Minimal sublinear functionals

by

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0. INTRODUCTION

In Section 1 we consider a class \( \mathcal{F} \) of sublinear functionals on a real linear space \( E \) and show that \( \mathcal{F} \) contains elements minimal with respect to the pointwise ordering on \( E^0 \). The general existence theorem in Theorem 15 and involves the definition of a “boundary” for \( \mathcal{F} \) in Notation 13.

In Section 2 we give conditions for an element of \( \mathcal{F} \) to dominate a unique minimal element of \( \mathcal{F} \).

In Section 3 we give a Shilov theorem for sublinear functionals on \( E \).

Under certain conditions (Theorem 1, Notation 5 and Lemma 27(b)) the minimal elements of \( \mathcal{F} \) coincide with the linear elements of \( \mathcal{F} \).

In Section 6 we deduce various forms of the Hahn–Banach theorem and generalizations of results of Kolmogorov and Slobodkin (see Remark 29).

In Section 7 we deduce, with a number of improvements over the known results, Shilov theorems and conditions for the existence and uniqueness of balayages defined by a cone in \( \mathcal{F}(X) \) (compact Hausdorff) (see Remark 32). There is also a short discussion of the Choquet boundary of a subspace of \( \mathcal{F}(X) \) (see Remark 35).

In Section 8 we suppose that \( X \) is a compact convex set in a Hausdorff locally convex space and deduce, with a number of improvements, results of Milman, Bauer and Choquet–Meyer (see Remark 38) as well as the Choquet–Bishop–deLeeuw theorem.

We use mainly linear space techniques — the only places where any measure theory is mentioned are Theorem 30(a), Theorem 33(a) and Theorem 36(c). In Section 9 we apply our results to a “non-\( \mathcal{F}(X) \)” situation, replacing \( \mathcal{F}(X) \) by the set of continuous affine functions on a compact convex set (in a Hausdorff locally convex space).

In Section 10 we make some further observations about the uniqueness problem.

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Sections 1–10 of this paper are at quite a high level of abstraction. In Section 11 we present a proof of the linear space part of the Choquet–Bishop–deLeeuw theorem that uses the same ideas as our general results but is completely self-contained. This proof does not use the Hahn–Banach theorem or Tychonoff’s theorem. Section 11 might well be read before Sections 1–10 to provide an insight into the techniques we use.

1. THE EXISTENCE OF CERTAIN SUBLINEAR FUNCTIONALS

1. Notation. We suppose that $E$ is a nonzero real linear space. We say that $F$ (⊂ $E$) is a cone if $0 \in F$, $F + F \subset F$ and, for all $\lambda > 0$, $\lambda F \subset F$. If $F$ (⊂ $E$) is a cone we say that $v$ is sublinear (resp. linear) on $E$ if $v(x) = \lambda v(x)$ for all $f, g \in F$, $\alpha > 0$, $\beta > 0$, $\gamma \in F$, $\delta \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ imply that $\lambda f, \alpha g, \beta f, \gamma g, \delta f, \lambda g$ in $F$ and, for all $x \in F$, $\delta = 0$ implies that $\lambda f, \alpha g, \beta f, \gamma g, \delta f, \lambda g$ in $F$. We write

$$\mathcal{F} = \{ S; \text{ $S$ is sublinear on } E \}.$$ 

We suppose that $\preceq$ is a relation on $E$ such that, for all $g \in E$, there exists $d \in E$ such that $\delta \preceq g$, for all $\lambda, \delta > 0$, $\lambda \delta \preceq \delta \gamma g$, and $\lambda \delta \preceq \alpha \delta g$ imply that $\lambda \gamma \delta \preceq \lambda \alpha \delta g$, and, for all $x \in E$, $d \preceq \delta$ implies that $\delta \preceq \delta$. We write

$$D = \{ d; \text{ $d \in E$, there exists } g \in E \text{ such that } d \preceq g \}.$$ 

$D$ is a cone (the set of “dominators”). We write

$$\mathcal{D} = \{ S; \text{ $S \in \mathcal{F}$, for all } g \in D, d \in E, d \preceq g \text{ implies that } S(g - d) \leq 0 \},$$

and, for all $g \in E$, $S(g)$ is the infimum of $\{ S(d); d \in E, d \preceq g \}$. We write $\preceq$ for the pointwise ordering on $\mathbb{R}^E$ and, if $\mathcal{P} \subseteq \mathbb{R}$, $\preceq$, for the pointwise ordering on $\mathbb{R}^E$. If $\mathcal{S} \subseteq \mathcal{F}$ (resp. $\mathcal{P} \subseteq \mathcal{F}$) we write $\mathcal{S} \preceq \mathcal{P}$ (resp. $\mathcal{P} \preceq \mathcal{S}$) for $\{ T; T \preceq (\text{resp. } \preceq) \mathcal{P} \}$. We observe that if $\mathcal{S} \subseteq \mathcal{F}$ and $\mathcal{P} \subseteq \mathcal{F}$ then

$$g, h \in E \text{ and } S(g - h) \leq 0 \text{ imply that } P(g) = P(h) \text{ and } g \in E \text{ implies that } P(g) \geq -S(g).$$

2. Example. If $d \preceq g$ means “$d = g$” then $D = E$ and $\mathcal{D} = \mathcal{F}$.

3. Example. $X$ is a compact Hausdorff space, $E = C(X)$, $C$ is a cone in $E$ containing the positive constants and $C - C$ is norm-dense in $E$. $d \preceq g$ means “$d \leq g$ and $d \in C$”. Then $D = C$. If $g \in E$ we write $S(g) = \sup \{ S(d); d \in C \text{ and } d \preceq g \}$. Then $S \in \mathcal{F}$.

4. Example. $X$ is a compact convex subset of a real Hausdorff locally convex space $V$, with dual $V'$, $E = C(X)$ and $C = \{ d; d \in E \text{ and } d \text{ is convex} \}$, from the Stone–Weierstrass theorem, $C - C$ is norm-dense in $E$. The remainder of the notation is as in Example 3.

5. Example. $X$ is as in Example 4, $E = \{ g; \text{ $g \in C(X)$, $g$ is affine} \}$, $C$ is a cone in $E$ containing the positive constants and $C - C$ is norm-dense in $E$. The remainder of the notation is as in Example 3.

6. Lemma. Let $P, Q \in \mathcal{F}$. Then $P \preceq Q \Rightarrow P \preceq P \preceq Q \preceq D \preceq Q \preceq D$.

Proof. ($\Rightarrow$) is trivial and ($\Leftarrow$) follows from the definition of $\mathcal{F}$.

