and, for $n = 2, 3, \ldots$

$$d_n \geq d_{n-1} - g_{n-1} + g_n \quad \text{and} \quad M(d_n) \leq M(d_{n-1} - g_{n-1} + g_n) + \varepsilon/2^n,$$

from which it follows that $d_1 \leq d_2 \leq \ldots$ and, for $n = 1, 2, \ldots$

$$(14) \quad d_n \geq g_n \quad \text{and} \quad M(d_n) - M(g_n) \leq \varepsilon \left( \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \right) \leq \varepsilon.$$

For all $x \in X$, $\lim_n d_n(x) \geq \lim_n g_n(x) = 0$ hence, from (d), $\lim_n \inf d_n(x) > 0$ and so $\lim_n M(d_n) > 0$. Combining this with (14), $\lim_n M(g_n) = -\varepsilon$ and the required result follows since $\varepsilon$ is arbitrary.

References


On the function $g_r^+$ and the heat equation

by

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INTRODUCTION AND NOTATIONS

In the present paper, a function analogous to the $g_r^+$ function of Littlewood, Paley, Zygmund, Stein (see [13] and [10]) is introduced for functions $u(x, t, y)$ which are solutions of the boundary problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^r u}{\partial x_r^2} + \frac{\partial^s u}{\partial y^s}, \quad y > 0$$

and

$$\lim u(x, t, y) = f(x, t).$$

The definition of $g_r^+$ is given in section 3, (2.1), and its properties concerning the preservation of $L^p$ classes are discussed in theorems (2.2), (2.3), and (2.4). The method used here is an adaptation to the parabolic case of the one found in C. L. Pefferman's doctoral dissertation [2]. In section 3, theorem (3.1), the function $g_r^+$ is applied to obtain a characterization of the $L^p$ spaces introduced by B. F. Jones in [4] and [5]. This characterization is suggested by those given by Hirschman [3] and Stein [9]. Also, a generalization of the $g$-function of Littlewood-Paley involving fractional derivatives is considered (theorem (2.25)). For an analogue in the case of analytic and harmonic functions, see [3] and [8].

We shall denote by $E_{n+1}$ the set of all $(a+1)$-tuples $(x_1, \ldots, x_n, t) = (x, t)$ of real numbers, with the explicit intention of distinguishing the last variable. $E_{n+1}$ denotes the set of all $(a+3)$-tuples $(x_1, \ldots, x_n, t, y)$ of real numbers with $y > 0$. By $|x|$ we denote the absolute value of $(x_1, \ldots, x_n)$, which is given by $(\sum_{i=1}^{n} |x_i|^a)^{1/a}$. The complement of a set $A$ is denoted by $A'$ and its Lebesgue measure by $|A|$. The definition of Fourier
transform of a function \( f(x,t) \) we use is
\[
\hat{f}(x,t) = \int_{E_{n+1}} \exp(-i(y_1x_1 + \cdots + y_nx_n + tx)) f(x, t) dx dt.
\]

Therefore, Plancherel's theorem reads as follows:
\[
\int_{E_{n+1}} \hat{f}(x, t) \hat{g}(x, t) dx dt = \frac{1}{(2\pi)^{n+1}} \int_{E_{n+1}} \hat{f}(x, t) \hat{g}(x, t) dx dt.
\]

By \( \mathcal{S} \) we denote the space of all infinitely differentiable functions defined on \( E_{n+1} \) with derivatives decreasing at infinity faster than any polynomial. The dual space \( \mathcal{S}' \) is the so-called space of tempered distributions.

\section{Preliminaries}

Let us consider the kernel \( \Gamma(x, t, y) \), \( y > 0 \), introduced by B. F. Jones in [4], and defined as
\[
\Gamma(x, t, y) = \begin{cases} \frac{y}{(4\pi t)^{n/2} \rho^2} \exp \left( -\frac{|x|^2 + y^2}{4t} \right), & t > 0, \\ 0, & t \leq 0. \end{cases}
\]

We list here some properties of this kernel that will be used later:
1. The kernel \( \Gamma(x, t, y) \) is positive and
\[
\int_{E_{n+1}} \Gamma(x, t, y) dx dt = 1
\]
for every \( y > 0 \).
2. If \( y_1 \) and \( y_2 \) are positive numbers then
\[
\int_{E_{n+1}} \Gamma(x, t, y_1) \Gamma(x, t, y_2) dx dt = \int_{E_{n+1}} \Gamma(x, t, y_1 + y_2) dx dt.
\]
3. If \( f(x, t) \in L^p(E_{n+1}), 1 \leq p < \infty \), then
\[
u(x, t, y) = \int_{E_{n+1}} \Gamma(x, t, y-s) f(x, s) dx ds
\]
satisfies
(a) \( \|\nu(x, t, y)\|_p \leq \|f\|_p \), for every \( y > 0 \), and
(b) \( \lim_{y \to 0} \nu(x, t, y) = f(x, t) \) a.e. and in norm.
4. If \( y' = y/(y-1) \), then
\[
\int_{E_{n+1}} \Gamma(x, t, y) dx dt \leq A_{n, p} y^{-(n+1)/p'}
\]
for \( y > 0 \).

5. The Fourier transform of \( \Gamma(x, t, y) \) in the variables \( x, t \) is given by
\[
\hat{\Gamma}(x, t, y) = \exp(-iy^2/4t) \exp(-iy^2/4t)
\]
for \( y > 0 \). The branch of \( \hat{\Gamma} \) chosen is that assigning positive values to \( \hat{\Gamma} \) for positive values of the variable \( x \).

6. For the partial derivatives of \( \Gamma(x, t, y) \) we have the estimates:
\[
\begin{align*}
|\partial_x^m \partial_t^n \partial_y^r \Gamma(x, t, y)| &\leq C t^{-n+2n+|y|^2/4t} e^{-y^2/4t} \exp(-|y|^2/4t)
\end{align*}
\]
and
\[
|\partial_x^m \partial_t^n \partial_y^r \Gamma(x, t, y)| \leq C e^{-y^2/4t} \Gamma(x, 2t, y).
\]

7. The function
\[
\left( |x| + |t|^{2n} + y^2 \right) \left( \frac{1}{|t|^{2n}} \exp(-|y|^2/4t) \right)
\]
is bounded on \( E_{n+2} \) for every \( r > 0 \).

For the proofs of these and other properties of the kernel \( \Gamma(x, t, y) \), see [4] and [3].

The following change of coordinates was introduced by E. Fabes and N. Riviere in [1]. Let
\[
x = \varphi_1 \sin \varphi_1 \cdots \sin \varphi_{n-1} \sin \varphi_n,
\]
(1.3)
\[
x = \varphi_1 \sin \varphi_2 \cdots \cos \varphi_{n-1},
\]
\[
\varphi_1 \cos \varphi_1,
\]
where \( 0 \leq \varphi_n \leq 2\pi \) and \( 0 \leq \varphi_i \leq \pi \) for \( i = 1, \ldots, n-1 \). We shall refer to this as the parabolic polar coordinates on \( E_{n+1} \). Its Jacobian satisfies
\[
\left| \frac{\partial(x, t)}{\partial(x, y)} \right| = \varphi^{n+1} (1 + \cos \varphi_1) \cdots (1 + \cos \varphi_{n-1})
\]
or also
\[
dx dt = \varphi^{n+1} (1 + \cos \varphi_1) \cdots (1 + \cos \varphi_{n-1})
\]
d\Omega denotes the element of area of the unit sphere in \( E_{n+1} \). From the definition of parabolic polar coordinates it follows that
\[
\frac{|x|^2}{\varphi^2} + \frac{t^2}{\varphi^2} = 1,
\]
which implies that
\[
\frac{t}{\varphi} \leq \sqrt{|x|^2 + |t|^2} \leq 2\varphi.
\]
We define the function \( \mathcal{F}^s(x, t) \) as:
\[
\mathcal{F}^s(x, t) = \begin{cases} 
\frac{1}{(4\pi|z|^2)^{1/4}} \exp\left(-\frac{|x|^2}{4t}\right), & t > 0, \\
0, & t < 0
\end{cases}
\]
for \( 0 < \alpha < n+2 \).

By (1.2), this function is locally integrable, and if \( f(x, t) \) is a well-behaved function, say belonging to \( \mathcal{F}^s \), the integral
\[
\int_{\mathbb{R}^{n+1}} \mathcal{F}^s(x, t) \cdot f(x, t) \, dx \, dt
\]
is absolutely convergent and defines an operator that will be denoted by \( \mathcal{F}_s(f) \) and called the \textit{parabolic fractional integral of order} \( \alpha \). It is easy to see that the function \( \mathcal{F}^s(x, t) \) defines a distribution belonging to \( \mathcal{F}^s \) and we shall compute its Fourier transform. Let \( \varphi(x, t) \in \mathcal{F}^s \). We have
\[
\int_{\mathbb{R}^{n+1}} \overline{\varphi}(x, t) \mathcal{F}(x, t) \, dx \, dt = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \overline{\varphi}(x, t) \exp\left(-y\sqrt{|x|^2 + 4t}\right) \, dx \, dt.
\]

Multiplying both members by \( y^\alpha \), integrating \( y \) from 0 to \( \infty \) and changing the order of integration, we get
\[
\int_{\mathbb{R}^{n+1}} \overline{\varphi}(x, t) \, dx \, dt = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \overline{\varphi}(x, t) \exp\left(-y\sqrt{|x|^2 + 4t}\right) \, dx \, dt = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \varphi(x, t) \exp\left(-y\sqrt{|x|^2 + 4t}\right) \, dx \, dt,
\]
or since
\[
\int_{\mathbb{R}^{n+1}} \mathcal{F}(x, t) \varphi(x, t) \, dx \, dt = \int_{\mathbb{R}^{n+1}} \mathcal{F}(x, t) \varphi(x, t) \, dx \, dt = \int_{\mathbb{R}^{n+1}} \mathcal{F}(x, t) \varphi(x, t) \, dx \, dt,
\]
we get
\[
\mathcal{F}^s(x, t) = \left\{\mathcal{F}(x, t)\right\}^{-\alpha/2}.
\]
This expression for the Fourier transform of \( \mathcal{F}^s \) and the next theorem justify calling (1.4) the \textit{parabolic fractional integral of order} \( \alpha \).

Parabolic version of the Sobolev theorem on fractional integrals (see also [17]):
\[
(1.6) \quad \text{Theorem. Let} \quad f(x, t) \ast \mathcal{F}(E_{\alpha+1}) \quad \text{and} \quad 1/p = 1/q - \alpha/(n+2), \quad 0 < \alpha < n+2, \quad 1 < p, q < \infty. \quad \text{The parabolic fractional integral operator of order} \ \alpha \ \text{satisfies}
\]
\[
\|\mathcal{F}_s(f)\|_q \leq C_{\alpha, n} \|f\|_p
\]
where the constant \( C_{\alpha, n} \) does not depend on \( f \).

