) and \( \lambda \neq \infty \) then \( \lambda \in \overline{\Gamma_n} \) whenever \( \lambda \in \sigma(a_{n+1}) \). Thus, \( \lambda \sigma(a_{n+1}) \) for each \( n = 1, 2, \ldots \), and \( \lambda D \). If \( \lambda = \infty \) then clearly \( \lambda \in D \) and thus \( \Gamma \subseteq D \). Conversely, if \( \lambda D \) and \( \lambda \neq \infty \) then \( \lambda \sigma(a_n) \) for each \( n = 1, 2, \ldots \). By Proposition 3.3, \( \Gamma_a \subseteq C - \bigcap_{n=1}^{\infty} \sigma(a_n) \) implies that \( \lambda \in \overline{\Gamma_n} \). Thus \( \lambda \in \overline{\Gamma_n} \) and the proposition is proven.

It follows from the above proposition that the closure of \( \Gamma_a \) in \( C \) is a Euclidean perfect set without any bounded components.

In [3], Birtel stated that if for some choice of \( m \)-base, \( \bigcap_{n=1}^{\infty} \sigma(a_n) = \emptyset \), then \( \sigma(a) = C \). [3, Theorem 3.2]. His proof assumed that \( \mathcal{A} \) admits square roots as previously noted. This assumption is not needed, nor is the theorem dependent on the choice of \( m \)-base for \( \mathcal{A} \).

**Proposition 3.5.** If \( \mathcal{A} \) is a singly generated Liouville \( \mathcal{F} \)-algebra with generator \( a \) and \( \Gamma = \emptyset \), then \( \sigma(a) = C \). Moreover, \( \tilde{\alpha} : \mathcal{M}_a \rightarrow C \) is a homeomorphism of \( \mathcal{M}_a \) with the direct limit topology onto \( C \) with the Euclidean topology.

**Proof.** Proposition 3.3 implies that \( C = \bigcup_{n=1}^{\infty} \sigma(a_n) \). The remainder of the proof follows as in [3, Theorem 3.2].

In fact, when \( \Gamma = \emptyset \), Birtel's proof in Theorem 3.3 together with Proposition 3.5 now yields the following characterization of the algebra \( \mathcal{F} \) of entire functions on \( C \).

**Proposition 3.6.** (Birtel) If \( \mathcal{A} \) is a singly generated Liouville \( \mathcal{F} \)-algebra with identity then \( \Gamma = \emptyset \) if and only if \( \mathcal{A} \) is topologically isomorphic to the algebra \( \mathcal{E} \) of entire functions on the complex plane with the compact-open topology.

If \( \mathcal{A} \) is a singly generated \( \mathcal{F} \)-algebra, then the existence of Liouville \( \mathcal{F} \)-algebras with \( \Gamma = \emptyset \) shows that our \( \Gamma \) is not a maximizing set for any \( a \in \mathcal{A} \) where \( a \neq \lambda \). In Section 4 we show that the two known concepts of topological divisors of zero for \( \mathcal{F} \)-algebras differ. For one of these concepts the non-topological divisors of zero may have a zero on \( \Gamma \). We now construct an example which shows that \( \Gamma \) is in general not a determining set for a singly generated \( \mathcal{F} \)-algebra.

**Example 3.7.** For each positive integer \( n \) and \( k \), define:

\[ E_n = \{z = x + iy : 1/n \leq y \leq n, -n+1/n \leq x \leq n-1/n\} \]

\[ E_n = \{z = x + iy : -n \leq y \leq -n+1/n \leq x \leq n-1/n\} \]

\[ w_n^+ = \{z = x + iy : y = \frac{2k+1}{2(k+1)}, -n+1/n \leq x \leq n-1/n\} \]

\[ w_n^- = \{z = x + iy : y = -\frac{2k+1}{2(k+1)}, -n+1/n \leq x \leq n-1/n\} \]