7. Remark. Lemma 6 shows that in Example 4 the ordering induced by $\preceq$ on $\mathcal{S}$ coincides formally with that usually used in the Choquet’s theorem. The behavior here is different because the functions we are considering are not necessarily linear.

8. Lemma. We suppose that $S \in \mathcal{F}$ and $E (\subset D)$ is a cone. We say that $v$ is $S$–$D$–admissible if $v$ is sublinear on $E$ and, for all $f \in E$, $v(f) \geq -S(-f)$. If this is the case then, for all $g \in E$, $Sv(g) = \inf \{ S(g - f) + v(f); f \in F \}$.

Then the following results are true.

(a) $S \preceq \mathcal{S}$,

(b) $S \preceq S \preceq S$,

(c) if $Q \preceq Q \preceq S \preceq S$.

(2) $Q \preceq S$ and $Q \preceq S \preceq S$.

Proofs. These results all follow from routine computations with infima. We give in detail only the nontrivial one that involves $\preceq$, the proof, in (a), that $S \preceq S \preceq S \preceq S$ implies $\preceq$. If $g \in E, f \in F$ and $d \preceq g$, then, since $f \preceq f, (h + f) \preceq g$. We write $d = h + f$ then $d \preceq g$ and $h - f$. Hence

$$Sv(g) = \inf \{ S(g - f) + v(f); f \in F \} = \inf \{ S((h + f) - f) + v(f); f \in F \} \geq \inf \{ S(h) + v(f); f \in F, h \preceq f \} \geq \inf \{ S(v(d)) + v(f); f \in F, d \preceq g \}.$$
10. Lemma. We suppose that \( S \notin \mathcal{P} \).

(a) If \( \emptyset \neq B \subseteq D \),

(3) for all \( b, b' \in B \) there exists \( b'' \in B \) such that \( S(b - b') \not\subseteq 0 \) and \( S(b' - b'') \not\subseteq 0 \),

\[ \beta = \sup -S(-B) < \infty \]

then \( S \) is \( F \)-admissible and, for all \( b, b' \in B \), \( S \ast y(b) \subseteq \beta \).

(b) If \( a \in D \) then there exists \( S \in \mathcal{P} \) such that \( S(a) \subseteq -S(-d) \).

Proof. (a) If \( f = \lambda_1 b_1 + \ldots + \lambda_n b_n \), we choose \( b, b' \in B \) such that, for all \( i = 1, \ldots, n \), \( S(b_i - b) \subseteq 0 \). Then

\[ -S(-f) = -S(-\lambda_1 b_1 - \ldots - \lambda_n b_n) \subseteq -S(-\lambda_1 b_i - \ldots - \lambda_n b_i) \]

It follows that \( y(f) \supseteq -S(-f) \). It is immediate that \( y \) is sublinear on \( F \).

Finally, if \( b \in B \) then, from Lemma 8(b), \( S \ast y(b) \subseteq y(b) \subseteq \beta \).

(b) follows from (a) with \( B = \{d) \) and \( S = S \ast y \).

11. Notation. We write \( \mathcal{M} \) for the set of all minimal elements of \( (\mathcal{P}, \subseteq) \) and, if \( S \in \mathcal{P} \), \( S \in \mathcal{M} \) for \( \{M \subseteq S : M \subseteq S \} \).

12. Theorem. If \( M \not\subseteq \mathcal{P} \) then the following conditions are equivalent:

(a) For all \( a \in D \), \( M(a) \not\subseteq -M(-d) \).

(b) \( M(D - D) \) is linear on the subspace \( D - D \).

(c) \( M \in \mathcal{M} \).

Proof. It is immediate that (a) implies (b). If (a) is true and \( Q \subseteq \mathcal{P} \), then, for all \( a \in D \), \( Q(a) \not\subseteq -M(-d) \), hence, from Lemma 6, \( M \not\subseteq Q \).

We have proved that (a) \( \Rightarrow \) (c). If \( a \in D \) then, as in Lemma 10(b), \( M \subseteq \mathcal{M} \) so, if (c) is true, \( M \not\subseteq M \) hence \( M(D - D) \subseteq -M(-d) \). We have proved that (c) \( \Rightarrow \) (a).

13. Notation. If \( \mathcal{A} \) is a convex set in a real linear space, we say that \( \mathcal{B} \) is a face of \( \mathcal{A} \) if \( \emptyset \neq \mathcal{A} \subseteq \mathcal{A} \), \( \mathcal{B} \) is convex and \( A, A' \in \mathcal{A} \), \( 0 < u < 1 \) and \( uA + (1 - u)A' \in \mathcal{A} \) imply that \( A, A' \in \mathcal{A} \). If \( Q, S \in \mathcal{P} \) we say that \( Q \) edges \( S \) if \( S \in \mathcal{F} \) (which implies that \( Q \not\subseteq S \)). We write \( \mathcal{M}(S) = \{M : M \in \mathcal{M}, M \notin \mathcal{S} \} \).

14. Lemma. We suppose that \( S, M \in \mathcal{P} \).

(a) If \( B, F, y \) are as in Lemma 10(a), then \( S \ast y \) edges \( S \).

(b) If \( a \in D \) then \( S \) edges \( S \).

(c) If \( a \in D \) and \( M \in \mathcal{S} \) then \( M \notin \mathcal{S} \).

Proofs. (a) If \( Q \subseteq \mathcal{P} \), \( 0 < u < 1 \) and \( aQ + (1 - u)Q \not\subseteq S \), then, given \( b, b' \in B \), we choose \( b'' \in B \) as in (3). Then

\[ \beta \supseteq S \ast y(b'') \supseteq aQ + (1 - u)Q \not\subseteq S \ast y \]

Then, for all \( b, b' \in B \), \( S \ast y(b') \subseteq \beta \supseteq S \ast y \).

Taking the sup over \( b' \) yields that \( \beta \supseteq Q \) hence

for all \( b, b' \in B \), \( Q(b') \subseteq \beta \).

If now \( f = \lambda_1 b_1 + \ldots + \lambda_n b_n \) then, from (4), \( Q(f) \subseteq (\lambda_1 + \ldots + \lambda_n) \beta \) and, taking the inf, \( \inf Q(f) \subseteq \beta \). We have proved that \( Q \subseteq \mathcal{P} \) and so, from Lemma 8(e), \( Q \in \mathcal{S} \). A similar argument shows that \( Q \subseteq \mathcal{S} \). Hence \( S \ast y \subseteq \mathcal{S} \).

(b) is a special case of (a).

(c) follows from (b) and the transitivity of the relation "is a face of".