Proof. We shall show that \( \mathcal{F}_s \) is an operator of weak type \( (p, q) \) for every \( p \) satisfying \( 1 < p < (n+2)/\alpha \). Let \( K_\alpha \) and \( K_{\alpha, n} \) be the restrictions of \( \mathcal{F}_s(x, t) \) to the sets \( \alpha \leq \mu \) and \( \mu \geq x \) respectively. Then
\[
(1.7) \quad \|K_\alpha\|_1 \leq c \mu^{\sigma - 2},
\]
\[
(1.8) \quad \|K_{\alpha, n}\|_p \leq C \int_\mathbb{R} \mu^{n-2\sigma + \alpha + 1} \, d\mu = C \mu^{n-2\sigma + \alpha + 1},
\]
where \( \sigma' = p/(p-1) \), and
\[
(1.9) \quad \|K_{\alpha, n}\|_p \leq C \mu^{n-2\sigma - 2}.
\]
Moreover,
\[
(1.10) \quad \left|\{(x, t) : |\mathcal{F}_s(f)(x, t)| > 2\lambda\}\right| \leq \left|\{(x, t) : |K_\alpha \ast f(x, t)| > \lambda\}\right| + \left|\{(x, t) : |K_{\alpha, n} \ast f(x, t)| > \lambda\}\right|.
\]
Let us consider first the case \( p = 1, \|f\|_1 = 1 \). If we choose \( \mu \) to be a constant times \( \lambda^{-(n+1)} \) then (1.9) implies that the second term on the right of (1.10) is zero. By Young's theorem and (1.7), the first term is majorized by \( \alpha^{n+1} \lambda^{-1} = \alpha^{-(n+2)(n+2)} \). Therefore,

\[
[\{(x, t): |I_{\gamma}(f)(x, t)| > 2\} \lesssim \alpha^{-(n+2)(n+2)},
\]

which shows that \( I_{\gamma} \) is an operator of weak type \( (1, (n+2)/(n+2)) \).

The case \( 1 < p < (n+2)/a \) is similar if we choose \( \mu \) to be constant times \( \lambda^{-(n+1)} \) and use (1.8) instead of (1.9). The theorem then follows from the Marcinkiewicz interpolation theorem. (See (14),)

Let us consider now the function \( \mathcal{F}^\mu(x, t), a > 0 \), defined as

\[
\mathcal{F}^\mu(x, t) = \begin{cases} 
\frac{1}{(4\pi t)^n} \exp \left\{ -t - \frac{|x|^2}{4t} \right\}, & t > 0, \\
0, & t < 0.
\end{cases}
\]

It is shown in [4] that \( \mathcal{F}^\mu(x, t) \) is integrable over \( \mathbb{E}_{a+1} \) with integral 1, and that its Fourier transform is \( (1 + |\omega|^2)^{-\frac{n+1}{2}} \). For \( f \in L^p(\mathbb{E}_{a+1}), 1 \leq p < \infty \), the operator

\[
\mathcal{F}_{\gamma}(f)(x, t) = \int_{\mathbb{E}_{a+1}} \mathcal{F}^\mu(x, t-s)f(s, t)dsds
\]

is therefore well-defined and maps \( L^p(\mathbb{E}_{a+1}) \) into \( L^p(\mathbb{E}_{a+1}) \).

(1.11) Definition. The space \( \mathcal{F}^\mu(\mathbb{E}_{a+1}), a > 0 \) and \( 1 \leq p < \infty \), is defined as the image under \( \mathcal{F}^\mu \) of \( L^p(\mathbb{E}_{a+1}) \); that is, \( f(x, t) \in \mathcal{F}^\mu \) if and only if

\[
f(x, t) = \mathcal{F}^\mu(f)(x, t)
\]

for some \( f \in L^p(\mathbb{E}_{a+1}) \). The norm in \( \mathcal{F}^\mu \) is defined by

\[
\|f\|_{\mathcal{F}^\mu} = \|f\|_{L^p(a+1)}.
\]

Now, let \( f \in L^p(\mathbb{E}_{a+1}) \) and denote by \( M(f) \) the function

\[
M(f)(x, t) = \sup_I \left\{ \frac{1}{|I|} \int_I |f(x, t)|dsdt \right\}
\]

where \( I \) denotes a parallelepiped of the form

\[
I = \{(x, s): |x-x_0| < h, \ldots, |x_n-x_0| < h, |s-s_0| < h^3\}
\]

containing the point \( (x, t) \). The transformation from \( f \) to \( M(f) \) is of weak type \( (p, p) \) for \( 1 < p < \infty \). (See (12).)

We now state a theorem whose proof is similar to that of theorem 6 of [0] and will therefore be omitted.

\[
\text{(1.12) Theorem. Let } \mathcal{G}(x, t) \text{ be a function defined on } \mathbb{E}_{a+1} \text{ which satisfies } \mathcal{G}(x, t) \leq \mathcal{G}(s), \text{ where } \phi(x, t) \geq 0, \text{ is non-negative, decreasing and such that}
\]

\[
\int_{(t)} \mathcal{G}(x, t)dxdt < \infty.
\]

Then the function \( g(\mathcal{G}(x, t)) \) given by

\[
g(\mathcal{G}(x, t)) = \int_{\mathbb{E}_{a+1}} f(x, s)k\left(\frac{x-x}{\epsilon}, \frac{t-s}{\epsilon}\right)dsds
\]

satisfies

\[
\sup_{x, t} |g(\mathcal{G}(x, t))| \leq cM(f)(x, t),
\]

where the constant \( c \) is independent of \( f \).

The function \( g(f)(x, t) \) defined as

\[
g(f)(x, t) = \int_{\mathbb{E}_{a+1}} f(x, s) \frac{\partial u}{\partial y}(x, s, t) dy
\]

where \( u(x, t, y) = f(x, t) \Gamma(x, t, y) \), was introduced in [2]. In that paper, it was shown that if \( f \in L^p(\mathbb{E}_{a+1}), 1 < p < \infty \), then (1.13)

\[
\alpha \|f\|_p \leq \|g(f)\|_p \leq \alpha \|f\|_p,
\]

the constants being positive and independent of \( f \).

\section*{Section 2}

\begin{center}
Parabolic Analogue of the \( \mathcal{G}^\mu \) Function of Littlewood, Paley, Zygmund, Stein,
\end{center}

We now define our analogue of the \( \mathcal{G}^\mu \) function and study its properties regarding the preservation of \( L^p \). The methods we use are those developed by C. L. Fefferman in his thesis [2]. We also discuss a variant of the \( \mathcal{G} \)-function which involves derivatives of a fractional order (see [3] and [8]) which will be needed in section 3.

Let \( u(x, t, y) \) be the convolution of a function \( f \in L^p(\mathbb{E}_{a+1}), 1 < p < \infty \), and \( \Gamma(x, t, y) \). For \( \lambda > 1 \), we define \( \mathcal{G}^\mu(f)(x, t) \) as

\[
\mathcal{G}^\mu(f)(x, t) = \left( \int_{\mathbb{E}_{a+1}} \left| \frac{y}{|x-y|+|t-s|+y}\right|^{(n+2)} \frac{\partial u}{\partial y}(x, s, t) \right)^{1/2} dydydy.
\]
We shall prove

(2.3) Theorem. Let $1 < p < \infty$ and $p > 2$. There exist two positive constants $c_1$ and $c_2$ such that

$$c_1 \|f\|_p \leq \|g_1^*(f)\|_p \leq c_2 \|f\|_p$$

for every $f \in L^p(E_{n+1})$. The constants $c_1$, $c_2$ depend only on $\lambda$, $p$, and $n$.

The part of the theorem asserting that $c_1 \|f\|_p \leq \|g_1^*(f)\|_p$ will be discussed at the end of this chapter. The part $\|g_1^*(f)\|_p \leq c_2 \|f\|_p$ is a consequence of the Marcinkiewicz interpolation theorem (see (14)) and the following two theorems.

(2.3) Theorem. Let $\lambda > 1$ and $p > 2$. There exists a constant $c$ such that

$$c \|g_1^*(f)\|_p \leq \|f\|_p$$

for every $f \in L^p(E_{n+1})$. The constant $c$ depends only on $\lambda$, $p$, and $n$.

(2.4) Theorem. Let $1 < p < 2$ and $\lambda = 2/p$. There exists a constant $c > 0$ such that

$$|(s, t); g_1^*(f)(s, t) > \mu| \leq c \mu^{-1} \|f\|_p$$

for every $\mu > 0$ and $f \in L^p(E_{n+1})$. The constant $c$ depends only on $n$ and $p$.

Proof of theorem (2.3). For $p \geq 2$, let $q = p/(p-2)$ and $h(x, t) \equiv \lambda$ and $L^q(E_{n+1})$. From the definition of $g_1^*(f)(x, t)$ and a change in the order of integration, we have

$$\int_{\mathbb{R}_{n+1}} h(x, t) \left[ g_1^*(f)(x, t) \right]^2 \, dx \, dt = \int_{\mathbb{R}_{n+1}} \frac{\partial^2}{\partial y^2} (x, s, t) \left[ \frac{y}{x-s} \right] \left[ \frac{y}{|x-s|^2 + y} \right]^{(n+1)/2} h(x, t) \, dx \, dt.$$

Since $(|x| + |y|^{1/2})^{(n+1)/2}$ is integrable over $E_{n+1}$, for $\lambda > 1$, we obtain from theorem (1.12) that

$$\sup_{p \geq 2} \left[ \frac{1}{y^{n+1}} \int_{\mathbb{R}_{n+1}} \left[ \frac{y}{|x-s| + |s-y|^3 + y} \right]^{(n+2)/2} h(x, t) \, dx \, dt \right] \leq c M(h)(x, s).$$

Thus

$$\int_{\mathbb{R}_{n+1}} h(x, t) \left[ g_1^*(f)(x, t) \right]^2 \, dx \, dt \leq c \int_{\mathbb{R}_{n+1}} \frac{\partial^2}{\partial y^2} (x, s, t) \left[ \frac{y}{x-s} \right] \left[ \frac{y}{|x-s|^2 + y} \right]^{(n+1)/2} M(h)(x, s) \, dx \, dy$$

$$= c \int_{\mathbb{R}_{n+1}} M(h)(x, s) \left[ g_1^*(f)(x, s) \right]^2 \, dx \, dy.$$

The last member above is less than or equal to

$$c \|h\|_1 \|g_1^*(f)\|_p^2,$$

and since $h$ is an arbitrary non-negative function in $L^q(E_{n+1})$, it follows that

$$\|g_1^*(f)\|_p \leq c \|f\|_p.$$

Applying (1.13) we get $\|g_1^*(f)\|_p \leq c \|f\|_p$, which is the statement of the theorem.