\[ w_n^0 = \{z = x + iy : y = 0, -n+1/n \leq x \leq n-1/n\} \]
Let $D_n = E_n \cup E'_n \cup \bigcup_{k=1}^{n} E_k^n \cup \bigcup_{k=1}^{n} E'_k^n$. Now $D_n$ is a compact non-separating subset of $C$, $E_k^n \subseteq E_n$ and $E_k'^n \subseteq E'_n$ for $k = 1, 2, \ldots, n-1$, and $C = \bigcup_{n=1}^{\infty} D_n$. Let $x_0$ be a fixed point in $E_1$ and set

$$V_n = \{ x = x + iy : \frac{-4n+3}{4n(n+1)} < y < \frac{4n+3}{4n(n+1)}, -n < x < n \}.$$

Note that $\sum_{k=n}^{\infty} E_k^n \subseteq V_n$ and $\bigcup_{k=n}^{\infty} E'_k^n \subseteq V_n$. Moreover, $V_n \cap E_n = V_n \cap E'_n = \emptyset$ and if $m > n$ then $V_m \cap D_n \subseteq V_n$. Also $E_n \subseteq E_{n+1}$ and $E'_n \subseteq E'_{n+1}$ for each $n = 1, 2, \ldots$

We now define a sequence of polynomials inductively as follows: Let $p_1(x)$ be a polynomial in $x$ such that $p_1(x_0) = 1$, and $\|p_1\|_{L_{\infty}([0,1])} < 1$ ($\|p\|_{L_{\infty}([0,1])} = \text{Sup}_{x \in [0,1]} |p(x)|$). Such a polynomial exists by Rang's theorem. Assume $p_1(0), \ldots, p_{n-1}(x)$ have been constructed where $p_1(x_0) = 1$, $|p_1|_{L_{\infty}([0,1])} < 1/2^t$ and $|p_i - p_{i-1}|_{L_{\infty}([0,1])} < 1/2^i$ for $i = 1, 2, \ldots, n - 1$. We now show how to pick $p_n(x)$. Since $p_1, E_n$, and $E'_n$ are disjoint, non-separating compact sets in $C$ we can find a polynomial $q(x)$ such that $|q|_{L_{\infty}([0,1])} < 1/2^{n+1}$ and $|q - p_{n-1}|_{E_n} < 1/2^{n+1}$. Let $p_n(x) = q(x) + [1 - q(x_0)]$. Then $p_n(x_0) = 1$, and

$$|p_n - p_{n-1}|_{E_n} = |q - (1 - q(x_0))q|_{L_{\infty}([0,1])} + |q - p_{n-1}|_{E_n} < 1/2^{n+1} + 1/2^{n+1} = 1/2^n.$$

Since $x \in E_n$. Furthermore,

$$|p_n - p_{n-1}|_{E'_n} = |q - p_{n-1} + 1 - q(x_0)|_{E'_n} < 1/2^{n+1} + |p_{n-1} - q(x_0)|_{E'_n} \leq 1/2^{n+1} + 1/2^{n+1} = 1/2^n.$$

Thus, there exists a sequence of polynomials $(p_n(x))_{n=1}^{\infty}$ satisfying $p_n(x_0) = 1$, $|p_n|_{E_n} < 1/2^n$ and $|p_n - p_{n-1}|_{E_n} < 1/2^n$. Let $A$ denote the projective limit of $(\text{Hol}(D_n))_{n=1}^{\infty}$ using our previous construction. We view $A$ as an algebra of functions on $C = \bigcup_{n=1}^{\infty} D_n$. Then the boundary $\Gamma$ is $\{ x \in C : x \text{ real} \}$.

**Proposition 3.8.** The algebra $A$ is a singly generated Liouville F-algebra. Moreover, the sequence $(p_n(x))_{n=1}^{\infty}$ determines a non-zero $f \in A$ satisfying $f \mid \Gamma = 0$. Furthermore, $f(x) = 0$ whenever $\text{Im}(x) \leq 0$.