15. Theorem. We suppose that \( S \in \mathcal{P} \).

(a) If \( F \) is as in Lemma 5 and \( S \) is \( F \)-admissible then there exists \( M \in \mathcal{M}(S) \) such that

\[ \mathcal{M}(S) \]

(b) If \( Q \in \mathcal{P} \) and \( Q \subseteq \mathcal{S} \) then there exists \( M \in \mathcal{M}(S) \) such that \( M \subseteq \mathcal{S} \).

(c) If \( \emptyset \neq B \subseteq \mathcal{P} \) and \( B \in \mathcal{S} \) then there exists \( M \in \mathcal{M}(S) \) such that \( \sup M(B) \subseteq S \).

Proofs. (a) We write \( \mathcal{S} = \{P : P \subseteq \mathcal{P}, P \not\subseteq \mathcal{S} \} \). If \( \mathcal{S} \) is \( (-\subseteq) \)-chain in \( \mathcal{S} \) and, for \( g \in \mathcal{S} \), we write \( T(g) = \inf \{P(g) : P \subseteq \mathcal{S} \} \). The result now follows from Zorn's Lemma and Lemma 8(c).

(b) The nonempty intersection of a decreasing chain of faces is a face and so, by an argument similar to that in (a), \( \mathcal{S} = \{P : P \subseteq \mathcal{P}, P \not\subseteq \mathcal{S} \} \) has a minimal element \( M \). If \( a \in D \) then, from Lemma 10(b) and Lemma 11(c), \( M = M \subseteq M \) and \( M \subseteq \mathcal{S} \) and so \( M(d) \subseteq M(d) \subseteq -M(-d) \). From Theorem 12, \( M \in \mathcal{M}(S) \).

(c) If \( M \in \mathcal{S} \) and \( b \in B \) then \( M(b) \subseteq S \) so \( \sup M(B) \subseteq S \). If \( \sup M(B) \subseteq S \) then the result follows from (b) with \( Q = S \). If, on the other hand, \( \sup M(B) \subseteq S \) then \( M \in \mathcal{S} \) and Lemma 11(a), there exists \( M \in \mathcal{M}(S) \) such that \( M \subseteq \mathcal{S} \). Then, for all \( b, b' \in B \), \( M(b') \subseteq S \) hence \( \sup M(B) \subseteq S \).

(d) is a special case of (c).
16. Theorem. We write $E'$ for the algebraic dual of $E$ and, if $S \in \mathcal{S}$, $E'_S = E' \cap \mathcal{S}_S$.

(a) The linear functionals on $E$ are the (extension) minimal sublinear functionals on $E$.

(b) If $S \in \mathcal{S}$ and $g \in E$ then there exists $L \in \mathcal{S}_S$ such that $L(g) = S(g)$.

(c) Any sublinear functional on $E$ is the upper envelope of the linear functionals on $E$ that it dominates.

Proofs. (a) follows from Theorem 12, (b) follows from Theorem 15(d) applied to Example 2. (c) is immediate from (b).

17. Theorem. We suppose that $S \in \mathcal{S}$ and $Q \in \mathcal{Q}_S$. Then the following conditions are equivalent:

(a) $P \in \mathcal{S}_S$ and $P|D \rightarrow Q|D$ then $P = Q$.

(b) $Q \in \mathcal{S}_S$ and if $P \in \mathcal{S}_S$ and $P|D \rightarrow Q|D$ then $P = Q$.

(c) $Q \in \mathcal{Q}_S \cap \mathcal{E}'$.

Proof. We first observe from Lemma 8(a) that if $P \in \mathcal{S}_S$ then $P|D \rightarrow Q|D$ is a $\mathcal{S}$-order on $Q(D)$, where $Q$ is $S \in \mathcal{Q}_S$. Hence this is the case of Theorem 15(d), (a) is true. For the other cases, (a) is true and, from Theorem 12, (c) is true.

2. The uniqueness of certain sublinear functionals

18. Notation. We write $\mathcal{N} = \{S \in \mathcal{S}_S \mid \mathcal{S}_S \text{ contains exactly one element}\}$.

19. Theorem. If $S \in \mathcal{S}$ the following conditions are equivalent:

(a) $S \in \mathcal{N}$.

(b) There exists $M \in \mathcal{M}$ such that $M|D = S|D$.

(c) $S|D$ is a $\mathcal{N}$-order on $D$.

(d) For all $g \in D$, $S(g) \leq \inf \{-S(-d) : d \in D\}$.

(e) If $Q \in \mathcal{Q}_S$ then $Q|D = S|D$.

(f) If $M \in \mathcal{M}_S$ then $M|D = S|D$.

Proofs. If (a) is true, $\mathcal{N} = \{M\}$ and $d \in D$ then, from Theorem 15(d), there exists $N \in \mathcal{M}_S$ such that $N|D = S|D$ hence, since $N \in \mathcal{M}$, $M|D = S|D$. We have proved that (a) $\Rightarrow$ (b). It is immediate from Theorem 12 that (b) $\Rightarrow$ (c). If $g \in D$ and $d \in D$ then $-d \in D$ hence, if (c) is true, $S(g) + S(-d) = S(g-d) \leq 0$ and so (d) is true. If (d) is true and $Q \in \mathcal{Q}_S$ then, for all $g \in D$,

$Q(g) \leq S(g) \leq \inf \{-S(-d) : d \in D\} \leq \inf \{Q(d) : d \in D\} = Q(g)$

and so (e) is true. It is trivial that (e) $\Rightarrow$ (f). If, finally, (f) is true and $M, N \in \mathcal{M}_S$ then $M|D = N|D$. From Theorem 12 and Lemma 6, $M - N$ and so (a) is true.

20. Theorem. We suppose that $S \in \mathcal{S}$ and $Q \in \mathcal{Q}_S$. We write $\psi = Q|D$. Then $S \ast \psi|D$ is linear on $-D$.

Proof. (⇒) $S \ast \psi(g) + S \ast \psi(h) = S \ast \psi(g+h) \leq S(g+h)$ from Lemma 8(a).

(⇐) $S(g+h) - d \in Q(d)$

$S(g+h-d) + Q(d) \geq S \ast \psi(g) + S \ast \psi(h-d) + Q(d)$

(⇒) $S \ast \psi(g) + S \ast \psi(h-d) + S \ast \psi(d)$

(⇐) $S \ast \psi(g) + S \ast \psi(h)$.

Taking the inf over $d$ yields

$S \ast \psi(g+h) \geq S \ast \psi(g) + S \ast \psi(h)$

as required.