Proof of theorem (2.4). Let $I(x, t)$ be any parallelepiped containing $(x, t)$, not necessarily as its center, of the type

$$I_{x, t} = \left( \begin{array}{c} (s, t) : |s-x| < \lambda, \ldots, |s-x| < \lambda, |t-y| < \lambda \end{array} \right),$$

and let $I(x, t)$ be the function

$$\tilde{f}(x, t) = \max \left\{ \frac{1}{|I_{x, t}|} \int_{I_{x, t}} \left[ f(x, s) \right]^2 \, ds \, dt \right\}.$$

where $f \in L^p(E_{n+1})$. Since $\tilde{f}(x, t)$ belongs to $L^p(E_{n+1})$, we have from section 1 that

$$\|(s, t); \tilde{f}(x, t) < \mu| \leq \frac{A}{\mu^2} \int_{E_{n+1}} \left[ f(x, s) \right]^2 \, ds \, dt.$$

Let us denote $(s, t); \tilde{f}(x, t) < \mu|$ by $\Omega$. Since $\Omega$ is open and has finite measure the distance from a point $(x, t) \in \Omega$ to $\Omega^c$ is bounded. Denote by $G_{x, t} = \infty < k < \infty$, the grid of non-overlapping parallelepipeds of the form

$$\left\{ (s, t) ; \frac{m_1}{2^k} < s < \frac{m_1+1}{2^k}, \frac{m_2}{2^k} < t < \frac{m_2+1}{2^k} \right\},$$

where $m_1$ and $m_2$ are arbitrary integers. These parallelepipeds have sides of lengths $1/2^k$ in the $x_1$ directions and $1/2^k$ in the $t$ direction. In general, if $I$ denotes a parallelepiped with sides of lengths $\lambda$ in the $x_i$ directions and $\lambda$ in the $t$ direction, then $I, I^*$, and $I^{**}$ denote parallelepipeds with the same center as $I$ and sides parallel to those of $I$ with lengths $2\lambda$, $4\lambda$, and $8\lambda$, respectively. We define $I_{x, t}$ to be the family of all $I \subset \Omega_{x, t}$ not contained in a parallelepiped belonging to $I_{k'}$ with $k' < k$, and such that $I \subset \Omega$. Observe that all the $I_{x, t}$ with indices less than a certain number are empty. We claim that the parallelepipeds in all the $I_{x, t}$ are not covered, then for every $k$ we have at least one parallelepiped $I_{x, t}$ such that $(s, t) \in I_{x, t}$ and $I^{**} \subset \Omega$. Pick a point $(s_0, t_0) \in \Omega$ and $(s_1, t_1) \in \Omega$. Clearly the sequence $(s_0, t_0)$ converges to $(s_1, t_1)$, and hence $(s_1, t_1) \in \Omega$, which is a contradiction.

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the parallelepipeds belonging to all the \( \Omega_k \) can be ordered in a sequence, which we shall denote by \( \{I_i\} \). These parallelepipeds are non-overlapping and

\[
\sum_i |I_i| = |\Omega| \leq \frac{A}{\mu^p} \|f\|_p.
\]

If \( (x, t) \notin \Omega \) then \( \bar{x}(x, t) \leq \mu^p \), so, by differentiation, we get

\[
|f(x, t)| \leq \mu \quad \text{a.e.}
\]

for \( (x, t) \in \Omega \).

Since \( I_i^{(x)} \subset \Omega_i \), there exists \( (x, t) \in I_i^{(x)} \cap \Omega_i \). Thus,

\[
\frac{1}{|I_i^{(x)}|} \int_{I_i^{(x)}} |f(x, s)|^p \, dx \, ds \leq \mu^p
\]

so

\[
\frac{1}{|I_i^{(x)}|} \int_{I_i^{(x)}} |f(x, s)|^p \, dx \, ds \leq \mu^p,
\]

and since \( |I_i^{(x)}| = (16\pi)^{n+3} |I_i| \), we obtain

\[
(2.7) \quad \frac{1}{|I_i^{(x)}|} \int_{I_i^{(x)}} |f(x, s)|^p \, dx \, ds \leq A \mu^p.
\]

Let \( l_1 \) and \( l_2 \) be the lengths of the sides of \( I_i \). Then, if \( (s, t) \in I_1 \) and \( s, s_1 \) are the lengths of the sides of the largest parallelepiped with center at \( (x, t) \) contained in \( \Omega_i \), there exist finite constants \( c_1, c_2 \) which are independent of \( i \) and such that

\[
(2.8) \quad c_1 l_1 \leq s \leq c_2 l_1.
\]

In particular, if \( I_j \) intersects \( I_i \), we get the fact that the ratio of \( I_i \) and \( I_j \) is bounded above and below by positive constants which are independent of \( i \) and \( j \).

To show (2.8), observe that since \( I_1 \subset \Omega_i \), then \( s \geq c_1 l_1 \). Also, since \( I_i^{(x)} \subset \Omega_i \), we have \( s \leq c_2 l_1 \).

We define \( f'(x, t) \) as

\[
f'(x, t) = \begin{cases} \frac{1}{|I_i^{(x)}|} \int_{I_i^{(x)}} f(x, s) \, dx \, ds, & \text{if } (x, t) \in I_i, \\ f(x, t), & \text{if } (x, t) \notin \Omega, \end{cases}
\]

and \( f''(x, t) = f(x, t) - f'(x, t) \). Note that \( |f'(x, t)| \leq A \mu \) a.e. (by (2.7)),

\[
|f''|_p \leq \frac{|f''|}{p} \quad \text{and} \quad |f''|_p \leq 2 |f|_p,
\]

\( f'' \) is supported on \( \Omega \),

\[
(2.10) \quad \frac{1}{|I_i^{(x)}|} \int_{I_i^{(x)}} |f''(x, s)|^p \, dx \, ds \leq A \mu^p \quad \text{(by (2.7))},
\]

\[
(2.11) \quad \int_{I_i^{(x)}} |f''(x, s)| \, dx \, ds = 0.
\]

To prove the theorem, it is enough to show that (2.5) holds for both \( f' \) and \( f'' \) instead of \( f \). For \( f' \) we have from theorem (2.3) that

\[
|\{x, t: g_\Omega(x, t) > \mu\}| \leq \frac{A}{\mu^p} \|f\|_p \leq \frac{A}{\mu^p} \mu^{p-\gamma} |f|^p,
\]

and by (2.9), the last member is less than or equal to \( \frac{A}{\mu^p} |f|^p \).

Consider now \( f'' \). Let

\[
f''(x, t, y) = f''(x, t) \ast \Gamma(x, t, y)
\]

and

\[
\frac{\partial u''}{\partial y} = f'' \ast \frac{\partial \Gamma}{\partial y} - \sum_i (f'' \cdot x_i \frac{\partial \Gamma}{\partial y_i})
\]

where \( x_i \) denotes the characteristic function of \( I_i \). Hence denoting \( (f'' \cdot x_i) \frac{\partial \Gamma}{\partial y_i} \) by \( h_i \), we have

\[
\frac{\partial u''}{\partial y} = \sum_i h_i
\]

and

\[
g''(x, t) = \left( \int_{K_n^{(x)} + y} \left| \frac{y}{g^{(x)}(x, t) + g^{(t)}(x, t)} \right|^{(n+2)\gamma} \right) \left( \sum_i h_i(x, s, y) \frac{dxdy}{y^{n+1}} \right)^{1/2}
\]

where

\[
g^{(x)}(x, t) = \left( \int_{K_n^{(x)} + y} \left| \frac{y}{g^{(x)}(x, t) + g^{(t)}(x, t)} \right|^{(n+2)\gamma} \right) \sum_{i, s \notin I_i} h_i(x, s, y) \frac{dxdy}{y^{n+1}}
\]

and

\[
g^{(t)}(x, t) = \left( \int_{K_n^{(x)} + y} \left| \frac{y}{g^{(x)}(x, t) + g^{(t)}(x, t)} \right|^{(n+2)\gamma} \right) \sum_{i, t \notin I_i} h_i(x, s, y) \frac{dxdy}{y^{n+1}}
\]

On the function \( g_\Omega^* \) and the heat equation
Consider first \( g^{(0)}(x, t) \). We claim that
\[
(2.12) \quad \left| \sum_{(n, \omega) \in I_i} h_i(x, t, y) \right| \leq \frac{A \mu}{y}.
\]
To prove this, we note

\[
\left| \sum_{(n, \omega) \in I_i} h_i(x, t, y) \right| \leq \sum_{(n, \omega) \in I_i} \int f''(\xi, \eta) \frac{\partial G}{\partial y}(z - \xi, s - \eta, y) \, d\xi \, d\eta \\
\leq \sum_{(n, \omega) \in I_i} \left[ \sup_{(n, \omega) \in I_i} \left| \frac{\partial G}{\partial y}(z - \xi, s - \eta, y) \right| \right] \int I_i f''(\xi, \eta) \, d\xi \, d\eta,
\]

which by (2.10) is less than
\[
(2.13) \quad A \mu \sum_{(n, \omega) \in I_i} \left| \frac{\partial G}{\partial y}(z - \xi, s - \eta, y) \right| \left| I_i \right|.
\]
We shall show later that for \((x, s) \notin I_i\), the inequality
\[
(2.13a) \quad \sup_{(n, \omega) \in I_i} \left| \frac{\partial G}{\partial y}(z - \xi, s - \eta, y) \right| \leq \frac{A}{|I_i|} \int I_i \left| z - \omega \right|^{-\alpha - n} \exp \left\{ -c \left| \frac{w^1 + y}{|z - \omega|} \right|^\alpha \right\} \, dw \, dy
\]
holds for positive constants \( A \) and \( c \) not depending on \( i \). If so, our last sum (2.13) is at most equal to
\[
A \mu \int_{|z - \omega|^\alpha} \left| z - \omega \right|^{-\alpha - n} \exp \left\{ -c \left| \frac{w^1 + y}{|z - \omega|} \right|^\alpha \right\} \, dw \, dy = \frac{A \mu}{y},
\]
which proves (2.12).

Hence,
\[
\left[ g^{(0)}(x, t) \right]^2 \leq A \mu \int_{|z - \omega|^\alpha} \left| z - \omega \right|^{-\alpha - n} \exp \left\{ -c \left| \frac{w^1 + y}{|z - \omega|} \right|^\alpha \right\} \left| \sum_{(n, \omega) \in I_i} h_i(x, t, y) \right| \, dw \, dy \\
= A \mu J(x, t).
\]

In order to show that
\[
(2.14) \quad \left| (x, t): \left[ g^{(0)}(x, t) \right] \geq \frac{A}{y} \right| \leq \frac{A}{\mu} \int I_i f''(u, \omega) \, du \, dw,
\]
it is enough to prove that
\[
(2.15) \quad \left| (x, t): \left[ f''(u, \omega) \right] > \frac{A}{\mu} \right| \leq \frac{A}{\mu} \left| f''(x, t) \right|^2.
\]
We claim that \( \left| J \right| \leq \frac{A}{y^2} \left| f''(x, t) \right|^2 \). If so, we get
\[
\left| (x, t): \left[ f''(u, \omega) \right] > \frac{A}{\mu} \right| \leq \frac{A}{\mu} \left| J \right| \leq \frac{A}{\mu} \left| f''(x, t) \right|^2,
\]
which proves (2.15). That \( \left| J \right| \leq \frac{A}{y^2} \left| f''(x, t) \right|^2 \) can be seen as follows:
\[
\left| J \right| = \int \int \left| \sum_{(n, \omega) \in I_i} h_i(x, t, y) \right| \, dw \, dy \int \left( \left| \frac{w^1 + y}{x - \omega} \right| \right)^{n+1} \, dw \, dt \\
= A \int \int \left| \sum_{(n, \omega) \in I_i} h_i(x, t, y) \right| \, dw \, dy \leq A \int \int \left| h_i(x, t, y) \right| \, dw \, dy.
\]

Now,
\[
\int \left| h_i(x, t, y) \right| \, dw \, dy = \int \int \left( \int f''(u, \omega) \frac{\partial G}{\partial y}(z - u, s - \omega, y) \, du \, dw \right) \, dw \, dy
\]
which, by (2.11), is
\[
(2.16) \quad \int \int \left( \int f''(u, \omega) \frac{\partial G}{\partial y}(z - u, s - \omega, y) - \frac{\partial G}{\partial y}(z - u, s - \omega, y) \right) \, du \, dw \, dy \left| x - u \right| \left| s - \omega \right| \left| y \right| \, dw \, dy
\]
where \((u, \omega)\) is the center of \( I_i \).