**Proof.** A is clearly a singly generated semisimple F-algebra. The proof that $A$ satisfies the Liouville property is the same as in [3]. It suffices to prove that $\lim_{x \to \infty} \|p_n(x)\|_{L_{\infty}([0,1])}$ exists for each $x \in C$ and $f \mid D_n \in \text{Hol}(D_n)$ for each $n = 1, 2, \ldots$. Let $l$ denote a fixed positive integer. We prove that $(p_n(x))_{n=1}^{\infty}$ is a Cauchy sequence on $D_l$. Let $\varepsilon > 0$ be given. Choose $j > 0$ such that $j > l$ and $1/2^{j+1} < \varepsilon$. Let $m > j$. We will show that $|p_m(x) - p_j(x)| < \varepsilon$ for each $x \in D_l$. Let $x$ be a fixed point in $D_l$.

If $x \in D_l \cap V_m$ then

$$|p_m(x) - p_j(x)| \leq |p_m(x)| + |p_j(x)| \leq |p_m|_{L_{\infty}([0,1])} + |p_j|_{L_{\infty}([0,1])} < 1/2^m + 1/2^j < 1/2^{j+1}$$

since $x \in V_m$. If $x \not\in D_l \cap V_m$ then there exists some $k > 0$ such that $x \in E_k \cup E'_k$ but $x \not\in E_n \cup E'_n$. Now $m > k$ since $x \in E_k \cup E'_k$.

**Case I.** Let $j > k$. Then $x \in D_j$, $E_j \subseteq E_k \cup E'_k$ since $j > 1$. If $x \in E_j$ then

$$|p_m(x) - p_j(x)| \leq \sum_{n=1}^{j-1} |p_n(x) - p_{n+1}(x)| \leq \sum_{n=1}^{m-1} |p_n(x) - p_{n+1}(x)|$$

$$\leq \sum_{n=1}^{m-1} 1/2^{n+1} < 1/2^j.$$

If $x \not\in E_j$ then

$$|p_m(x) - p_j(x)| \leq |p_m(x)| + |p_j(x)| \leq |p_m|_{L_{\infty}([0,1])} + |p_j|_{L_{\infty}([0,1])} < 1/2^m + 1/2^j < 1/2^{j+1}.$$
Case II. Let $k \geq j$. Recall that $x \in E_{k+1} \cup E_{j+1}$. Let $x \in E_{k+1}$. Since $x \in E_k$ for $k > j$ and since $x \in \mathcal{V}_k \cap D_k \in \mathcal{V}_j$ for $k > j$ we have:

$$|p_n(x) - p_j(x)| \leq \sum_{k=n}^{j} \left| p_n(x) - p_k(x) \right| + \left| p_k(x) + |p_j(x)| \right| \leq \sum_{k=n}^{j-1} \left| p_n(x) - p_k(x) \right| + \left| p_k(x) + |p_j(x)| \right| \leq \sum_{k=n}^{j-1} 1/2^{k+1} + 1/2^{j} < 1/2^{j-1}.$$

Now let $x \in E_{k+1}$. Then $x \in \mathcal{V}_k$. Since $x \in V_k \cap D_k \in \mathcal{V}_j$ for $k \geq j$ and

$$|p_n(x) - p_j(x)| \leq \left| p_n(x) \right| + \left| p_j(x) \right| \leq \left| p_n(x) \right| + \left| p_j(x) \right| < 1/2^{k} + 1/2^{j} < 1/2^{j-1}$$