3. A general Shilov theorem

21. Theorem. We suppose that $S \in \mathcal{S}$, $\mathcal{A} \subset \mathcal{A}_S$ and that either

(a) $\mathcal{A}$ is closed in $\mathcal{S}_S$ in the (possibly non-Hausdorff) topology (t) of pointwise convergence on $D$.

(b) $\mathcal{A}$ is closed in $\mathcal{S}_S$ in the topology (p), induced from the product topology of $\mathcal{B}$ and $D$ is dense in $\mathcal{B}$ in the topology given by the seminorm $g \rightarrow S(g) \ast S(-g)$.

Then $\mathcal{A} \supset A(S) \Rightarrow$ for all $d \ast D$, inf $\{P(d) : P \in \mathcal{A} \} \leq -S(-d)$.

Proof. The density condition in (b) implies that $(\mathcal{S}_S, t)$ is Hausdorff hence, since $t \subset (p)$ and $(\mathcal{S}_S, p)$ is compact, (t) $⇒$ (p). So (b) is a special case of (a). We shall establish (a).

(⇒) is immediate from Theorem 15(d).

(⇐) We suppose that $M \ast (S)|D \notin \mathcal{A}$. Then there exist $f_1, \ldots, f_n \in D$ such that if $P \ast \mathcal{A}$ then there exists $i = 1, \ldots, m$ such that $|P(f_i)| - M(f_i) \geq 1$. Now $P(f_i) - M(f_i) \geq 1$. New $P \ast (f_i) - M(f_i)$ and so $f_i \in D$ and $M \in \mathcal{M}$ and $f_i \in \mathcal{M}$ and $M(f_i) = -M(-f_i)$. Consequently, if $g, \ldots, g_n \in \{\pm f_1, \ldots, \pm f_n\}$ and, for $i = 1, \ldots, m$,

$\mathcal{A} = \{P \in \mathcal{S}_S : P(g_i) \geq M(g_i) + 1\}$

then $\mathcal{A} \subset \mathcal{A} \cup \ldots \cup \mathcal{A}_n$. 

We write
\[ \alpha' = \bigcup \{ \lambda_1 \alpha_1 + \ldots + \lambda_n \alpha_n : \lambda_1, \ldots, \lambda_n \geq 0, \lambda_1 + \ldots + \lambda_n = 1 \}. \]
\( \alpha' \) is a face of \( \alpha' \) and, for each \( i = 1, \ldots, n \), \( \alpha_i' \cap \alpha' = \emptyset \) hence \( \alpha' \cap \alpha' = \emptyset \), i.e.,
for all \( P \in \alpha' \) there exists \( h \in E \) such that \( M(h) < P(h) \).

By the usual “continuous image” argument, \( \alpha' \) is compact in \( E^d \) hence there exist \( \delta > 0, h_1, \ldots, h_k \in E \) such that
for all \( P \in \alpha' \) there exists \( j (= 1, \ldots, k) \) such that \( M(h_j) + \delta < P(h_j) \).

The sets
\[ A = \{ P(h_j) : j = 1, \ldots, k \} \]
and
\[ B = \{ x : x \in E^d, x \leq M(h_j) + \delta \} \]
are convex and disjoint in \( E^d \), hence there exists a nonzero linear functional \( \varphi \) on \( E^d \) such that \( \text{sup}_P(B) \leq \text{inf}_P(A) \). It is immediate that \( \varphi \) is of the form \( x \mapsto \lambda_1 x_1 + \ldots + \lambda_n x_n \), where \( \lambda_1, \ldots, \lambda_n \geq 0 \). We write \( h = \lambda_1 h_1 + \ldots + \lambda_n h_n \). If \( P \in \alpha' \), we choose \( i (= 1, \ldots, n) \) such that \( P \leq \alpha_i' \). From Theorem 16(b), there exists \( L \in E^d \) such that \( L(g) = P(g) \). Then \( L \in \alpha' \) and hence
\[ P(h) \geq L(h) = \lambda_1 L(h_1) + \ldots + \lambda_n L(h_n) \geq \lambda_i(M(h_i) + \delta) + \ldots + \lambda_n(M(h_n) + \delta) \geq M(h) + (\lambda_1 + \ldots + \lambda_n) \delta, \]
from which \( M(h) < \text{inf}_P(P(h) : P \in \alpha') \). Since \( \alpha \neq \emptyset \), there exists \( d \in E \) (hence \( d \in D \)) such that
\[ M(d) < \text{inf}_P(P(h) : P \in \alpha') \leq \text{inf}_P(d : P \in \alpha'). \]

The proof is completed by the observation that \( M(d) > -S(-d) \).

22. REMARKS. In the above proof we can use a separation theorem in \( E^d \) rather than in \( E^d \). This needs an extra application of the axiom of choice. See also Remark 29. The appeal to Theorem 16(b) (and the axiom of choice) can also be avoided. What we need for (6) is \( Q \in \alpha' \) such that \( Q(g) = P(g) \) and \( Q(h_j) = Q(-h_j) \), for all \( j = 1, \ldots, n \). Such a \( Q \) can be constructed explicitly by using the reducing operation of Lemma 10(b) a finite number of times, taking first \( d = -g \) and then \( d = h_1, \ldots, h_k \) in sequence (imagining for this construction that we are in the case of Example 2).
5. RESULTS ON SUBSPACES AND LINEARITY

26. LEMMA. We suppose that $S \subseteq \mathcal{A}$ is a subspace of $E$ and $\psi \in \mathcal{F}_{\mathcal{B}_{\mathcal{B}}}$. (We write "ex" for "extreme points of".)

(a) $\psi$ is $S$-admissible.
(b) $\psi$ is $S$-admissible.
(c) $\psi$ is $S$-admissible.
(d) $\psi$ is $S$-admissible.
(e) $\psi$ is $S$-admissible.
(f) $\psi$ is $S$-admissible.

Proofs. (a) is immediate, (b) follows from Theorem 16(a) applied to $E$ and $\psi$, (c) follows from (d) and Theorem 15(a). (d) follows from (c) and Lemma 8(c) ($\supset$), (e) follows from (d), Lemma 9(a) and Lemma 8(b). (f) follows from (e), Theorem 15(b) and Lemma 8(c) ($\supset$).

27. LEMMA. We suppose that $S \subseteq \mathcal{A}$.

(a) If $S$ linearizes then $\mathcal{A}$ is $\mathcal{F} \cap \mathcal{F}_{\mathcal{B}}$.
(b) If $D = D$ is dense in $E$ in the topology given by the seminorm $g = S(g) \vee S(-g)$ then $S$ linearizes $\supset$.