We shall prove later that for \((u, \omega) \notin I_i\), the integral
\[
(2.17) \quad \int \left| \frac{\partial G}{\partial y}(z - u, s - \omega, y) - \frac{\partial G}{\partial y}(z - u, s - \omega, y) \right| \, dw \, dy
\]
is less than a constant independent of \( i \). If so, changing the order of integration in (2.16) we have that
\[
A \int \int \left| f''(u, \omega) \right| \, du \, dw = A \int \left( \frac{1}{|I_i|} \int I_i \int f''(u, \omega) \, du \right)^2 \, dw \, dy.
\]
majorizes (2.16). By (2.10), this is smaller than $A \mu |I_i|$. Hence

$$
||f||_2 \leq A \mu \sum |I_i| = A \mu |\Omega| \leq \frac{A}{p-2} ||f||_p^3
$$

which proves our claim and therefore also (2.14).

Consider now $g_0(x, t)$. We shall show that

$$
||([x, t]; g_0(x, t) > \mu)|| \leq \frac{A}{p} ||f||_p^3
$$

where $p = 2/\lambda$. By (2.6) we may consider only points $(x, t) \in \Omega$. Then,

$$
[g_0(x_0, t)]^2 \leq \int_0^\infty \int_0^\infty \left( \sum_{(\alpha, \beta) \in I_i} h_i(x, s, y)^2 \right)^{\frac{p}{2}} ds dy \leq \sum_{(\alpha, \beta) \in I_i} \int_0^\infty \int_0^\infty \left( \sum_{(\alpha, \beta) \in I_i} h_i(x, s, y)^2 \right)^{\frac{p}{2}} ds dy.
$$

Now, for $(x, t) \in \Omega$ so $(x, t) \in I_i$ and $(x, s) \in I_i$, we have

$$
|x - s| + |t - s|^\alpha \geq c |x - s| + |t - s|^\beta
$$

where $(s, t)$ is the center of $I_i$. Hence

$$
[g_0(x_0, t)]^2 \leq c \sum_{(\alpha, \beta) \in I_i} \left( \frac{1}{|x - s| + |t - s|^\beta} \right)^{\frac{p}{2}} \int_0^\infty \int_0^\infty \left( \sum_{(\alpha, \beta) \in I_i} h_i(x, s, y)^2 \right)^{\frac{p}{2}} ds dy.
$$

For the inner sum, we have

$$
\sum_{(\alpha, \beta) \in I_i} h_i(x, s, y)^2 \leq \sum_{(\alpha, \beta) \in I_i} \int f''(u, w)^2 dy \left( |x - u, s - w, y| \right) du dw.
$$

If $(x, y, z, t) \in \Omega$, we have from (2.8) that $I_i \subset I_i$. Hence, denoting by $D_j$ the union of those $I_i$ which are contained in $I_i$, we obtain that (2.19) is less than or equal to

$$
\int f''(u, w)^2 \left| \frac{\partial f}{\partial y} (x - u, s - w, y) \right| du dw \leq c \int f''(u, w)^2 (s - w)^{-(n+3)/2} \exp \left( -\frac{|x - w|^2 + y^2}{8(s - w)} \right) du dw,
$$

where $f''(u, w)$ denotes the restriction of $|f''(u, w)|$ to $D_j$. This last integral is equal to a constant times

$$
\frac{1}{y} (f''(u, w))(x, s, y).
$$

Thus,

$$
\int \int g^{n+2l-1-n} \left( \sum_{(\alpha, \beta) \in I_i} h_i(x, s, y)^2 \right)^{\frac{p}{2}} ds dy \leq c \int g^{n+2l-1-n} (f''(u, w))^2 (x, s, y)^2 ds dy.
$$

By Plancherel's theorem, the last expression is equal to

$$
c \int g^{n+2l-1-n} \int \int f''(u, w)^2 \left( \exp \left( -y \left( |x|^2 + t \right) \right) \right) ds dy = c \int g^{n+2l-1-n} \int \int f''(u, w)^2 \left( \exp \left( -2y \text{Re} \left( \bar{w} + \bar{v} \right) \right) \right) dy ds
$$

and

$$
c \int g^{n+2l-1-n} \int \int f''(u, w)^2 \left( \exp \left( -2y \text{Re} \left( \bar{w} + \bar{v} \right) \right) \right) dy ds.
$$

But since $\text{Re} \left( |\bar{w} + \bar{v}| \right) \leq |\bar{w}| + |\bar{v}| \leq 2 |\bar{w}| + |\bar{v}|$, the last integral is majorized by

$$
c \int f''(x, s)^2 \left( \frac{n+2l-1}{2} \right) \int ds dy.
$$

Now, for $p = 2/\lambda$ we have the identity

$$
\frac{1}{p} \left( \frac{n+2l-1}{2} \right) \gamma \leq \frac{1}{p} \left( \frac{n+2l-1}{2} \right) \gamma + m/2.
$$

Therefore, by theorem (1.6), we have

$$
\left( \delta^m |(f''(u, w)|^2 \right) \leq c ||f''||_p^2.
$$

But by (2.10),

$$
||f''||_p^2 \leq \int f''(x, s, y)^2 ds dy \leq A \left( \sum_{i=0}^m \mu^p |I_i| \right)^{1/p}
$$

which by (2.8) is majorized by

$$
A \mu |I_i|^{1/p}.\]
Therefore, from (2.20) we get
\[ |g^{10}(x, t)|^2 \leq A_\beta^2 \sum_{i \in I_i} \sum_{j \in J_j} \int_{I_i^{0\beta}} \int_{J_j^{0\beta}} |I_i^{0\beta}| \cdot |J_j^{0\beta}| \cdot |I_i^{0\beta} + J_j^{0\beta}| \cdot A_\beta^2 H(x, t), \]
where \((x_i, t_i)\) is the center of \(I_i\) and \((x_j, t_j)\) of \(J_j\).

To complete the proof of (2.18) it is enough to show that
\[ |H(x, t)|^2 \cdot (x, t) \cdot H(x, t) \geq A_\beta^2 \cdot |f|^2. \]
This will follow if we can prove that
\[ \int_{\mathbb{R}^2} H(x, t) \cdot dx \cdot dt \leq A_\beta^2 \cdot |f|^2. \]
Now
\[ \int_{\mathbb{R}^2} H(x, t) \cdot dx \cdot dt = \sum_{i \in I_i} \sum_{j \in J_j} \int_{I_i^{0\beta}} \int_{J_j^{0\beta}} |I_i^{0\beta}| \cdot \int_{J_j^{0\beta}} \int_{J_j^{0\beta}} |I_i^{0\beta} + J_j^{0\beta}| \cdot A_\beta^2 H(x, t). \]
If \(I_i\) and \(J_j\) are the lengths of the sides of \(I_i\), we get
\[ \int_{\mathbb{R}^2} H(x, t) \cdot dx \cdot dt \leq A_\beta^2 \sum_{i \in I_i} \sum_{j \in J_j} |I_i^{0\beta} + J_j^{0\beta}| \cdot A_\beta^2 H(x, t). \]
which implies that
\[ \int_{\mathbb{R}^2} H(x, t) \cdot dx \cdot dt \leq A_\beta^2 \sum_{i \in I_i} \sum_{j \in J_j} |I_i^{0\beta} + J_j^{0\beta}| \cdot A_\beta^2 H(x, t). \]
This completes the proof of (2.18).

To complete the proof of the theorem we must show that (2.13a) and (2.17) are valid. Changing variables, we see it is enough to prove the following two lemmas.

(2.21) LEMMA. Let \(I\) denote a parallelepiped with center at the origin and edges parallel to the axes with lengths \(h\) and \(h^2\) for the \(x\) and \(y\) directions respectively. Then, if \((x, t) \in I\), we have
\[ \sup_{(x, t) \in I} |s - \eta|^{-\alpha + \beta / 2} \exp \left\{ - \frac{a}{|s - \eta|} \right\} \cdot \left| \frac{s - \xi - y}{|s - \eta|} \right|^2 \leq \frac{c}{|I|} \int_{\mathbb{R}^2} \left| \frac{s - \eta}{|s - \eta|} \right|^{-\alpha + \beta / 2} \exp \left\{ - \frac{b}{|s - \eta|} \right\} \cdot \left| \frac{s - \xi - y}{|s - \eta|} \right|^2 \cdot dx \cdot dy, \]
where \(b\) and \(c\) are constants depending on \(a\) and \(v\) only.

Proof. If \((x, t) \in I\), then \(|s| > \eta\) for at least one \(s\) or else \(|s| > 2a^2 h^2\). This implies \(|s| > \eta|\) or \(|s| > 2a^2 h^2\). We shall consider three cases: 1) \(|s| > \eta|\) and \(|s| > 2a^2 h^2\); 2) \(|s| > \eta|\) and \(|s| > 2a^2 h^2\); and 3) \(|s| \leq \eta|\) and \(|s| > 2a^2 h^2\).

We observe that under the assumption \((x, t) \in I\), the conditions \(|s| > \eta|\) and \(|s| > 2a^2 h^2\) imply that for a constant \(c > 1\)
\[ \frac{|s|}{c} < |s - \xi| < c \cdot |s| \quad \text{and} \quad \frac{|s|}{c} < |s - \eta| < c \cdot |s|, \]
respectively. Also, we remark that the function \(t^{-\alpha + \beta / 2} \exp \left\{ - \frac{a}{|s - \eta|} \right\} \cdot \left| \frac{s - \xi - y}{|s - \eta|} \right|^2 \)
has its maximum at \(t = 2a^2 (\alpha - 1)\), and that the value of this maximum is
\[ \left( \frac{n + 3}{2} \right)^{\alpha + \beta / 2} e^{-\alpha / 2}. \]

Case 1. Since \(|s| > \eta|\) and \(|s| > 2a^2 h^2\) we have
\[ |s - \eta|^{-\alpha + \beta / 2} \exp \left\{ - \frac{a}{|s - \eta|} \right\} \cdot \left| \frac{s - \xi - y}{|s - \eta|} \right|^2 \leq \left( \frac{n + 3}{2} \right)^{\alpha + \beta / 2} e^{-\alpha / 2}. \]

Also,
\[ |s - \eta|^{-\alpha + \beta / 2} \exp \left\{ - \frac{a}{|s - \eta|} \right\} \cdot \left| \frac{s - \xi - y}{|s - \eta|} \right|^2 \geq \frac{1}{\left( \frac{n + 3}{2} \right)^{\alpha + \beta / 2} e^{-\alpha / 2}} = \left( \frac{n + 3}{2} \right)^{\alpha + \beta / 2} e^{-\alpha / 2}. \]

which proves the lemma in this case.