Thus in all cases, if $x \in D_k$ then $|p_n(x) - p_j(x)| < 1/2^{j-1} < \epsilon$. Then $(p_n(x))_{n=0}^{\infty}$ converges uniformly on $D_k$ to a function $f_k$ in $\text{Hol}(D_k)$. Since $f_{k+1} \mid D_k = f_k$ then $f(x) = \lim_{n \to \infty} p_n(x)$ exists for each $x \in C$ and defines an element $f$ of $A$. We note that $f(x) = \lim_{n \to \infty} p_n(x) = 1$. Also, if $x$ is a real number then $x \cap \mathcal{V}_n$ for some $k \geq 1$ and $|p_n(x)| < 1/2^{k}$ for $n \geq k$. Thus $f(x) = \lim_{n \to \infty} p_n(x) = 0$ if $x \notin \mathcal{V}_k$ i.e., $f \mid \mathcal{V}_k = 0$. Similarly, $|p_n(x)| < 1/2^{k}$ implies $f(x) = 0$ whenever $n \geq k$. This completes the proof of the proposition.

The algebra of entire functions and the published examples of Liouville $F$-algebras are all integral domains. Example 3.7 illustrates that this is not a consequence of the Liouville property. By reversing the roles of $E_x$ and $E_y$ we may construct a $g \in A$ with $g \neq 0$ and $g(x) = 0$ whenever $x \notin \mathcal{V}_k$. Then $f \mid \mathcal{V}_k = 0$. If $f \mid \mathcal{V}_k = 0$ and $g \neq 0$ and thus $A$ has divisors of zero.

4. Algebraically principal closed maximal ideals in singly generated $F$-algebras. For Banach algebras we have the following characterization of principal maximal ideals. This answers a question posed in [12].

Proposition 4.1. Let $A$ be a commutative semisimple Banach algebra with an identity element. If $x \in M_A$ and $h^{-1}(0) = xA$ for some $x \in A$ then $h \mid \mathcal{V}_k = 0$ if and only if $x$ is isolated in $M_A$.

Proof. If $x \in M_A$ and $h^{-1}(0) = xA$ where $x$ is isolated in $M_A$ then we may assume that $x$ is an idempotent element in $A$ with $h(x) = 0$ and $h \mid \mathcal{V}_k = 0$. It follows that $h \mid \mathcal{V}_k$. Conversely, assume that $h$ is not isolated in $M_A$. Now $y(x) = xy$ for $x \in A$ is continuous and one-one since $h$ is not isolated in $M_A$. Moreover, $y(x)$ is bounded by the Inverse Mapping Theorem. Now $\|y(x)\| = \sup_{y \in A} \|\text{supp}(y)\| \leq 1/|y(x)|$ since $y \neq 0$ if and only if $y \neq 0$ using the fact that $h$ is not isolated in $M_A$ and $x(x') \neq 0$ for each $h \in M_A$ such that $h \neq 0$. Thus $\inf_{y \in A} \|y(x)\| = M \neq 0$. By a result of Arens’ [6] there exists an extension $B$ of $A$ in which $x \in A$ has an inverse. Thus $h \in B$ does not extend to a multiplicative linear functional on $B$ and it follows that $h \mid \mathcal{V}_k = 0$.

Proposition 4.2. If $A$ is a singly generated semisimple $F$-algebra without proper idempotent elements then $A$ contains a dense subalgebra $B$ such that:

1. $B$ is a singly generated semisimple $F$-algebra without proper idempotent elements in a stronger topology.

2. $M_B$ and $M_A$ are homeomorphic with respect to their direct limit topologies.

3. $\Gamma_B$ is homeomorphic with $\Gamma_A$ with respect to the direct limit topologies.

4. Each closed maximal ideal of $B$ is algebraically principal.

Proof. Let $A$ be a choice of generator for $A$. $A$ is the inverse limit of $\{A_i\}_{i=1}^{\infty}$. We assume without loss of generality that the seminorms are chosen on $A$ such that $|\tilde{a}|_n = |\tilde{a}|_m$ and $|\tilde{a}|_m = |\tilde{a}||\tilde{a}|_m$ for each $\tilde{a} \in A$ and $n \geq m$. We identify $M_A$ with $s(a)$ (pointwise) and $M_A$ with $s(a)$. As noted in Section 2, $M_A$ is homeomorphic with $\sigma_{\lambda}(a)$. Since $A$ is a semisimple algebra, we may obtain a concrete realization for elements in $A$ as functions on $s(a)$ by defining $s(\lambda) = A(h)$ where $\lambda = \sigma_{\lambda}(a)$ for each $h \in M_A$. In particular, $\lambda = \lambda$ for each $\lambda \in s(a)$.

