On some countably modulated spaces

by

J. MUSIELAK and A. WASZAK (Poznań)

1. The theory of countably modulated spaces was started by [1]. In that paper the notions of a countably modulated space and of a uniformly countably modulated space were defined, and investigated in case of atomless finite measures and purely atomic infinite measures. Also the problem of equality of these two spaces was solved in the above case.

Next, the theory of countably modulated spaces was developed in [3] and [4]. There are considered countably modulated spaces and uniformly countably modulated spaces defined by means of families of non-negative measures and by means of various sequences of pseudomodulars. The results of [3] generalize those of [1] concerning finite atomless measures.

In this paper we shall deal with countably modulated spaces and uniformly countably modulated spaces defined by means of families of infinite purely atomic measures. The problem of equality of the above two spaces is investigated.

1.1. In the sequel the following notations and terminology will be used:

Let a real linear space $X$ be given and let $\varphi$ be a functional defined on $X$ with values $0 \leq \varphi(x) < \infty$. This functional will be called a modular, if it satisfies the following conditions:

A.1. $\varphi(x) = 0$ if and only if $x = 0$.

A.2. $\varphi(-x) = \varphi(x)$.

A.3. $\varphi(\alpha x + \beta y) \leq \varphi(x) + \varphi(y)$ for every $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

If in place of A.1, $\varphi$ satisfies only the condition $\varphi(0) = 0$, then $\varphi$ is called a pseudomodular (see [2]).

By a $\varphi$-function we understand a continuous, non-decreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. 
Let \( q_i \) be \( \varphi \)-functions. In the sequel we shall often make use of the following conditions:

(a) \( q_i(u) \) are equicontinuous at \( u = 0 \);

(b) for every \( i \) there exist positive constants \( \lambda_i, \beta_i, \delta_i \) such that for every \( u \geq \delta_i \) and \( k \geq i \) the inequality \( q_k(\lambda_i u) \leq \beta_i q_k(u) \) holds;

(c) there exist positive constants \( \lambda_i, \alpha_i, \delta_i \) and an index \( i_k \) such that \( q_k(u) \leq \lambda_i q_k(\alpha_i u) \) for \( u \geq \delta_i \) and \( k \geq i_k \);

(d) for every \( \varepsilon > 0 \) there exist numbers \( u_0, \eta > 0 \) and \( \alpha_i > 0 \), depending on \( i \), such that \( q_k(\alpha_i u) < q_k(u) \) for \( 0 < \eta \leq \alpha_i, u \geq u_0 \) (see [1] and [2]).

1.2. The countably modular space \( X \) and the uniformly countably modular space \( X_0 \) are defined as follows. Let \( q_i, i = 1, 2, \ldots \) be a sequence of pseudomodulars in a real linear space \( X \), and let \( q_i(0) = 0 \) for all \( i \). Implies \( x = 0 \). First, we define the modulars

\[
\phi(x) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \frac{q_i(x)}{1+q_i(x)}, \quad \phi(0) = \sup_i q_i(0),
\]

and then we define the spaces

\[
X = \{ x : \phi(x) \to 0 \text{ as } \lambda \to 0, x \neq 0 \},
\]

\[
X_0 = \{ x : \phi(x) \to 0 \text{ as } \lambda \to 0, x \neq 0 \},
\]

In [1] the problem under which conditions the identity \( X = X_0 \) holds was investigated. Two cases were considered.

In the first case, the pseudomodulars are defined as

\[
\phi_i(x) = \int_{E} \varphi_i(|x(t)|) \, d\mu,
\]

where \( \mu \) is a finite atomless measure on a \( \sigma \)-algebra \( \mathcal{E} \) of subsets of a set \( E \), and \( X \) is the space of all \( \varphi \)-measurable functions defined on \( E \).

In the second case, the pseudomodulars are of the form

\[
\phi_i(x) = \sum_{j=1}^{\infty} \omega_j \varphi_i(|x_j|),
\]

where \( X \) is the space of all real sequences (or real bounded sequences), and \( \omega_j \) is a sequence of positive numbers such that \( \lim \omega_j > 0 \) (or \( \lim \omega_j = 0 \) and \( \sum_{j=1}^{\infty} \omega_j = \infty \) for a sequence of indices \( \{ j_k \} \)).