Proofs. It is easily seen that $\mathcal{F}_{\mathcal{B}} = E^* \cap \mathcal{F}$ (cf. the proof of Lemma 26(f) ($\supset$)) hence

$$S \cap \mathcal{F}_{\mathcal{B}} = \mathcal{F}_{\mathcal{B}} \cap \mathcal{F} \cap \mathcal{F} = \mathcal{F} \cap \mathcal{F}_{\mathcal{B}}.$$ 

Further, $\mathcal{A} = E^* \cap \mathcal{F}$, however, if $\mathcal{A}$ is dense, then from Theorem 16(a), $\mathcal{A} = (L \cap \mathcal{F}_{\mathcal{B}})$; the result follows.

(b) is immediate from Theorem 12 and the fact that if $Q \subseteq \mathcal{A}$ then $Q$ is continuous in the seminorm topology.

6. APPLICATIONS: THE HAHN–BANACH THEOREM

28. THEOREM. We suppose that $S \subseteq \mathcal{A}$.

(a) If $g \in E$ then there exists $L \in \mathcal{F}_{\mathcal{B}}$ such that $L(g) = S(g)$ (cf. Theorem 16(b)).
(b) If $E$ is a subspace of $E$ and $\psi \in \mathcal{F}_{\mathcal{B}_{\mathcal{B}}}$ then there exists $L \in \mathcal{F}_{\mathcal{B}}$ such that $L(g) = \psi$.
(c) If $E$ is a subspace of $E$ and $\psi \in \mathcal{F}_{\mathcal{B}_{\mathcal{B}}}$ then there exists $L \in \mathcal{F}_{\mathcal{B}}$ such that $L(g) = \psi$.
(d) If $\varnothing \neq \mathcal{A}$ (is $\varnothing$) is convex then there exists $L \in \mathcal{F}_{\mathcal{B}}$ such that $L(A) = \inf S(A)$.

29. REMARKS. In the above theorem, (a) is equivalent to the Krein–Milman theorem via the bipolar theorem. (If $X$ is a compact convex set in a Hausdorff locally convex space $E$ and $B$ is the dual of $E$, each $g \in E$ we write $S(g) = \sup \{X \in E \cap \mathcal{B} : S(X) \geq g\}$ can be identified with $E'$. (b) is the usual Hahn–Banach theorem. (c) can also be proved by using the usual proof of the Hahn–Banach theorem and preserving the essential property of the extension at each stage (cf. [3], Lemma 11, p. 171).

If $E = 0$, $E$ is the vector lattice of all bounded real functions on $X$ and, for $g \in E$, $S(g) = \sup \{X \in E \cap \mathcal{B} : S(X) \geq g\}$ then, from (d), if $\varnothing \neq \mathcal{A}$ (is $\varnothing$) is convex then there exists a positive linear functional $L$ on $E$ such that $L(1) = 1$ and $\inf L(A) = \inf S(A)$. This is a result used by Kelley in [1]; (b) is the familiar statement. (c) is an extension of the result in [1] in which it was assumed that $\inf S(B) > 0$ which, in turn, strengthened a result of Sikorski in [1] in which $S$ was a norm such that $0 \leq f \leq g$ implies that $S(f) < S(g)$ and $B$ was a set of positive elements.

From any of the above results we can deduce: if $\varnothing \neq \mathcal{A}$ (is $\varnothing$) is convex then there exist $\lambda_1, \ldots, \lambda_n > 0$, $\lambda_1 + \cdots + \lambda_n = 1$ such that

$$\inf \{X_1 \vee \cdots \vee X_n : \lambda_1 \epsilon A_1 + \cdots + \lambda_n \epsilon A_n\} = \inf \{X_1 \vee \cdots \vee X_n : \lambda_1 \epsilon A_1 + \cdots + \lambda_n \epsilon A_n\}.$$
This generalizes the result we deduced from a separation theorem in the proof of Theorem 21.

7. Applications: Cones of Continuous Functions

In this section we use the notation of Example 3. If \( \sigma \in X \) and \( x \in E \) we write \( \sigma_x \) for \( \sigma (x) \). We use the words "directed" and "envelope" with respect to the usual ordering on \( E \). (We do not need to assume that \( C \) is closed under \( \wedge \).)

30. Theorem.

(a) There is a \( \partial_0 X \) of \( X \) such that
\[
\mathcal{A} (S) = \{0\} \cup \{v_x : x \in \partial_0 X\}.
\]

(b) If \( \sigma \in X \) is a closed subset of \( X \) such that
for all \( \sigma \in \mathcal{A} \), \( \inf \{P (\sigma) : P \in \mathcal{A} \} < -1 \) implies that \( \inf \{P (\sigma) : P \in \mathcal{A} \} < -1 \) then \( \sigma \) is the \( \sigma \)-envelope of \( \mathcal{A} \).

(c) If \( A \subset X \) and
for all \( \sigma \in \mathcal{A} \), \( \inf \{P (\sigma) : P \in \mathcal{A} \} < -1 \) implies that \( \inf \{P (\sigma) : P \in \mathcal{A} \} < -1 \) then \( A \) is \( \sigma \)-envelope of \( \mathcal{A} \).

(d) If the extended real valued function \( f \) on \( E \) is the upper envelope of an upper directed \( \sigma \)-envelope \( B \) of \( X \) and either \( \inf \{f (x) : x \in X\} < 0 \) or there exists \( \sigma \in \mathcal{A} \) such that \( \inf \{f (x) : x \in \sigma \} < 0 \) then there exists \( \sigma \)-envelope \( \mathcal{A} \) such that \( \inf \{f (x) : x \in \mathcal{A} \} < 0 \).

(e) If \( \mathcal{A} \) and \( \mathcal{B} \) are \( \sigma \)-envelopes of \( \mathcal{A} \) and \( \mathcal{B} \) respectively then there exists \( \mathcal{A} \)-envelope \( \mathcal{C} \) such that \( \mathcal{A} \subset \mathcal{C} \subset \mathcal{B} \).

(f) If \( \sigma \)-envelope \( \mathcal{A} \) such that \( \mathcal{A} \subset \mathcal{B} \) and, for all \( f \in \mathcal{A} \), \( f (x) > 0 \) then there exists \( \sigma \)-envelope \( \mathcal{B} \) such that \( \mathcal{B} \subset \mathcal{A} \).

(g) If \( \mathcal{A} \), \( \mathcal{B} \), and \( \mathcal{C} \) are \( \sigma \)-envelopes of \( \mathcal{A} \), \( \mathcal{B} \), and \( \mathcal{C} \) respectively then there exists \( \sigma \)-envelope \( \mathcal{D} \) such that \( \mathcal{D} \subset \mathcal{A} \) and \( \mathcal{B} \subset \mathcal{D} \).