If $a(x)$ is a polynomial with complex coefficients in the complex variable $z$ then $a(x) = p(z) + (z-\lambda)g(z)$ where $g(z)$ is a polynomial in $z$. We note that $g(z)$ is uniquely determined and when $\lambda$ is allowed to vary, $g(z)$ becomes a polynomial in the two variables $\lambda$ and $z$. Now $p(a) \in A$ and $p(\lambda) = p(x) + (x-\lambda)g(x)$ where $g(x) \in A$ is a similar decomposition in $A$ for each $\lambda \in s(a)$. The semimultiplicity of $A$ and the fact that $M_A$ has no isolated points guarantees the uniqueness of the representation.

We define a new sequence of seminorms on the algebra of polynomials, $s(a) = A$, inductively as follows: Set $s_{\lambda}(a) = s_{\lambda}(a) = s(a)$ and write $p$ for $p(a)$. If $|\tilde{a}|_n$ has been defined for $i = 1, 2, \ldots, n-1$ then set $|\tilde{a}|_n = |\tilde{a}|_n$.
Then \((g_{0,1})_{n}^{\infty}\) converges uniformly in \(\lambda \sigma(a_{n})\) and
\[
[p_{n}-p_{m}] = [p_{n}-p_{m}] + \sup_{\lambda \in \sigma(a_{n})} |g_{n,1}-g_{m,1}|, \quad n, m \in \mathbb{N}.
\]
Thus \(p_{n}(a_{n}) \to b \in \mathcal{B}_{1}\). Now if \(\omega \in \sigma(a_{n})\), then there exists \(h \in M_{a_{n}}\) such that \(\lambda \sigma(a_{n}) = \omega\). But \(h(b_{n}) = \lim_{n \to \infty} p_{n}(a_{n}) = \lim_{n \to \infty} p_{n}(a)\) which is a contradiction since \(p_{n}(a) \not\to n\) for each \(n \in \mathbb{N}\).

Now assume inductively that \(\sigma(a_{n+1}) = \sigma(a_{n})\). It suffices to prove that \(\sigma(a_{n+1}) = \sigma(a_{n})\). Choose a simple closed rectifiable curve \(\gamma\) about \(\sigma(a_{n+1}) = \sigma(a_{n})\) as before, where \(\sigma(a_{n+1}) \cup \sigma(a_{n}) = \gamma\). Theorem 5.1 ([3]), with our induction hypothesis, guarantees constants \(K_{a_{n+1}}\) and \(K_{a_n}\) such that
\[
|p_{n}(a_{n})| \leq K_{a_{n+1}} \sup_{y} |p_{n}(a_{n})| \quad \text{and} \quad |p_{n}(a_{n})| \leq K_{a_{n}} \sup_{y} |p_{n}(a_{n})| \quad \text{for each polynomial}\ p_{n}.
\]
Next choose a particular sequence \(p_{n}(a_{n})\) where \(p_{n}(a_{n}) \equiv 1/\tau - n\) on \(\sigma(a_{n})\) and \(p_{n}(a_{n}) \equiv n\) for each \(n \in \mathbb{N}\). Following the previous argument we have:
\[
[p_{n_{0}} - p_{n_{0}}] = |p_{n_{0}} - p_{n_{0}}| + \sup_{\omega \in \sigma(a_{n})} |g_{n,1} - g_{m,1}| = 0.
\]