Case 2. Since \(|s| > \eta|\), we have
\[ |s - \eta|^{-\alpha + \beta / 2} \exp \left\{ - \frac{a}{|s - \eta|} \right\} \cdot \left| \frac{s - \xi - y}{|s - \eta|} \right|^2 \leq \left( \frac{n + 3}{2} \right)^{\alpha + \beta / 2} e^{-\alpha / 2}. \]

Now, since \(|s| > \eta|\) and \(|s| > 2a^2 h^2\), our second remark shows that the last expression is less than
\[ \frac{1}{\left( \frac{n + 3}{2} \right)^{\alpha + \beta / 2} e^{-\alpha / 2}} \leq \frac{2a}{|s - \eta|} \cdot \left( \frac{3a h^2}{\alpha} \right)^{\alpha + \beta / 2} \exp \left\{ - \frac{a}{|s - \eta|} \right\} \cdot \left| \frac{s - \xi - y}{|s - \eta|} \right|^2 \]
(2.23)
\[ \frac{a + 3}{2} \cdot \frac{\alpha + 1}{2} \cdot \left( \frac{3a h^2}{\alpha} \right)^{\alpha + \beta / 2} \exp \left\{ - \frac{a}{|s - \eta|} \right\} \cdot \left| \frac{s - \xi - y}{|s - \eta|} \right|^2 \leq \left( \frac{n + 3}{2} \right)^{\alpha + \beta / 2} e^{-\alpha / 2}. \]

Also, we have that
\[ |s - \eta|^{-\alpha + \beta / 2} \exp \left\{ - \frac{a}{|s - \eta|} \right\} \cdot \left| \frac{s - \xi - y}{|s - \eta|} \right|^2 \leq \left( \frac{n + 3}{2} \right)^{\alpha + \beta / 2} e^{-\alpha / 2}. \]
Integrating this inequality over $I$, we get
\[ \int |\eta - \eta'|^{-\alpha} \exp \left\{ -b |s - \xi|^2 + y^2 \right\} d\xi d\eta \geq \frac{1}{(3\lambda)^{n+2}\pi} \int I \exp \left\{ -bc^2 \frac{|s - \xi|^2 + y^2}{|s - \eta|} \right\} d\xi d\eta \]
\[ \geq \frac{1}{(3\lambda)^{n+2}\pi} \int I \exp \left\{ -bc^2 \frac{|s - \xi|^2 + y^2}{h^2} \right\} d\xi d\eta \]
\[ \geq \frac{1}{(3\lambda)^{n+2}\pi} \exp \left\{ -4bc^2 \frac{|s|^2 + y^2}{h^2} \right\} |I| \cap \{ |s - \eta| > h \} \]

Since $|I| \cap \{(s, \eta): |s - \eta| > h/3\}$ is greater than a constant times $|I|$, we obtain
\[ \frac{1}{|I|} \int |s - \eta|^{-\alpha} \exp \left\{ -b |s - \xi|^2 + y^2 \right\} d\xi d\eta \geq \frac{A}{(3\lambda)^{n+2}\pi} \exp \left\{ -4bc^2 \frac{|s|^2 + y^2}{h^2} \right\} \]

If $3\lambda^2 < \frac{2a}{(n+3)c^2} (|s|^2 + y^2)$, the last expression multiplied by a constant is greater than (2.33), provided that the constant and $b$ are suitably chosen.

If $3\lambda^2 > \frac{2a}{(n+3)c^2} (|s|^2 + y^2)$, then
\[ \frac{A}{(3\lambda)^{n+2}\pi} \exp \left\{ -4bc^2 \frac{|s|^2 + y^2}{h^2} \right\} \geq \frac{A}{(3\lambda)^{n+2}\pi} \exp \left\{ -6(n+3)bc^2 \right\} \]

But since $|s| > h$, (2.23) is less than
\[ \frac{n+3}{2} \frac{c^{(n+3)/2}}{a} e^{-6(n+3)bc^2} h^2, \]
and the lemma follows in this case too.

**Case 3.** For this case we have
\[ |s - \eta|^{-\alpha} \exp \left\{ -a |s - \xi|^2 + y^2 \right\} \leq \left( \frac{\alpha(n+\alpha)}{\alpha} \right)^{1/2} \exp \left\{ -a \frac{y^2}{|s|^2} \right\} \]

Also,
\[ |s - \eta|^{-\alpha} \exp \left\{ -b |s - \xi|^2 + y^2 \right\} \geq \frac{1}{|s|^{n+2}} \exp \left\{ -4bc^2 \frac{|s|^2 + y^2}{|s|^2} \right\} \]
\[ \geq \exp \left\{ -2bc^2 \right\} \frac{1}{|s|^{n+2}} \exp \left\{ -4bc^2 \frac{y^2}{|s|^2} \right\} \]

which shows the validity of the lemma for case 3.
where $m$ is an integer, $0 < \gamma \leq 1$ and $\gamma + \beta = m$. To show that $u^{(\theta)}(x, t, y)$ is well defined for $y > 0$, we use the estimate

$$\frac{\partial^m \Gamma}{\partial y^m}(x, t, y + \eta) \leq c(y + \eta)^{n-1} \gamma \Gamma(x, 2t, y + \eta)$$

(see (1.1)) together with (1.0) to obtain

$$\frac{\partial^m u}{\partial y^m}(x, t, y + \eta) \leq c \|f\|_p (y + \eta)^{-n+1 \over 2}.$$

Hence,

$$\int_\varepsilon^\infty \frac{\partial^m u}{\partial y^m}(x, t, y + \eta) \eta^{-1} \, d\eta \leq c \|f\|_p y^{-n+1 \over 2},$$

which shows that $u^{(\theta)}(x, t, y)$ is finite for $y > 0$.

(2.23) Theorem. Let $f \in L^p(\mathcal{B}_m, 1 < p < \infty, \beta > 0$ and $u(x, t, y) = f(x, t) \ast \Gamma(x, t, y)$. There exist two positive constants $c_1, c_2$ such that the function

$$g_\beta(f)(x, t) = \left( \int_0^\infty |u^{(\theta)}(x, t, y)|^2 \, dy \right)^{1/2}$$

satisfies

$$c_1 \|f\|_p \leq \|g_\beta(f)\|_p \leq c_2 \|f\|_p.$$

The constants do not depend on $f$.

This function $g_\beta(f)(x, t)$ may be called the parabolic Littlewood–Paley function of fractional order $\beta$. To prove the theorem we need the following lemma.

(2.26) Lemma. Let $0 < \gamma < \beta'$. Then $g_{\beta'}(f)(x, t) \leq c g_{\beta}(f)(x, t)$, where the constant $c$ is independent of $f$.

Proof. It is easy to see that

$$u^{(\theta)}(x, t, y) = \int_0^\infty u^{(\theta)}(x, t, y + \eta) \eta^{-1} \, d\eta.$$

Hence, using Schwarz’s inequality and changing variables, we have

$$|u^{(\theta)}(x, t, y)|^2 \leq \int_0^\infty |u^{(\theta)}(x, t, \eta)|^2 \eta^{-1} \, d\eta \int_0^\infty \eta^{-n+1 \over 2} \eta^{-\beta} \, d\eta$$

$$= c \eta^{-n+1 \over 2} \int_0^\infty |u^{(\theta)}(x, t, \eta)|^2 \eta^{-1} \, d\eta \leq c \eta^{-n+1 \over 2} \int_0^\infty |u^{(\theta)}(x, t, \eta)|^2 \eta^{-1} \, d\eta.$$

provided that $\beta' - \beta < m$. Multiplying both members by $y^{m-1}$ and integrating with respect to $y$ we get

$$\int_0^\infty \gamma y^{m-1} \int_0^\infty |u^{(\theta)}(x, t, \eta)|^2 \eta^{-1} \, d\eta \leq c \int_0^\infty \gamma y^{m-1} \int_0^\infty |u^{(\theta)}(x, t, \eta)|^2 \eta^{-1} \, d\eta.$$

Interchanging the order of integration, we obtain

$$\int_0^\infty \gamma y^{m-1} \int_0^\infty |u^{(\theta)}(x, t, \eta)|^2 \eta^{-1} \, d\eta \leq c \int_0^\infty |u^{(\theta)}(x, t, \eta)|^2 \eta^{-1} \, d\eta \int_0^\infty \gamma y^{m-1} \eta^{-1} \, d\eta.$$

Now, if $\beta' < m$, the inner integral on the right equals $\eta^{-m+1}$ times the value of $\int_0^\infty \gamma y^m (1 - y)^{m-1} \eta^{-1} \, d\eta$. Therefore, we get

$$\int_0^\infty |u^{(\theta)}(x, t, y)|^2 \, dy \leq c \int_0^\infty \gamma y^{m-1} \int_0^\infty |u^{(\theta)}(x, t, \eta)|^2 \eta^{-1} \, d\eta \leq c \int_0^\infty |u^{(\theta)}(x, t, \eta)|^2 \eta^{-1} \, d\eta = c |g_{\beta'}(f)(x, t)|^2,$$

and the lemma is proved.

Proof of theorem (2.23). By lemma (2.26), we have

$$g_{\beta'}(f)(x, t) \leq c g_{\beta}(f)(x, t),$$

where $m > \beta + 2n + 2$. Consider the inequality,

$$\left| \frac{\partial^m u}{\partial y^m}(x, t, y / 2) \right| \leq c \int_{X_{m+1}} \left| \frac{\partial^m f}{\partial y^m}(\xi, \eta, y / 2) \right| \left| \frac{\partial^m \Gamma}{\partial y^m}(x, t, \eta) \right| \, d\xi \, d\eta.$$

By (1.1), we can write

$$\left| \frac{\partial^m u}{\partial y^m}(x, t, y) \right| \leq c \int_{X_{m+1}} \left| \frac{\partial^m f}{\partial y^m}(\xi, \eta, y / 2) \right| \left| \frac{\partial^m \Gamma}{\partial y^m}(x, t, \eta) \right| \, d\xi \, d\eta.$$

Applying Schwarz’s inequality, we obtain

$$\left| \frac{\partial^m u}{\partial y^m}(x, t, y) \right| \leq c \int_{X_{m+1}} \left| \frac{\partial^m f}{\partial y^m}(\xi, \eta, y / 2) \right| \left| \frac{\partial^m \Gamma}{\partial y^m}(x, t, \eta) \right| \, d\xi \, d\eta \times \int_{X_{m+1}} \left| \frac{\partial^m \Gamma}{\partial y^m}(x, t, \eta) \right| \, d\xi \, d\eta.$$
where the last integral converges to a constant times \( y^{m+1} \). Then, multiplying by \( y^{m+1} \) and integrating \( y \) from 0 to \( \infty \) we get

\[
\int_{\mathbb{R}^{n+2}} y^{m+1} \frac{\partial^m u}{\partial y^m}(x, t, y) \, dy,
\]

\[
\leq \int_{\mathbb{R}^{n+2}} y^m \exp \left\{ -\frac{|x|}{|t|} \right\} \frac{\partial u}{\partial y}(x, t, y) \, dy.
\]

By (1.2), the second member is less than

\[
c \int_{\mathbb{R}^{n+2}} \frac{y^m}{|x|^{n+1}} \exp \left\{ -\frac{\xi}{|t|} \right\} \frac{\partial u}{\partial y}(x, t, y) \, d\xi \, dy \leq c \int_{\mathbb{R}^{n+2}} \frac{y^m}{|x|^{n+1}} \frac{\partial u}{\partial y}(x, t, y) \, dx \, dy.
\]

Since \( \lambda = \frac{n+1+m}{n+2} > \frac{3n+3}{n+2} > \frac{2m+4}{n+2} = 2 \), theorem (2.2) implies

\[
\|u(x, t)\|_{L^p} \leq c \|u(x, t)\|_{L^p} \leq c \|u(x, t)\|_{L^p}.
\]

for every \( 1 < p < \infty \). To complete the proofs of theorems (2.2) and (2.23), it remains for us to show

\[
\sup_{n \geq 1} \|u(x, t)\|_{L^p} \leq c \|u(x, t)\|_{L^p}.
\]

Let \( \varphi \) and \( \psi \) be \( C^\infty \) on \( \mathbb{R}^{n+1} \) and \( u = \varphi \ast \Gamma \), \( v = \psi \ast \Gamma \). Consider the expression

\[
\int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} y^{m+1} \varphi(y, t) \psi(y, t) \, dy \, dx.
\]