Proofs. If \( \mathcal{A} \) is as in (b) then \( \mathcal{A} \) is \( \sigma \)-envelope of \( \mathcal{A} \). Then, for all \( \sigma \in X \),
\[
\inf \{P (\sigma) : P \in \mathcal{A} \} = 0 \wedge \inf \{P (\sigma) : P \in \mathcal{A} \} = 0 \wedge \inf \{P (\sigma) : P \in \mathcal{A} \} = 0.
\]

From Theorem 21(b), \( \mathcal{A} \) is \( \sigma \)-envelope of \( \mathcal{A} \). Taking \( \mathcal{A} = \{v_x : x \in \partial_0 X\} \),
\[
\mathcal{A} (S) = \{0\} \cup \{v_x : x \in \partial_0 X\}.
\]

It is easily seen that \( \sigma \) is \( \sigma \)-envelope of \( \mathcal{A} \) and so (a) follows. If, again, \( \mathcal{A} \) is as in (b),
\[
\{0\} \cup \{v_x : x \in \partial_0 X\} \subset \mathcal{A} \subset \{0\} \cup \{v_x : x \in \partial_0 X\}.
\]

and (b) follows. (c) follows from (b) by writing \( \mathcal{A} \) as \( \{v_x : x \in \partial_0 X\} \),
\[
\mathcal{A} (S) = \{0\} \cup \{v_x : x \in \partial_0 X\}.
\]

We first observe that \( \sup_{x \in \partial_0 X} \inf \{f (x) : x \in X\} = 0 \), so that conditions ensure that \( \sup_{x \in \partial_0 X} \inf \{f (x) : x \in X\} = 0 \). The result follows from (a) and Theorem 15(c), (e) is a special case of (d).
\( \mathcal{M}_0 \) coincides with the family of "maximal measures on \( X \) of mass \( \leq 1 \). It follows that \( \partial X \) coincides with \( \partial_0 X \) as defined in [5], Definition 42, p. 240. If \( x \in \partial_0 X \) then there exists \( x \in \mathcal{M}_0 \) and, arguing as in [3], p. 441, \( x \in \text{ext} F_0 \) hence, from Lemma 27(a), \( x \in \partial_0 X \). If, conversely, \( x \in \partial_0 X \) then \( x \in \mathcal{M}_0 \) then \( x \in \partial_0 X \). Theorem 30(c) is then [5], Theorem 48(b), p. 241, and Theorem 30(b) permits generalizations to (for instance) Shilov sets of measures of mass \( \leq 1 \). Theorem 30(e) strengthens [5], Theorem 48(a), p. 241, in that it replaces \( a < b \) by \( a \neq b \). Substituting \( P = \delta_x \) in Theorem 31 gives a variety of conditions for \( \delta_x \) to have a unique maximal balayage — we observe that \( \gamma (\delta_x) = \gamma (\delta) \) as defined in [5], Definition 44, p. 240. These results do not appear in [5].

33. Theorem. Suppose that \( \lambda \in C \). We write \( C^0 \) for the family of extended real functions on \( X \) that are bounded below and the lower envelope of a subset of \( C^0 \).

(a) In Theorem 30(g) we now have \( \mu (X) = 1 \).

(b) If \( f \in C^0 \) then there exists \( x \in \partial_0 X \) such that \( f(x) = \inf f(X) \).

(c) If \( f \in C^0 \) then \( \inf f(\partial X) = f(\partial X) \).

(d) If \( f \in C^0 \) and \( g \in C^0 \) and, for all \( x \in X \), \( f(x) \geq g(x) \) then, for all \( x \in X \), \( f(x) \geq g(x) \).

(e) If \( f \in C^0 \) and \( g \in C^0 \) and \( f (\partial X) = g(\partial X) \) then \( f = g \).

Proofs. (a) follows from applying Theorem 30(d) to \( f \) minus a sufficiently large positive constant. (c), (d) and (e) follow in sequence from (b).

34. Theorem. Suppose that \( F \) is a subspace of \( E \) such that \( \mathcal{M} \subset C \) and each \( \mathcal{M} \) is the lower envelope of a subset of \( F \).

(a) For all \( x \in X \), \( \mathcal{M}(\{x\}) = S_x(\{x\}) \).

(b) \( \partial X = x : x \in X \), \( x \in \text{ext} F^p \) and \( \partial X = \text{ext} F^p \).

(c) If \( \mathcal{M} \) is a closed subset of \( \mathcal{M} \) such that for all \( f \in F \) \( \inf f(\partial X) \geq -1 \) implies that \( \inf f(\partial X) \geq -1 \) then \( \mathcal{M} \) is closed.

(d) If \( F \) is closed and \( x \in \partial X \) then \( P \{ | P \} \langle \delta \rangle \langle x \rangle \}

Proofs. (a) is immediate from the definitions.

(b) If \( x \in \partial X \) and \( x \in \mathcal{M} \) and, from (a) and Lemma 9(b), \( \mathcal{M}(\{x\}) = \mathcal{N}(\{x\}) = \mathcal{N} \) which edges \( S \). Hence, from Lemma 26(f), \( x \in \text{ext} F^p \). If, conversely, \( x \in \text{ext} F^p \) then, from Lemma 26(g), there exists \( x \in \partial_0 X \) such that \( x = x \). The result follows since \( F \) separates the points of \( X \).

(e) is immediate.

(f) follows from the argument used in the proof of the second assertion of Theorem 30(f).

(g) follows from (a) and Lemma 8(c).

35. Remark. If \( F \) is a subspace of \( E \) such that \( F \) and \( F \) separates points we write \( C = \{ f_i \ldots f_n : n > 1, f_i \ldots, f_n \} \). The Stone-Weierstrass theorem then shows that all the results of this section are valid. From Theorem 34(b), \( \partial X \) is the Choquet boundary of \( F \) as defined in [6], p. 38. \( \mathcal{F}_0 = \{ \lambda x : 0 \leq \lambda < 1, \lambda \in F \} \) in the notation of [6].

This little problem can be avoided by defining \( \mathcal{F}(F) = \{ x : x \in X \} \) in which case \( \mathcal{F}_0 = \{ x : x \in X \} \). We can now deduce the real versions of [6], Proposition 6.1, p. 40, [6], Proposition 6.6, p. 40, and [6], p. 43, from Theorem 33(b), Theorem 34(c), and Theorem 30(g) and Theorem 33(a), respectively. The remaining parts of Theorems 30, 31, 33 and 34 give further results, including various conditions equivalent to the statement "there exists a unique maximal measure \( \mathcal{M} \) on \( X \) such that \( \mathcal{M}(\{x\}) = \mathcal{M}_0(\{x\}) \).