Then \((g_{0,1})_{n}^{\infty}\) converges uniformly in \(\lambda \sigma(a_{n})\) and
\[
[p_{n}-p_{m}] = [p_{n}-p_{m}] + \sup_{\lambda \in \sigma(a_{n})} |g_{n,1}-g_{m,1}|, \quad n, m \in \mathbb{N}.
\]
Thus \(p_{n}(a_{n}) \to b \in \mathcal{B}_{1}\). Now if \(\omega \in \sigma(a_{n})\), then there exists \(h \in M_{a_{n}}\) such that \(\lambda \sigma(a_{n}) = \omega\). But \(h(b_{n}) = \lim_{n \to \infty} p_{n}(a_{n}) = \lim_{n \to \infty} p_{n}(a)\) which is a contradiction since \(p_{n}(a) \not\to n\) for each \(n \in \mathbb{N}\).

Now assume inductively that \(\sigma(a_{n+1}) = \sigma(a_{n})\). It suffices to prove that \(\sigma(a_{n+1}) = \sigma(a_{n})\). Choose a simple closed rectifiable curve \(\gamma\) about \(\sigma(a_{n+1}) = \sigma(a_{n})\) as before, where \(\sigma(a_{n+1}) \cup \sigma(a_{n}) = \gamma\). Theorem 5.1 ([3]), with our induction hypothesis, guarantees constants \(K_{a_{n+1}}\) and \(K_{a_n}\) such that
\[
|p_{n}(a_{n})| \leq K_{a_{n+1}} \sup_{y} |p_{n}(a_{n})| \quad \text{and} \quad |p_{n}(a_{n})| \leq K_{a_{n}} \sup_{y} |p_{n}(a_{n})| \quad \text{for each polynomial}\ p_{n}.
\]
Next choose a particular sequence \(p_{n}(a_{n})\) where \(p_{n}(a_{n}) \equiv 1/\tau - n\) on \(\sigma(a_{n})\) and \(p_{n}(a_{n}) \equiv n\) for each \(n \in \mathbb{N}\). Following the previous argument we have:
\[
[p_{n_{0}} - p_{n_{0}}] = |p_{n_{0}} - p_{n_{0}}| + \sup_{\omega \in \sigma(a_{n})} |g_{n,1} - g_{m,1}| = 0.
\]
Applying \( \psi \), we obtain \( \psi(b) = b(\lambda)e + (a - \lambda)e \psi(b) \) since \( b(\lambda) = \psi(b)(\lambda) \) for each \( \lambda \in \sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a) \). If \( \psi(b) = 0 \) then \( 0 = (a - \lambda)e \psi(b) \). If \( \lambda, \lambda' \in \sigma_{\mathcal{A}}(a) \) and \( \lambda' \neq \lambda \) then \( ((a - \lambda)e \psi(b))(\lambda') = 0 \) implies \( \psi(b)(\lambda') = 0 \) since \( (a - \lambda)e(\lambda') = \lambda' - \lambda \neq 0 \). The fact that \( A \) contains no proper idempotent elements implies that \( \lambda \) is not an isolated point in \( \sigma_{\mathcal{A}}(a) \) (c.f. the proof of 2.5). Thus \( \psi(b)(\lambda) = 0 \) for each \( \lambda \in \sigma_{\mathcal{A}}(a) \), i.e., \( \psi(b) = 0 \). The semisimplicity of \( A \) implies that \( \psi(b) = 0 \).

By our definition of \( \|b\|_{n+1} \) we have:

\[
|b|_n = \|\psi(b)\|_n + \sup_{\lambda \in \sigma_{\mathcal{A}}(a)} \|a - \lambda e\|_n \sup_{\lambda \in \sigma_{\mathcal{B}}(a)} \|\psi(b)\|_n.
\]

In general, for \( n > 1 \) we have:

\[
|b|_n = \|\psi(b)\|_n + \sup_{\lambda \in \sigma_{\mathcal{A}}(a)} \|a - \lambda e\|_{n-1} \sup_{\lambda \in \sigma_{\mathcal{B}}(a)} \|\psi(b)\|_{n-1}.
\]

By applying (1) we obtain \( |b|_0 = 0 \). Since \( \psi(b) = 0 \), the previous argument applied to \( b_0 \) implies that \( |b|_0 = 0 \). In general, if \( |b|_{n-1} = 0 \) then the above argument implies \( |b|_{n-1} = 0 \) for each \( \lambda \in \sigma_{\mathcal{A}}(a) \). By applying (2) we obtain \( |b|_n = 0 \). Since \( B \) is an \( F \)-algebra we have that \( b = 0 \). Thus \( \psi: B \rightarrow A \) is an isomorphism.