1.3. A pseudomodular more general than (1) was considered in [3] namely

\[
\phi_i(x) = \sup_{E} \int_{E} \varphi_i(|x(t)|) \, d\mu,
\]

where \( \mu_i \) is a family of non-negative measures on \( \mathcal{E}_i \) and \( \tau \in \mathcal{E}_i \), where \( \mathcal{F} \) is an abstract set. In this case the following two properties of the family \( \{ \mu_i \} \) are needed in order to investigate the problem of the identity \( X = X_0 \):

(a) the family of measures \( \{ \mu_i \} \) is called uniformly bounded, if there exists a constant \( K > 0 \) such that \( \mu_i(B) \leq K \) for all \( x \in \mathcal{F}^i \);

(b) the family of measures is called equistable, if there exists \( \eta > 0 \) such that for each sequence of numbers \( q_1, q_2, \ldots \) for which \( q_i \leq \eta = \frac{1}{2} \), \( \frac{q_{i+1}}{q_i} \leq \frac{1}{2} \) for all \( i \), there exist constants \( K > 0 \) and a sequence of pairwise disjoint sets \( \mathcal{A} \in \mathcal{F} \) satisfying the condition \( 0 < \sum_{\mathcal{A}} \mu_i \leq 1 \).

The following theorem holds (see [3]):

If the family of measures \( \{ \mu_i \} \) possesses the properties (a) and (b) and if the \( \varphi \)-functions \( q_i \) satisfy conditions (a), (b), and (c), then the identity \( X = X_0 \) holds if and only if \( q_i \) satisfy condition (c).

In case when \( \mathcal{F} = \mathcal{F} \cup \{ t_0 \} \), where \( t_0 \in \mathcal{F} \), is a topological space, a sequence of pseudomodulars \( \{ \phi_i \} \) may be defined as follows (see [4]):

\[
\phi_i(x) = \lim_{\tau \to 0} \int_{E} \varphi_i(|x(t)|) \, d\mu,
\]

or, equivalently,

\[
\phi_i(x) = \inf \sup \int_{U \subseteq \mathcal{F}} \varphi_i(|x(t)|) \, d\mu,
\]

where \( U \) runs over the set of all neighborhoods of \( t_0 \) in \( \mathcal{F} \).

In this case, in order to investigate the identity \( X = X_0 \), the family of measures \( \{ \mu_i \} \) must pass besides the property (b), the following property (t.e.b.) called topological equistability:

The family of measures \( \{ \mu_i \} \) is called topologically equistable in \( \mathcal{F} \), if there exists \( \eta > 0 \) such that for any sequence of numbers \( q_1 \leq 0 \) satisfying the inequalities \( q_i \leq \eta, \frac{q_{i+1}}{q_i} \leq \frac{1}{2} \) for all \( i \), there exist constants \( \lambda > 0 \) and a sequence of pairwise disjoint sets \( \mathcal{A} \in \mathcal{E}_i \) for which \( \lambda < \sum_{\mathcal{A}} \mu_i \leq 1 \).

The following theorem holds (see [4]):

If the family of measures \( \{ \mu_i \} \) possesses the properties (a) and (b) and if the \( \varphi \)-functions \( q_i \) satisfy conditions (a), (b), and (c), then the identity \( X = X_0 \) holds if and only if \( q_i \) satisfy condition (c).

It is easily seen that if we take in \( \mathcal{F} \) the coarsest topology, then the pseudomodulars (4) are reduced to the pseudomodulars (3), and the property (t.e.b.) is identical with the property (a).

1.4. Now, let us consider the case of purely atomic finite measures defined in the following manner by means of a non-negative matrix \( (a_{ij}) \) containing no column consisting of zeros only:

\[
\mu_i = \sum_{t_0 \in \mathcal{F}} a_{ii} \quad \text{for } A = (a_{ij}) \in \mathcal{E}_i, \quad \mu_i(0) = 0.
\]
In [3], 5.1, there are given sufficient conditions in order that the family of measures \( \mu_n \), defined by (2), be equispellatable.

Also in [3] and [4] there are given examples of matrices for which the respective families of measures possess property (a) or (t.o).

2.1. Now, we shall investigate the case of a family of infinite atomic measures \( \mu_n \) defined by means of formula (5). Then the pseudomodulars (3) are of the form

\[
\phi_i(s) = \sup_{u \in I_i} \sum_{n=1}^{\infty} a_{i,n} \phi_i(u_i).
\]

Here, the following conditions will be of use in place of 1.1 (5) and (γ):

(8') for every index \( i \) there exist positive constants \( \lambda_i, \beta_i, \theta_i \) such that for every \( u \in \mathcal{B}_i \) and \( h > i \) the inequality \( \phi_i(\lambda_i u) \leq \beta_i \phi_i(u) \) holds;

(γ′) there exist positive constants \( \lambda, \gamma, \kappa, \nu \) and an index \( i_0 \) such that \( \phi_i(\nu u) \leq \kappa \phi_i(u) \) for \( 0 \leq u \leq u_0 \) and \( i \geq i_0 \).