Changing the order of integration and applying Plancherel's theorem, we see this is

\[
\int_{\mathbb{R}^{n+1}} y^{m+1} \varphi(y, t) \psi(y, t) \, dy \, dx.
\]

Changing the order of integration once more, we get that this equals

\[
\int_{\mathbb{R}^{n+1}} \varphi(y, t) \psi(y, t) \, dy \, dx,
\]

and since \( \left( \frac{|x|^{n+1}}{|y|^{n+1}} \right)^{\frac{1}{p}} \) is the symbol of an invertible parabolic singular integral (see [1]), we obtain that for a positive constant \( c \),

\[
\int_{\mathbb{R}^{n+1}} \varphi(y, t) \psi(y, t) \, dy = c \int_{\mathbb{R}^{n+1}} \varphi(x, t) \psi(x, t) \, dx.
\]

On the other hand, we have by Schwarz's inequality

\[
\int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} y^{m-1} \varphi(y, t) \psi(y, t) \, dy \leq \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \varphi(x, t) \psi(x, t) \, dx.
\]

H"older's inequality plus the part of theorem (2.23) already proved shows that if \( p' = p/(p-1) \), then

\[
\int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} y^{m-1} \varphi(y, t) \psi(y, t) \, dy \leq \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \varphi(x, t) \psi(x, t) \, dx.
\]

We conclude that for every \( \varphi, \psi \in C^\infty \) (\( \mathbb{R}^{n+1} \)),

\[
\int_{\mathbb{R}^{n+1}} \varphi(x, t) \psi(x, t) \, dx \leq \int_{\mathbb{R}^{n+1}} \varphi(x, t) \psi(x, t) \, dx
\]

which implies \( \|\varphi\| \leq \|\varphi\|_{L^p} \), and so \( \|u\|_{L^p} \leq \|u\|_{L^p} \). The case of an arbitrary \( \varphi \) follows by continuity.

The proof for \( g^*_0 \) is similar and will not be given.

SECTION 3
CHARACTERIZATION OF \( \mathbb{S}^s_\alpha \)

This chapter is devoted to a characterization of the \( \mathbb{S}^s_\alpha \) spaces defined in [1]. Our characterization is suggested by those given in [3] and [9]. See also theorem 3 of [4].

Let \( \varphi(x, t) \) be an infinitely differentiable function belonging to \( L^p \) for every \( 1 < r < \infty \), together with all its derivatives, and let \( x_\alpha(x, t) \) denote its parabolic fractional integral of order \( \alpha \), \( 0 < \alpha < 1 \); that is, \( x_\alpha(x, t) = F_\alpha(x)(x, t) \). We shall be concerned first with the function \( T(x)(x, t) \) defined by

\[
T(x)(x, t) = \int_{\mathbb{R}^{n+1}} \frac{|x_\alpha(x)|}{|x_\alpha(x)|^{n+1+\frac{1}{2}}}
\]

and will prove the following theorem:

(3.1) Theorem. Let \( \varphi(x, t) \) be an infinitely differentiable function which together with its derivatives of all orders belongs to all \( L^p(\mathbb{R}^{n+1}) \), \( 1 < r < \infty \), and let \( 0 < \alpha < 1 \). There exist two positive constants \( c_1, c_2 \) such that

\[
c_1 \|u\|_{L^p} \leq \|u\|_{L^p} \leq c_2 \|u\|_{L^p}.
\]
provided that $p > 2(n+2)/(2\alpha+n+2)$. The constants $c_1, c_2$ depend on $\alpha, p$ and $\eta$, but not on $\varphi$.

Proof. Let

$$u(x, t, y) = \varphi(x, t + \epsilon) + \eta(x, t, y)$$

We denote $|\varphi - \eta|^{\beta}$ by $\varepsilon$ and observe that this $\varepsilon$ is equivalent to the one introduced in the parabolic change of coordinates (1.3). Adding and subtracting $u(x, t, \varphi)$, we have

$$|\varphi|_{x-s,t-s} - \eta(x, t, 0)$$

$$\leq |u(x-s,t-s,0) - u(x,t,\varphi)| + |u(x, t, \varphi) - u(x, t, 0)|.$$

For the first term on the right we have

$$u(x-s,t-s,0) - u(x,t,\varphi)$$

$$= - \int \frac{d}{dr} \left[ \frac{u'(x-s+r \frac{\partial u}{\partial y} t-s + r \frac{\partial u}{\partial y}, r)}{r} \right] dr.$$

Integrating by parts, we obtain

$$u(x-s,t-s,0) - u(x,t,\varphi) = - \int \frac{d}{dr} \left[ u(x-s+r \frac{\partial u}{\partial y}, t-s + r \frac{\partial u}{\partial y}, r) \right]_{t}^{+}$$

$$+ \frac{r^{2}}{2} \frac{\partial^{2} u}{\partial r^{2}} \left[ u(x-s+r \frac{\partial u}{\partial y}, t-s + r \frac{\partial u}{\partial y}, r) \right]_{t}^{+}$$

$$- \int \frac{r^{2}}{2} \frac{\partial^{2} u}{\partial r^{2}} \left[ u(x-s+r \frac{\partial u}{\partial y}, t-s + r \frac{\partial u}{\partial y}, r) \right]_{t}^{-} dr.$$

Performing the indicated differentiations, we obtain

$$|u(x-s,t-s,0) - u(x,t,\varphi)|$$

$$\leq \varepsilon \left\{ \sum_{\alpha + \beta = 3} \frac{\partial^{\alpha + \beta} u}{\partial x^{\alpha} \partial y^{\beta}} (x, t, \varphi) \right\}$$

$$+ \sum_{\alpha + \beta = 3} \int \frac{d}{dr} \left[ \frac{\partial^{\alpha + \beta} u}{\partial x^{\alpha} \partial y^{\beta}} (x-s+r \frac{\partial u}{\partial y}, t-s + r \frac{\partial u}{\partial y}, r) \right] dr$$

$$+ \sum_{\alpha + \beta = 3} \int \frac{d}{dr} \left[ \frac{\partial^{\alpha + \beta} u}{\partial x^{\alpha} \partial y^{\beta}} (x-s+r \frac{\partial u}{\partial y}, t-s + r \frac{\partial u}{\partial y}, r) \right] dr.$$
Multiplying both members by $e^{-(2n+1)|z|}$, integrating in $(z, s)$ over $E_{n+1}$, we obtain

\begin{equation}
(3.4) \quad \int_{E_{n+1}} \left| T_n(v)(z, s) \right|^2
dr ds = \int_{E_{n+1}} \frac{\partial^2 u_n}{\partial y^2}(x, t, \xi) dx ds + \sum_{k=1}^n \int_{E_{n+1}} \frac{\partial^2 u_n}{\partial x^2}(x, t, \xi) dx ds + \sum_{k=1}^n \int_{E_{n+1}} \frac{\partial^2 u_n}{\partial x \partial y}(x, t, \xi) dx ds \leq c \int_{E_{n+1}} \frac{\partial^2 u_n}{\partial y^2}(x, t, \xi) dx ds.
\end{equation}

For the first group of terms on the right, we pass to parabolic polar coordinates and obtain

\begin{equation}
(3.5) \quad \int_{E_{n+1}} e^{2\pi n-2|z|} \left| \frac{\partial^2 u_n}{\partial y^2}(x, t, \xi) \right|^2 dx ds \leq c \int_{E_{n+1}} e^{2\pi n-2|z|} \left| \frac{\partial^2 u_n}{\partial y^2}(x, t, \xi) \right|^2 dx ds.
\end{equation}

By theorem (2.25), $g_{k-1}(\theta)$ has $L^p$ norm less than a constant times $|\partial g| \leq c_0 |\partial g|$. For the second group of terms on the right of (3.4), we apply Schwarz's inequality and obtain the majorization

\begin{equation}
\int_{E_{n+1}} e^{2\pi n-2|z|} \left| \frac{\partial^2 u_n}{\partial x \partial y}(x, t, \xi) \right|^2 dx ds \leq c \int_{E_{n+1}} e^{2\pi n-2|z|} \left| \frac{\partial^2 u_n}{\partial x \partial y}(x, t, \xi) \right|^2 dx ds,
\end{equation}

where $\alpha$ is an arbitrary number between 0 and $2n$.

We claim that

\begin{equation}
(3.6) \quad \left| \frac{\partial^2 u_n}{\partial y^2}(x, t, \xi) \right| \leq c \int_{E_{n+1}} \left| \frac{\partial u_n}{\partial y}(\xi, \eta, r/2) \right| |\xi - \eta|^{-n} e^{-n|\xi - \eta|/2} |\xi - \eta|^{-n} e^{-n|\xi - \eta|/2} |\xi - \eta|^{-n} e^{-n|\xi - \eta|/2} d\xi d\eta.
\end{equation}

This can be seen as follows. By definition of fractional integration, we have

\begin{equation}
\int_{E_{n+1}} \left| \frac{\partial^2 u_n}{\partial y^2}(x, t, \xi) \right|^2 dx ds \leq \int_{E_{n+1}} e^{2\pi n-2|z|} \left| \frac{\partial^2 u_n}{\partial y^2}(x, t, \xi) \right|^2 dx ds.
\end{equation}

By (1.1), the norm is smaller than a constant times

\begin{equation}
\int_{E_{n+1}} \left| \frac{\partial u_n}{\partial y}(\xi, \eta, r/2) \right| |\xi - \eta|^{-n} e^{-n|\xi - \eta|/2} |\xi - \eta|^{-n} e^{-n|\xi - \eta|/2} |\xi - \eta|^{-n} e^{-n|\xi - \eta|/2} d\xi d\eta.
\end{equation}

which proves (3.6).

Now let $0 < a, b, a + b > 2n + k - n + k - a$ and $b > n + k - a$. We can choose such $a$ and $b$ since

\begin{equation}
(3.7) \quad \frac{2a + n + b - n + k - a}{n + k - a} = \frac{n + 2}{n + k - a} < \frac{2n + 2}{n + 3} = 2.
\end{equation}

Applying Schwarz's inequality in (3.6), we get

\begin{equation}
\int_{E_{n+1}} \left| \frac{\partial^2 u_n}{\partial y^2}(x, t, \xi) \right|^2 dx ds \leq c \int_{E_{n+1}} \left| \frac{\partial u_n}{\partial y}(\xi, \eta, r/2) \right| |\xi - \eta|^{-n} e^{-n|\xi - \eta|/2} |\xi - \eta|^{-n} e^{-n|\xi - \eta|/2} |\xi - \eta|^{-n} e^{-n|\xi - \eta|/2} d\xi d\eta.
\end{equation}


The last integral is a constant times \( r^{-\alpha(n+k+4)+\alpha+3} \). Therefore (3.5) is less than a constant times

\[
\int_{E_{n+1}} \int_{E_{n+1}} \frac{\partial \xi}{\partial \eta} \left( \xi, \eta, r/2 \right) \left| \eta - s + r^2 \frac{\xi}{\eta} \right|^{\alpha(n+k+4)+\alpha+3} d\xi d\eta \times 
\times \exp \left\{ -a \left( |\xi - x + r^2 \frac{\xi}{\eta}|^{1/4} \right) \right\} \left( |\xi - x + r^2 \frac{\xi}{\eta}|^{-1/4} \right) d\xi d\eta.
\]

A change of variables in \( \xi, \eta \) gives

\[
\int_{E_{n+1}} \int_{E_{n+1}} \frac{\partial \xi}{\partial \eta} \left( x-r^2 \xi, t-r^2 \eta, r/2 \right) \left| \eta - s + r^2 \eta \right|^{\alpha(n+k+4)+\alpha+3} d\xi d\eta \times 
\times \exp \left\{ -a \left( |\xi - x + r^2 \xi/\eta|^{-1/4} \right) \right\} \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) d\xi d\eta.
\]

Changing the order of integration and then changing variables in \( z, s \), we obtain

\[
(3.9) \int_{A} \frac{d\xi}{d\eta} \left( x-r^2 \xi, t-r^2 \eta, r/2 \right) |\eta - s + r^2 \eta|^{\alpha(n+k+4)+\alpha+3} \times 
\times \exp \left\{ -a \left( |\xi - x + r^2 \xi/\eta|^{-1/4} \right) \right\} \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) d\xi d\eta.
\]

We claim that the innermost integral in (3.9) is smaller than a constant times

\[
(\xi + |\eta|^{1/2} + 1)^{-\alpha(n+k+4)-\alpha}. 
\]

We consider two cases: (1) \( |\xi + |\eta|^{1/2} | < 4m \), and (2) \( |\xi + |\eta|^{1/2} | > 4m \) where \( m \) is a large positive constant to be chosen.