Now \( b = b(\lambda)e + (a - \lambda)e \psi(b) \) is a unique representation for \( b \in B \) since \( \psi \) is an isomorphism and \( A \) is semisimple. Thus each closed maximal ideal in \( B \) is algebraically principal. This completes the proof of the proposition.

**Corollary 4.3.** If \( A \) is a singly generated Liouville \( F \)-algebra, then \( A \) contains a dense subalgebra \( B \) such that:

1. \( B \) is a singly generated Liouville \( F \)-algebra in a stronger topology.
2. \( A \) is isomorphic with \( B \) if and only if \( B \) is isomorphic with \( A \).
3. Each closed maximal ideal in \( B \) is algebraically principal.

**Proof.** Let \( B \) be the algebra constructed in the previous proposition. We identify \( B \) with its image \( \psi(B) \subseteq A \). If \( b \in B \subseteq A \) has a bounded spectrum with respect to \( B \), then \( M_{\lambda} = M_{\lambda'} = \{x \in A : \lambda x = x\} \). Each \( M_{\lambda} \) has a bounded spectrum with respect to \( A \). Hence \( B \) is a Liouville \( F \)-algebra. Moreover, \( A \) is isomorphic with \( B \) if and only if \( J_A = 0 \) (Proposition 3.6). The corollary now follows from the fact that \( J_A = 0 \).

**Definition 4.4.** (Arena) Let \( A \) be a topological algebra and \( x \in A \). Then \( x - \lambda e \) is a strong topological divisor of zero if either \( R_{x - \lambda e} \) or \( L_{x - \lambda e} \) is not a topological isomorphism into \( \overline{A} \) (c.f. p. 46).

**Definition 4.5.** (Michael) Let \( A \) be a locally \( m \)-convex algebra and let \( x \in A \). Then \( x + \lambda e \) is a topological divisor of zero in \( A \) if, whenever \( \{T_n\} \subseteq A \), there exists an \( n \) such that \( x + \lambda e \) is a topological divisor of zero in \( A \) (c.f. p. 47).

For Banach algebras, these two definitions are equivalent (c.f. Michael). Michael [9] notes in Proposition 11.3 that Definition 4.5 is stronger than Definition 4.4, and raises the question of their equivalence. Keshva (7) has recently given an example where there are topological divisors of zero that are not strong topological divisors of zero, but his algebra is not semisimple. Our Proposition 4.2 provides a class of semisimple algebras in which the two definitions are different. We note that the conditions of the following proposition are satisfied if \( A \) is the algebra obtained by applying Proposition 4.2 to Example 2.4.

**Proposition 4.6.** Let \( A \) denote a singly generated semisimple \( F \)-algebra without proper idempotent elements, and \( h, h' \in M_A \). Then \( a - \lambda e \) is not a strong topological divisor of zero in \( A \) if and only if \( h^{-1}(0) \) is algebraically principal. Moreover, if \( h \in \Gamma_A \) then \( a - \lambda e \) is a topological divisor of zero in \( A \).