2.2. The following conditions for the identity \( X_s = X_{s0} \) will be proved now in case of pseudomodulars (6):

THEOREM 1. Let \( \lim \inf_{n \to \infty} a_{a_i^+} > 0 \) for a fixed \( n_i \). If the \( \phi_i \) satisfy conditions (8') and (γ'), then \( X_s = X_{s0} \).

Proof. From (γ') we conclude that there are positive constants \( \lambda, \gamma, \kappa, \nu \) and an index \( i_0 \) such that \( \phi_i(\nu u) \leq \kappa \phi_i(u) \) for \( 0 \leq u \leq u_0 \) and \( i \geq i_0 \). Let \( s \in X_s \); then there are \( \lambda_i > 0 \) such that \( \phi_i(\lambda_i u) < \infty \) for all \( i \). Hence

\[
\lim_{n \to \infty} a_{a_i^+} \phi_i(\lambda_i u) = 0 \quad \text{for each } i,
\]

and so

\[
\lim_{n \to \infty} \phi_i(\lambda_i u) = 0.
\]

By continuity of \( \phi_i \), \( \lambda_i \to 0 \) as \( s \to \infty \), and so (ti) is bounded. Taking \( \lambda_i \) sufficiently small, we have \( |\lambda_i| \leq \nu_0 \) for all \( i \). Hence \( \phi_i(\lambda_i u) \leq 2 \kappa \phi_i(\nu u) \) for \( i \geq i_0 \) and all \( u \), i.e., \( \phi_i(\lambda_i u) \leq \kappa \phi_i(u) \) for \( i \geq i_0 \). Thus \( X_s = X_{s0} \).

THEOREM 2. Let \( \sum_{j=1}^{\infty} a_{a_j^+} = \infty \) for a fixed index \( n_j \), and \( \sup a_{a_j^+} \leq M < \infty \) for all \( j \), i.e., \( \sum_{j=1}^{\infty} a_{a_j^+} \leq M < \infty \).

Proof. Let \( s \in X_s \), but for every \( \lambda_i, \gamma, \kappa_0 > 0 \) and every \( i_0 \), there exist \( 0 \leq u \leq u_0 \) and \( i \geq i_0 \) such that \( \phi_i(\kappa u) \geq \delta \phi_i(u) \). Given \( h > 0 \), we choose \( \delta = 3 \). Then there exist \( u_{n,m,h} \geq n \) and \( u_{n,m,h} \leq 1 / \delta \) such that

\[\phi_{n,m,h}(2^{-h} u_{n,m,h}) > 2^{h} \phi_{n}(u_{n,m,h})\]

for \( n, m, h = 1, 2, \ldots \). Now, we define an increasing sequence of indices \( m_n \) as follows. We choose \( m_1 \) so large that \( m_1 \geq 1 / \delta \) and \( \phi_{n}(1/m_n) \leq \min (1, 1/2 \delta M) \) and we put \( u_1 = u_{n,m_1} \). Let us suppose the numbers \( m_1, m_2, \ldots, m_{n-1} \) are defined in such a manner that \( \phi_{n}(1/m_n) \leq 1, m_n \geq 1 / \delta \) and \( \phi_{n}(1/m_n) \leq \phi_{n-1}(u_{n-1}) \), where \( u_n = u_{n,m_n} \), \( i = 2, 3, \ldots, n-1 \). Then we take \( m_n \) so large that \( \phi_{n}(1/m_n) \leq \min (1, 1/2 \delta M) \) and \( m_n \geq 1 / \delta \) and \( \phi_{n}(1/m_n) \leq \phi_{n-1}(u_{n-1}) \), and we put \( u_n = u_{n,m_n} \). Then \( \phi_{n}(u_k) \leq \phi_{n}(1/m_n) \leq \phi_{n-1}(u_{n-1}) \) for \( k = 2, 3, \ldots \).

We show now that there exists a sequence \( \{n_i\}, \{\nu_i\}, \ldots \) of pairwise disjoint sets of indices such that

\[
\sup_{n} \sum_{i=1}^{n} a_{a_{n_i^+}} \phi_{i}(u_{i}) \leq 1/2\delta - 1
\]

for all \( k \). It is sufficient to construct one set \( \{n_i\}, \{\nu_i\}, \ldots \) in such a manner that \( \nu_i \) is arbitrarily large. Let \( \nu_i \) be fixed. We shall prove indirectly that such a set \( \{n_i\} \) exists. In other case, there are three possibilities:

1° the inequality

\[
\sup_{n} \sum_{i=1}^{n} a_{a_{n_i^+}} \phi_{i}(u_{i}) \leq 1/2\delta - 1
\]

holds for all sequences \( \{j_i\} \) and all \( m \geq 1 \);

2° there is a sequence \( \{j_i\} \) and \( m \geq 1 \) such that (9) holds, but

\[
\sup_{n} \sum_{i=1}^{n} a_{a_{n_i^+}} \phi_{i}(u_{i}) \geq 1/2\delta - 1
\]

3° (10) holds always.