Case 1. Since the function

\[
|\xi - x + r^2 \xi/\eta|^{-1/4} \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) \left| \eta - s + r^2 \eta \right|^{\alpha(n+k+4)+\alpha+3} 
\]

is bounded on \( E_{n+1} \), we see that the integral under consideration is less than

\[
c \int_{E_{n+1}} \exp \left\{ -a \left( |\xi - x + r^2 \xi/\eta|^{-1/4} \right) \right\} \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) d\xi d\eta.
\]

and since the function \((\xi + |\eta|^{1/2} + 1)^{-2n-n-2-x} \) is bounded below by a positive number for \( |\xi + |\eta|^{1/2} | < 4m \), our claim follows in this case.

Case 2. We write our integral as

\[
\int_{E_{n+1}} \exp \left\{ -a \left( |\xi - x + r^2 \xi/\eta|^{-1/4} \right) \right\} \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) d\xi d\eta.
\]

Consider first the integral \( A \). If \( |\xi + |\eta|^{1/2} | \geq 2 |\xi + |\eta|^{1/2} | \) and since \( 1 < q < 1/((\xi + |\eta|^{1/2})^2) \), we obtain \( |\xi + |\eta|^{1/2} | \leq 1/3 \). Thus

\[
|\xi - x + r^2 \xi/\eta|^{-1/4} \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) \left| \eta - s + r^2 \eta \right|^{\alpha(n+k+4)+\alpha+3} \times 
\times \exp \left\{ -a \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) \right\} \left( |\xi - x + r^2 \xi/\eta|^{-1/4} \right) d\xi d\eta.
\]

Hence if \( 1 < q < 1/((\xi + |\eta|^{1/2})^2) \) and \( |\xi + |\eta|^{1/2} | > 8 \), we have

\[
|\xi - x + r^2 \xi/\eta|^{-1/4} \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) \left| \eta - s + r^2 \eta \right|^{\alpha(n+k+4)+\alpha+3} \times 
\times \exp \left\{ -a \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) \right\} \left( |\xi - x + r^2 \xi/\eta|^{-1/4} \right) d\xi d\eta.
\]

With this inequality and the fact that

\[
(\xi + |\eta|^{1/2} + 1)^{\alpha(n+k+4)+\alpha} \left| \eta - s + r^2 \eta \right|^{\alpha(n+k+4)+\alpha+3} \times 
\times \exp \left\{ -a \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) \right\} \left( |\xi - x + r^2 \xi/\eta|^{-1/4} \right) d\xi d\eta.
\]

is bounded on \( E_{n+1} \), see (1.2) we obtain

\[
A \leq \int_{E_{n+1}} \exp \left\{ -a \left( |\xi - x + r^2 \xi/\eta|^{-1/4} \right) \right\} \left( |\xi - x + r^2 \xi/\eta|^{1/4} \right) d\xi d\eta.
\]

The last inequality is true since by (3.7)

\[
a(n+4+k-a) > 2n+2n+2 - a.
\]
The value of \( \eta \) is less than

\[
(3.11) \quad \frac{1}{(|\xi| + |\eta|^{1/2} + 1)^{\frac{m+3}{2}} - m}\times \\
\times \int_{S^{m-1}} \left| \eta - \frac{\xi}{\sqrt{\eta}} \right|^{-a+2 \frac{m}{2}} \exp \left( -a \frac{|\xi - \eta| + \frac{1}{4}}{8 \eta - \frac{1}{4}} \right) d\sigma.
\]

Consider the change of variables

\[
x' = \frac{\xi}{\sqrt{\eta}}, \quad y' = \frac{\xi}{\sqrt{\eta}}
\]

for \( q > m \). We shall prove that this is one-to-one and that the absolute value of its Jacobian is greater than a positive constant. Let \( (x_1, x_2) \) and \( (y_1, y_2) \) be two points such that \( x_1 := x_{2}^{'} \) and \( y_{1} := y_{2}^{'} \), then

\[
|x_1| \left( 1 - \frac{1}{\sqrt{\eta}} \right) = |y_1| \left( 1 - \frac{1}{\sqrt{\eta}} \right)
\]

and since \( 1 - \frac{1}{\sqrt{\eta}} > 1 - \frac{1}{\sqrt{\eta}} \), it follows that \( |x_1| < |y_1| \). In the same way

we obtain \( |x_2| > |y_2| \) and therefore \( x_1 > y_1 \), which is a contradiction.

Thus, if two points have the same image \( x_1 = y_1 \), which in turn implies \( x_2 = y_2 \) and \( y_2 = y_2 \). Then to estimate the Jacobian \( \frac{\partial (x', y')}{\partial (x, y)} \), we observe that

\[
\frac{\partial x'}{\partial x} = 1 - \frac{1}{\sqrt{\eta}} + \frac{x}{\sqrt{\eta}^2} = 1 + O \left( \frac{1}{\sqrt{\eta}} \right), \quad \frac{\partial y'}{\partial x} = 1 + \frac{1}{\sqrt{\eta}} - \frac{x}{\sqrt{\eta}^2} = 1 + O \left( \frac{1}{\sqrt{\eta}} \right).
\]

\[
\frac{\partial x'}{\partial y} = \frac{y}{\sqrt{\eta}^2} \quad \text{and} \quad \frac{\partial y'}{\partial y} = \frac{x}{\sqrt{\eta}^2}.
\]

In particular,

\[
\frac{\partial (x', y')}{\partial (x, y)} = \left| x \right|^{\frac{1}{2}} \left| y \right|^{\frac{1}{2}} = O \left( \frac{1}{\sqrt{\eta}} \right).
\]

Therefore, in the expansion of the Jacobian, the term which arises from the product down the diagonal is \( 1 + O(1/\sqrt{\eta}) \) and every other term is \( O(1/\sqrt{\eta}) \). If we choose \( m \) sufficiently large, it follows from the condition \( \eta > m \) that the Jacobian exceeds a positive constant.

Therefore, changing variables and enlarging the domain of integration, we see that the integral in (3.11) turns out to be less than a constant times

\[
\int_{S^{m-1}} \left| \eta - \frac{\xi}{\sqrt{\eta}} \right|^{-\frac{a}{2}} \exp \left( - a \frac{|\xi - \eta| + 1/4}{8 |\xi - \eta|} \right) d\sigma.
\]

This converges since \( a > 2a + m + 2 - \varepsilon \quad \text{and} \quad 0 < \varepsilon < 2a \), and the proof of case 2 is complete.

We see that (3.10) implies that (3.9) is less than a constant times

\[
\int_{S^{m-1}} \left( \frac{r}{|\xi| + |\eta|^{1/2} + 1} \right)^{\frac{a}{2}} \left| \frac{\partial u}{\partial y} (x - r \xi, t - r \eta, \eta/2) \right| d\xi d\eta.
\]

A change of variables shows this is

\[
\int_{S^{n-1}} \left( |x - \xi| + |\eta|^{1/2} + r \right)^{\frac{a}{2}} \left| \frac{\partial u}{\partial y} (x, \eta, r) \right| d\xi d\eta.
\]

where \( \lambda = \frac{2a + m + 2 - \varepsilon}{n + 2 - \varepsilon} \).

By theorem (2.2) the \( L^p \)-norm of the square root of the second term on the right of (3.4) is less than a constant times \( |\xi|^p \), provided that

\[
p > 2a + m + 2 - \varepsilon.
\]

But since \( \varepsilon \) can be chosen arbitrarily small, the condition becomes \( p > 2a + m + 2 \).

Finally, for the third term on the right of (3.4), we have

\[
\int_{S^{m-1}} \left( \frac{r}{|\xi| + |\eta|^{1/2} + 1} \right)^{\frac{a}{2}} \left| \frac{\partial u}{\partial y} (x, t, r) \right|^p d\sigma d\eta d\xi d\sigma d\eta d\xi
\]

which by (3.6) is smaller than

\[
\int_{S^{m-1}} \left( \frac{r}{|\xi| + |\eta|^{1/2} + 1} \right)^{\frac{a}{2}} \left| \frac{\partial u}{\partial y} (x, t, r) \right|^p d\sigma d\eta d\xi d\sigma d\eta d\xi
\]

Therefore, we can drop the last term on the right of (3.4) to conclude that

\[
\int_{S^{m-1}} \left( \frac{r}{|\xi| + |\eta|^{1/2} + 1} \right)^{\frac{a}{2}} \left| \frac{\partial u}{\partial y} (x, t, r) \right|^p d\sigma d\eta d\xi d\sigma d\eta d\xi
\]
Let \( a > \frac{2a + n + 2 - \varepsilon}{n + 4 - a} \), \( b > \frac{n + 2}{n + 4 - a} \), \( a + b = 2 \). By Schwarz's inequality, the last integral is smaller than

\[
\varepsilon \int_{\mathbb{R}^n} e^{-\frac{a|\xi|^2 + b|\eta|^2}{4}} \, d\eta \times \int_{\mathbb{R}^n} \left| \frac{\partial u}{\partial y} (\xi, \eta, r^2) \right| \left| |\eta|^n - |\eta + (\eta - \xi)|^n \right| \exp \left( - \frac{a|\xi|^2 + b|\eta|^2}{8|\eta|^n} \right) \, d\xi \, d\eta.
\]

 Changing the order of integration and applying (1.2), we get

\[
c \int_{\mathbb{R}^n} \left( \frac{|x - \xi|^n}{|x|^n} \frac{\partial u}{\partial y} (\xi, \eta, r) \right) \frac{d\xi}{d\eta} \, d\eta \, dr \quad c |p_s(x)|^2,
\]

where \( \lambda = a(n + 4 - a) \). Then, since \( a > \frac{2a + n + 2 - \varepsilon}{n + 4 - a} \), it follows that

\[
\lambda = a(n + 4 - a) > \frac{2a + n + 2 - \varepsilon}{n + 2}.
\]

Therefore, the square root of the third integral on the right of (3.4) has \( L^p \)-norm less than a constant times \( |p_s|_p \), provided that \( p > \frac{2(n + 2)}{a + 8} \), as before.