**Proof.** Now \( R_{a - \lambda e} \) is one-one. If \( (a - \lambda e) \neq 0 \) for some \( f \in A \), then \( f(h) = 0 \) for each \( h' \neq h, h' \in M_A \). The algebra \( A \) has no proper idempotent elements and hence \( h \) is not isolated in \( M_A \). Thus \( f = 0 \) and the semisimplicity of \( A \) implies \( f = 0 \). Multiplication is continuous in \( A \), so \( R_{a - \lambda e} \) is continuous. If \( (a - \lambda e)A = h^{-1}(0) \) then \( (a - \lambda e)A \) is closed in \( A \) and \( R_{a - \lambda e} \) is a topological isomorphism by the Inverse Mapping Theorem. Conversely, if \( R_{a - \lambda e} \) is a topological isomorphism, then \( (a - \lambda e)A \) is complete and hence closed in \( A \). But \( (a - \lambda e)A \) is dense in \( h^{-1}(0) \) since \( A \) is singly generated by \( a \). Thus \( h^{-1}(0) = (a - \lambda e)A \).

Proposition 3.6 states that \( \Gamma_A \) is independent of the choice of \( m \)-base for \( A \). Now fix an \( m \)-base for \( A \). If \( h \in \Gamma_A \), then \( (a - \lambda e)h = h(\lambda) - \lambda = 0 \). But this implies that \( h_\lambda - \lambda e \) has a transform which vanishes on the Shilov boundary of \( A \), and \( a - \lambda e \) is a topological divisor of zero according to Definition 4.5.

**Bibliography**


On a class of operators on Orlicz spaces

by

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Abstract. Let $L^p$ be an Orlicz space over a $\sigma$-finite measure space. If $X$ is a Banach space and $\iota: L^p \to X$ is a linear operator, $\| \iota \|_\Phi = \sup \sum_{i=1}^n |a_i| \| \chi_{E_i} \|_\Phi$ where the supremum is taken over all measurable simple functions $f = \sum_{i=1}^n a_i \chi_{E_i}$, $\{E_i\}$ disjoint and $|a_i| \leq 1$.

Under fairly general assumptions on $X$ and $\iota$ it is shown that $\| \iota \|_\Phi < \infty$ if and only if $\iota(f) = \int f \mu$ where $\mu : \mathbb{B} \to \mathbb{X}$ is measurable and the above Bochner integral exists for all $f \in L^p$. Consequently it is shown that such operators are compact. Finally, under moderate assumptions on $X$, it is shown that $\iota: L^p \to L^q$ has $\| \iota \|_\Phi < \infty$ if and only if $\iota$ is adjoint is of finite double norm, thus providing a new characterization of Hilbert-Schmidt operators.

1. Introduction. Let $(\Omega, \Sigma, \mu)$ be a sigma-finite measure space, $\Phi$ and $\Psi$ be complementary Young’s functions and $L^\Phi(\Omega, \Sigma, \mu) (= L^\Phi)$ and $L^\Psi(\Omega, \Sigma, \mu) (= L^\Psi)$ be the corresponding Orlicz spaces of (equivalence classes of) measurable functions on $\Omega$. $L^\Phi$ is a Banach space under each of the equivalent norms $N_\Phi$ and $\| \cdot \|_\Phi$ defined for $f \in L^\Phi$ by $N_\Phi(f) = \inf \{ \int E \geq 0 : f \in L^\Phi \}$ and $\| f \|_\Phi = \sup \int f \mu$ if $g \in L^\Omega$, $N_\Phi(g) \leq 1$. If $X$ is a Banach space and $\iota$ is a bounded linear operator mapping $L^\Phi$ into $X$, Dixmieux has defined $\| \iota \|_\Phi$ by $\| \iota \|_\Phi = \sup \sum |a_i| \| \chi_{E_i} \|_\Phi$,

where the supremum is taken over all measurable simple functions, $f = \sum a_i \chi_{E_i}$, $\{E_i\} \subset \Sigma$ disjoint, such that $N_\Phi(f) \leq 1$. This norm for operators has been the subject of some study by Dixmieux in [1], [2], and [3]. The purpose of this note centers around proving a Bochner integral representation theorem for these operators, examining their compactness properties and looking at their rather close relationship with operators of finite double norm [8].