In the case 1°, we take in (9) \( j_i = j_{i+1} - (r-1) \) and \( m \to \infty \), and the assumption \( \sum_{i=1}^{\infty} a_{a_{n_i^+}} = \infty \) yields a contradiction.

In the case 2°, there exists an index \( n = n_k \) such that

\[
\sum_{i=1}^{n_k} a_{a_{n_i^+}} \phi_{i}(u_{i}) \leq 1/2\delta \quad \text{and} \quad \sum_{i=1}^{n_k} a_{a_{n_i^+}} \phi_{i}(u_{i}) \geq 1/2\delta - 1
\]

Hence \( a_{a_{n_k+1}^+} \phi_{i}(u_{i}) \geq 1/2\delta \), and so

\[
\frac{1}{2\delta} \geq M \phi_{i} \left( \frac{1}{m_n} \right) \quad \text{for} \quad M \phi_{i} (u_i) \geq a_{a_{n_k+1}^+} \phi_{i}(u_{i}) \geq \frac{1}{2\delta},
\]

a contradiction.
In the case 3', the contradiction is obtained as in 2', taking $m+1 = j_1$.

Now, we define $x = (t_k)$, where

$$
t_k = \begin{cases} 
  u_k & \text{for } k = 1, 2, \ldots, \\
  0 & \text{elsewhere.}
\end{cases}
$$

Then

$$
\epsilon \psi (\lambda; x) \leq \sup_{n} \sum_{k=1}^{n-1} \mu_k A_k \psi (\lambda, u_k) + \sup_{n} \sum_{k=n}^{\infty} \mu_k A_k \psi (\lambda, u_k).
$$

But, by (8), the first term at the right-hand side of the above inequality is less than or equal to

$$
(1-1) \psi (\lambda; u_k) \frac{1}{2^{2-1} \psi (u_k)},
$$

and the second one is less than or equal to $\beta \sum_{k=1}^{\infty} 2^{-k+1} < \infty$. Hence

$$
\epsilon \psi (\lambda; x) < \infty.
$$

Let us take in (8), $x = \epsilon/2 \psi (\lambda; \psi)$, and let $\lambda' > 0$ be so small that $\lambda' \leq \lambda$ and $\lambda' u_k \leq u_k$ for all $k$; this is possible because $u_k \leq 1/\mu_k$. Let $\epsilon' = \epsilon/2 \psi (\lambda; \psi) > \epsilon$; then $u_k \geq \epsilon u_k$ and so $\lambda' u_k \leq u_k$ for all $k$. Hence, if $0 < \lambda < a_k \lambda'$, then

$$
\epsilon \psi (\lambda; x) < \epsilon' \sum_{k=1}^{\infty} \mu_k A_k \psi (\lambda' u_k) = \epsilon' \psi (\lambda; x) = \frac{\lambda}{\psi}.
$$

Hence $x \in X_{\epsilon}$. Now, by (7) and (8),

$$
\epsilon \psi_{\lambda} 2^{1-\psi} \geq \sup_{n} \mu_k A_k \psi_{\lambda} 2^{1-\psi} \geq 2^{1-\psi} \psi (u_k) \sup_{n} \mu_k A_k > 1,
$$

and so $x \in X_{\epsilon}$, a contradiction.

Let us remark that the results of 3.2 and 3.3 in [1] follow from our theorems 1 and 2 if we put $a_n = \infty$ for $n, r = 1, 2, \ldots$; in case of 3.3, the assumption $\sum_{n=1}^{\infty} \psi_n = \infty$ must be replaced by $\sum_{n=1}^{\infty} c_n = \infty$, where $\lim_{r \to \infty} c_n = 0$, which was not mentioned in [1].

Theorems 1 and 2 imply the following

**Theorem 3.** Let us suppose that there exists a sequence of indices $(n_r)$ such that $\lim_{r \to \infty} \psi_{n_r} > 0$ for a fixed index $n$, and $\sup_{n} \psi_{n_r} = M$ for all $j$, where $M > 0$ is a constant. Let the $\gamma$-functions $\phi_k$ satisfy conditions (a), (3') and (8). Then $X_{\epsilon} = X_{\epsilon'}$ if and only if $\phi_k$ satisfy (9').

2.3. The results of 2.2 may be proved also if we replace the pseudo-modulars (6) by means of (4), where the measures are given by (5).