To complete the proof of theorem (3.1), we must show that \( c_1|p_s|_p \leq \| T_s(p) \|_p \). In order to prove this, we observe that

\[
\int_{\mathbb{R}^n} \frac{\partial u}{\partial y} (x, y, s) \, ds = 0 \quad \text{for} \quad y > 0.
\]

Using this, we have

\[
\frac{\partial u}{\partial y} (x, t, y) = \int_{\mathbb{R}^n} [p_s(x - s, t - s) - p_s(x, t)] \frac{\partial u}{\partial y} (x, y, s) \, ds.
\]

Applying (1.2), we get

\[
|p_s(x)|^2 = \int_{\mathbb{R}^n} \left[ |p_s(x - s, t - s) - p_s(x, t)|^2 \right] \frac{\partial u}{\partial y} \, ds \times \int_{\mathbb{R}^n} \frac{y^{-\left(\frac{n+2}{2}\right)}}{8|s|^n} \, dy.
\]

The last integral converges to a constant times \( y^{-\left(\frac{n+2}{2}\right)} \). Therefore

\[
\int_{\mathbb{R}^n} |y^{-\left(\frac{n+2}{2}\right)}| \frac{\partial u}{\partial y} (x, t, y) \, dy \quad c \int_{\mathbb{R}^n} \left| |x|^n - |x - \xi|^n \right| \exp \left( - \frac{a|\xi|^2 + b|\eta|^2}{8|\eta|^n} \right) \, d\xi \, d\eta.
\]

If \( a \) and \( b \) satisfy

\[
\frac{n + 2}{n + 3} < b < \frac{n + 4 - 2a}{n + 3}, \quad a = 2 - b,
\]

then, by Schwarz's inequality,

\[
\left| \frac{\partial u}{\partial y} (x, t, y) \right| \leq c \int_{\mathbb{R}^n} |p_s(x - s, t - s) - p_s(x, t)|^2 \frac{y^{-\left(\frac{n+2}{2}\right)}}{8|s|^n} \, ds \times \int_{\mathbb{R}^n} \frac{y^{-\left(\frac{n+2}{2}\right)}}{8|s|^n} \, dy.
\]

Theorem (2.25) gives the inequality

\[
c_1|p_s|_p \leq \| T_s(p) \|_p,
\]

which completes the proof of theorem (3.1).
Let \( P_s(f) \) be the function

\[
P_s(f)(x, t) = \left( \int_{|x-s|+|t|}^{\infty} \frac{|f(x-s, t-t)|}{|x-s|+|t-t|} \, ds \, dt \right)^{1/n}.
\]

We shall prove the following theorem, which is the main result of this section.

\[\text{(3.13) THEOREM. For } 0 < a < 1 \text{ and } \frac{2(n+2)}{2n+n+2} < p < \infty, \text{ the following two conditions are equivalent:}\]

1. \( f \in \mathcal{L}^p(E_{n+1}) \) and \( P_s(f) \in \mathcal{L}^p(E_{n+1}) \),

2. \( f \in \mathcal{L}_a^p(E_{n+1}) \),

Moreover, if \( f \in \mathcal{L}_a^p(E_{n+1}) \), there exist two positive constants \( c_1, c_2 \) independent of \( f \) such that

\[
c_1 \|f\|_{L^p} \leq \|P_s(f)\|_{L^p} \leq c_2 \|f\|_{L^p}.
\]

To prove this theorem we will need two lemmas.

\[\text{(3.17) LEMMA. For } 0 < a < 4, \text{ there exist three finite measures } \mu^{(0)}, \mu^{(1)}, \text{ and } \mu^{(2)} \text{ on } E_{n+1} \text{ satisfying }\]

1. \((|x|^2 + t)^{a/2} = \mu^{(0)}(x, t)(1 + |x|^2 + t)^{a/2},\)

and

2. \((1 + |x|^2 + t)^{a/2} = \mu^{(2)}(x, t) + \mu^{(0)}(x, t)(1 + t)^{a/2},\)

Proof. The proof of a similar lemma in [11] applies without change, provided that \( 0 < a < 4 \).

\[\text{(3.18) LEMMA. Let } \varphi \in \mathcal{L}^r(E_{n+1}) \text{ for every } 1 \leq r < \infty \text{ and let } 0 < a < n+2. \text{ Then } \]

\[
\mathcal{F}_s(\mu^{(0)} \ast \varphi) = \mathcal{F}_s(\varphi)
\]

almost everywhere on \( E_{n+1} \).

Proof. The term on the left of (3.19) is well defined and belongs to every \( L^p \) with \( p > \frac{n+2}{n+2-a} \). For \( \varphi(x, t) \in \mathcal{S}' \)

\[
\int_{E_{n+1}} \varphi(x, t) \mathcal{F}_s(\mu^{(0)} \ast \varphi)(x, t) \, dx \, dt
\]

\[
= \int_{E_{n+1}} \left[ \int_{E_{n+1}} \mathcal{F}_s(x-u, t-w) \varphi(x, t) \, dx \, dt \right] (\mu^{(0)} \ast \varphi)(u, w) \, du \, dw.
\]

We observed in section 1 that

\[
\Gamma(a)^{-1} \int \Gamma(x, t, y) y^{n+1} \, dy = \mathcal{F}_s(\varphi(x, t)).
\]

Thus, if we denote by \( \varphi(x, t, y) \) the convolution of \( \varphi(-x, -t) \) with \( \Gamma(x, t, y) \), we get that the integrals in (3.20) are equal to

\[
\int \mathcal{F}_s(\varphi(x, t)) \, dx \, dt
\]

By Plancherel's theorem, this is

\[
\int \mathcal{F}_s(\varphi)(x, t) \, dx \, dt
\]

\[
\times \int_{E_{n+1}} \mathcal{F}_s(\psi(u, w)) \exp (-y \sqrt{|u|^2 + |w|^2}) \left( |u|^2 + |w|^2 \right)^{a/2} \psi(u, w) \, du \, dw.
\]

Changing the order of integration and integrating in \( y \), we get

\[
\int_{E_{n+1}} \mathcal{F}_s(\psi(u, w)) \left( 1 + |u|^2 + |w|^2 \right)^{-a/2} \psi(u, w) \, du \, dw
\]

which by another application of Plancherel's theorem gives

\[
\int_{E_{n+1}} \mathcal{F}_s(\psi)(x, t) \, dx \, dt
\]

This proves the lemma.

Proof of theorem (3.13). Let \( f \in \mathcal{S}_0^\infty \); that is, \( f = \mathcal{F}_s(\varphi) \), where \( \varphi \in \mathcal{L}^r(E_{n+1}) \). Choose a sequence \( \{\varphi_k\} \) of functions in \( \mathcal{S} \) which converge to \( \varphi \) in \( D' \) and pointwise a.e. Since \( \mu^{(0)} \ast \varphi_k \) belongs to all \( L^r(E_{n+1}) \), \( 1 \leq r < \infty \), we have by theorem (3.21)

\[
c_1 \|\mu^{(0)} \ast \varphi_k\|_p \leq \|\mathcal{F}_s(\mu^{(0)} \ast \varphi_k)\|_p \leq c_2 \|\mu^{(0)} \ast \varphi_k\|_p.
\]

By lemma (3.18), however, \( T_s(\mu^{(0)} \ast \varphi_k) \) coincides a.e. with \( P_s(f_k) \), where \( f_k = \mathcal{F}_s(\varphi_k) = \mathcal{F}_s \ast \varphi_k \). In particular,

\[
\|P_s(f_k)\|_p \leq c_3 \|\mu^{(0)} \ast \varphi_k\|_p \leq c_2 \|\mu^{(0)} \|_p \|\varphi_k\|_p,
\]

and since \( f_k \) converges pointwise to \( f \) we obtain from Fatou's lemma

\[
\|P_s(f)\|_p \leq c \|\mu^{(0)}\|_p.
\]

This inequality and the sublinearity of \( P_s(f) \) as a function of \( \varphi \) imply that if \( \varphi_k \) converges to \( \varphi \) in \( D' \) then \( \|P_s(f_k)\|_p \) converges to \( \|P_s(f)\|_p \). From
the first inequality (3.21) we then obtain
\begin{equation}
\|\mathbf{u}\|_{L^p} \leq c \|P_n(f)\|_{L^p}.
\end{equation}

Since the identity (see lemma (3.17))
\[1 = \widehat{\mu_0^{(0)}}(x, t)(1 + |x|^2 + t)^{-\alpha} + \mu_0^{(0)}(x, t) \mathbf{\mu_0^{(0)}}(x, t)\]
implies
\[
\mathbf{\varphi}(x, t) = (\mu_0^{(0)} * f)(x, t) + (\mu_0^{(0)} * \mathbf{\mu_0^{(0)}}) \mathbf{\varphi}(x, t)
\]
a.e.,
we have
\[
\|\mathbf{\varphi}\|_{L^p} = \|\mathbf{\varphi}\|_{L^p} \leq c(\|f\|_{L^p} + \|\mu_0^{(0)} * \mathbf{\varphi}\|_{L^p}) \leq c(\|f\|_{L^p} + \|P_n(f)\|_{L^p}).
\]

On the other hand, by (3.1), we have
\[
\|f\|_{L^p} + \|P_n(f)\|_{L^p} \leq c\|\mathbf{\varphi}\|_{L^p},
\]
which proves
\[
c_\alpha \|f\|_{L^p} \leq \|\mathbf{\varphi}\|_{L^p} \leq c_\alpha \|f\|_{L^p},
\]
for \(f \in L^\alpha(E_{n,1})\).

To complete the proof of the theorem we have only to show that (3.14) implies (3.15). If \(V_{n,\alpha}(x, t, y) > 0\), is the function defined by
\[
V_{n,\alpha}(x, t, y) = \left(1 + |x|^2 + t\right)^{-\alpha/2} \exp\left(-y\sqrt{|x|^2 + t}\right),
\]
then \(V_{n,\alpha}\) is integrable and
\[
V_{n,\alpha} * \mathbf{\varphi} = \Gamma(x, t, y).
\]
Let \(f\) and \(P_n(f)\) belong to \(L^\alpha(E_{n,1})\). We have
\[
u(x, t, y) = (f * \Gamma)(x, t, y) = (\mathbf{\varphi} * (V_{n,\alpha} * f))(x, t),
\]
which shows that \(u(x, t, y)\) belongs to \(L^\alpha\) for every \(y > 0\). Hence, by (3.16),
\[
c_\alpha \|V_{n,\alpha} * f\|_{L^p} \leq \|P_n(u(x, t, y))\|_{L^p} \leq \|u(x, t, y)\|_{L^p}.
\]

Since
\[
P_n(u(x, t, y)) \leq P_n(f) * \Gamma(x, t, y)
\]
by Minkowski's inequality, we get
\[
c_\alpha \|V_{n,\alpha} * f\|_{L^p} \leq \|P_n(f)\|_{L^p} + \|f\|_{L^p} \leq c
\]
for every \(y > 0\). This shows that the family of functions \(V_{n,\alpha} * f, y > 0\), is bounded in norm. By a theorem of Banach and Saks (see [6]) there is a sequence \(y_k \to 0\) such that
\[
\lim_{n \to \infty} \frac{1}{m} \sum_{k=1}^m V_{n,\alpha} * f = \mathbf{\varphi}
\]

for some \(f \in L^\alpha\), convergence being in \(L^\alpha\). But since \(\mathbf{\varphi} * V_{n,\alpha} * f(x, t) = u(x, t, y_k)\) and \(u(x, t, y_k)\) converges in norm to \(f(x, t)\), we obtain
\[
\int_{E_{n,1}} f(x, t) = \lim_{k \to \infty} u(x, t, y_k) = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^m u(x, t, y_k)
\]

which shows that \(f \in L^\alpha\) and completes the proof of the theorem.

References