5. Applications: Compact Convex Sets

We suppose in this section that the notation is as in Example 4. We write \( F = \{ f : \lambda \in \mathcal{M} \} \) and \( x \in \mathcal{M} \) if for all \( f \in F \), \( f(x) = f(x) + \lambda \). Since \( F \) separates the points of \( X \), the map \( x \to x_\mathcal{M}(\{x\}) \) is injective; it is clearly affine. A simple application of the bipolar theorem shows that \( \mathcal{F}_0 = \{ x \in X : 0 \leq \lambda < 1, x \in X \} \). It follows from [5], Theorem 7(a), p. 222, that Theorem 34 is applicable. Then Theorem 36(a) follows from Theorem 34(b) and Theorem 30(b) follows from Theorem 33(c).

From [5], Theorem 7(b), p. 222, any extended real-valued concave lower semicontinuous function on \( X \) that is bounded below is in \( C^0 \) and any bounded affine semicontinuous function on \( X \) is in \( C^0 \cap C^0 \).

Then Theorem 36(c) follows from Theorem 33(b), (c). Finally, Theorem 36(e) follows from Theorem 30(g) and Theorem 33(a).

36. Theorem.

(a) \( \partial X = \{ x \in X \} \).

(b) If \( A \) is closed in \( A \) and, for all \( x \in X \),

\[ \inf f(\{ x \}) = f(\{ x \}) \]

then \( A \subset x \in \partial X \).

(c) If \( f \) is an extended real-valued concave lower semicontinuous function on \( X \) that is bounded below then there exists \( x \in \mathcal{M} \) such that \( f(x) = \inf f(X) \).

(d) \( f \) and \( g \) are bounded affine semicontinuous functions on \( X \) and \( f(\{ x \}) = g(x) \) then \( f = g \).

(e) If \( x \in X \) then \( x \) is the barycenter of a maximal measure \( \mu \) on \( X \).

If \( \mu \) is a maximal measure on \( X \) then \( \mu(X) = 1 \) and \( \mu(Y) = 0 \) whenever \( Y \) is a compact \( G \) in \( X \) such that \( Y \cap \mathcal{M} = \emptyset \).

\[ \text{ICM} \]
If $g \in E$ and $x \in X$ we write \( \hat{g}(x) = \inf \{ d(x, z) : x \in C, z \geq g \} \).

37. Theorem. If $x \in X$ then conditions (a)-(g) are equivalent.

(a) $x$ is the barycenter of a unique maximal measure on $X$.

(b) $x$ is the barycenter of a maximal measure $\mu$ on $X$ such that, for all $g \in C$,
\[ \int g \, d\mu = \hat{g}(x). \]

(c) If $g, h \in C$, \( \hat{g}(x) + \hat{h}(x) = \hat{g + h}(x) \).

(d) If $x$ is the barycenter of a maximal measure on $X$ then, for all $g \in C$,
\[ \int g \, d\mu = \hat{g}(x). \]

(e) If $g, h \in C$ then \( \hat{g}(x) + \hat{h}(x) \leq \hat{g + h}(x) \).

(f) If $g \in C$ and $g \leq h$ then \( \hat{g}(x) \leq \hat{h}(x) \).

(g) If, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $a_i \geq 0$, $\beta_j \geq 0$, $y_i \in X$, $x_1, x_2 \in X$, $\sum a_i = 1$, $\sum \beta_j = 1$, $\sum a_i y_i = x = \sum \beta_j x_j$ then there exist $\gamma_{ij} \geq 0$, $l_i \in \mathbb{R}$ such that, for each $i$, $\sum \gamma_{ij} l_i = a_i y_i$, and, for each $j$, $\sum \gamma_{ij} l_i = \beta_j x_j$, and $\sum l_i y_i = x_j$.

Proof. We apply Theorem 34(d) and then the equivalence of (a)-(f) follows from the corresponding statements in Theorem 31, with $P = C$.

(a) $\Rightarrow$ (g). If $a_i, \beta_j, y_i, x_j$ are as in (a) then, from Theorem 36(a), there exist maximal measures $\mu_i$ and $\nu_i$ with barycenters $y_i$ and $x_j$, respectively. Then $\sum a_i \mu_i$ and $\sum \beta_j \nu_j$ are both maximal measures with barycenter $x$. By (a), $\sum a_i \mu_i = \sum \beta_j \nu_j$. From the decomposition property, there exist $\gamma_{ij} \geq 0$ and probability measures $\nu_i$ and $\gamma_{ij}$ such that, for each $i$, $\sum \gamma_{ij} \nu_i = a_i \mu_i$, and, for each $j$, $\sum \gamma_{ij} \nu_j = \beta_j \nu_j$. The required result follows with $l_i$ the barycenter of $\nu_i$.

(g) $\Rightarrow$ (e). We suppose that $g, h \in C$ and $\sum a_i y_i = x = \sum \beta_j x_j$, where $a_i, \beta_j, y_i, x_j$ are as in (g). Since $g$ and $h$ are convex.
\[ \sum a_i \hat{g}(y_i) + \sum \beta_j \hat{h}(x_j) \leq \sum a_i \hat{g}(y_i) + \sum \beta_j (\hat{h}(x_j) + \hat{h}(x_j)) \leq S(g + h). \]

Hence, from [6], Lemma 9.6, $\hat{g}(x) + \hat{h}(x) \leq S(g + h)$, as required.

36. Remarks. Theorem 36(b) implies Milman’s theorem [3], Lemma V.8.5, p. 440. Theorem 36(c), (d) are due to Baser [1]. Of course, we can generalize (d) to “if $f, g \in C^m \cap -C^m$...” All the functions in $C^m \cap -C^m$ are affine - it might be interesting to find exactly which affine functions are in $C^m \cap -C^m$. Theorem 36(e) is the Choquet–Bishop–deLeeuw theorem [6], p. 24.

We can consider the ordering defined on $V \times X$ by the cone
\[ \mathcal{Y} = \{ (l_\delta, x) : x \in X, l_\delta > 0 \}. \]

Then Theorem 37(g) says simply that if, for all $i, j, a_i, b_i V \times X, a_i, b_i > 0$ and $\sum a_i = (x, 1) = \sum b_i$, then there exist $c_i \in \mathcal{Y}$ such that, for each $i$, $\sum c_i = a_i$ and for each $j$, $\sum c_j = b_j$. We can think of this as a “local decomposition property” at $(x, 1)$ (cf. [3], Theorem 39, p. 231). Consequently

(12) each $x \in X$ is the barycenter of a unique maximal measure on $X$ if, and only if, the ordering on $V \times X$ satisfies the decomposition property. This is, in turn, equivalent to the statement that $Y$ induces a lattice ordering on $Y - Y$, i.e., that $Y$ is a simplex. This is part of the Choquet–Meyer theorem ([8], p. 66, (5) $\Rightarrow$ (1)). [9], p. 66, (5) $\Rightarrow$ (3) is immediate from Theorem 37(a) $\Rightarrow$ (c) $\Rightarrow$ (d). Finally, (12) is equivalent to

for all $g \in C$, $\hat{g}$ is affine

(see [6], p. 66, (2)). It is immediate from Theorem 37(a) $\Rightarrow$ (f) that (12) $\Rightarrow$ (13). It follows from Lemma 9(b), Theorem 36(d) and Theorem 37(c) $\Rightarrow$ (a) that (13) $\Rightarrow$ (12). See Theorem 41 for a different approach.


In the context of Section 7 and Section 8, $E^2_1 = \{ 0 \} \cup \{ a_\epsilon : x \in X \}$ which is closed in $E^2$. In this section we discuss a generalization of the results of Section 7 in which $E^2_1$ is not necessarily closed in $E^2$.

We suppose that the notation is as in Example 5. If $x \in X$ and $g \in E$ we write $\epsilon_x(g)$ for $g(x)$. We use the words “directed” and “envelope” with respect to the usual ordering on $E$. Then (cf. the introductory remarks in Section 8)
\[ E^2_1 = \{ \lambda \epsilon_x : 0 \leq \lambda \leq 1, x \in X \} \quad \text{and} \quad \text{ex} E^2_0 = \{ 0 \} \cup \{ a_\epsilon : x \in X \}.

We note from Theorem 10(c) that $P \cdot \mathcal{S}_E = P$ is of the form $g \mapsto \sup \lambda_x \epsilon_x$ where $0 \leq \lambda_x \leq 1$ and $x \in X$.

39. Theorem.

(a) There exists $\delta \in \mathcal{E}'$ such that $\mathcal{E} = \{ 0 \} \cup \{ x \delta : x \in X \}$.

(b) Theorem 30(c), (d), (e) are true as stated.

(c) If $M \in \mathcal{S}_E$ there exists $\epsilon, 0 \leq \epsilon \leq 1$ and $y \in X$ such that $M = \lambda \epsilon$ and $g(y) < g_1 \leq \cdots \leq g_n$ and, for all $x \in X$, $\sup_{\epsilon_x} \epsilon(x) > 0$.

(d) Theorem 31 is true as stated.

Proofs. (a) follows from the remarks above and Lemma 27(b), (c) and (d) are proved by analogy with the proofs we already have.

40. Remark. It is well known that if $X$ is compact Hausdorff, $\mathcal{E}(X)$ can be identified with the set of continuous affine functions on a certain compact simplex with closed extreme points. Thus Theorem 39...
in fact represents a considerable generalization of Theorem 30 and Theorem 31.

We can also make the appropriate modifications and obtain the analogy of Theorem 33 and Theorem 34.

10. A DIFFERENT APPROACH TO THE UNIQUENESS PROBLEM

In this section we return to the general considerations of Section 1 and Section 2. We suppose that $S, F$ are as in Lemma 8 and that $\mathcal{A}$ is a compact convex subset of $E'$ such that each $p \in \mathcal{A}$ is $S - E$-admissible. Further, we suppose that $\mathcal{P}_\mathcal{A}$ is convex.

41. THEOREM. For all $p \in \mathcal{A}$, $S \epsilon \mathcal{A}$ for all $q \in \mathcal{A}$, $S \epsilon \mathcal{A}$ and for all $q \in S - F$, the map $p \rightarrow S \epsilon q$ is affine on $\mathcal{A}$. We write $\mathcal{A} = \mathcal{A}(X)$, $S \epsilon \mathcal{A}$ for all $q \in S - F$, the map $p \rightarrow S \epsilon q$ is affine on $\mathcal{A}$.

Proof. (a) We suppose that $p_1, p_2 \in \mathcal{A}$. From Theorem 19(a) $\Rightarrow$ (c)

$S \epsilon (q + (1 - a)q) = aS \epsilon q + (1 - a)S \epsilon q$.

By hypothesis, the left hand element is in $\mathcal{A}$ and the right hand one is in $S \epsilon \mathcal{A}$. Hence, from Theorem 19(a) $\Rightarrow$ (c)

$S \epsilon (q + (1 - a)q) = aS \epsilon q + (1 - a)S \epsilon q$.

and this gives the required result.

42. REMARK. We mention two applications of Theorem 41.

In the context of Theorem 28 if $S \epsilon \mathcal{A}$ and $\epsilon$ is a subspace of $E$ then for all $P \epsilon \mathcal{A}$ there is a unique $L \epsilon \mathcal{E}^G$ such that $L \epsilon E = \epsilon$ and for all $q \epsilon E$, the map $p \rightarrow S \epsilon q$ is affine on $\mathcal{E}^G$.

In the context of Theorem 34 (Remark 35), for all $q \epsilon \mathcal{E}^G$ there is a unique $\mathcal{A} \epsilon \mathcal{A}^G$ such that $\mathcal{A} \epsilon F = \epsilon$ for all $S \epsilon X$ there is a unique $\mathcal{A} \epsilon \mathcal{A}^G$ such that $\mathcal{A} \epsilon F = \epsilon$ and for all $q \epsilon E$, the map $p \rightarrow S \epsilon q$ is affine on $\mathcal{E}^G$.

11. THE CHOQUET-BISHOP-DE LEEUW THEOREM

One might consider Theorem 43 as the linear space part of The Choquet-Bishop-de Leeuw theorem. The measure theoretic result is deduced from Theorem 43(e) as in [5], p. 28, or [5], Theorem 32, p. 333.

43. THEOREM. Let $X$ be a compact convex subset of a real Hausdorff locally convex space and $X \epsilon X$. We write $E = \mathcal{E}(X)$ and $\epsilon = d \epsilon E$, $d \epsilon (X)$. If $g \epsilon E$ we write $S_g = \epsilon \inf (d \epsilon (X))$. We write $\epsilon \epsilon$ for the pointwise ordering on $E^G$ and $\mathcal{P}_\mathcal{A} = \epsilon (P \epsilon E^G, P \epsilon (X))$ is sublinear on $E$, $P \epsilon \mathcal{P}_\mathcal{A}$ and, for all $g \epsilon E$, $g \epsilon (X) = \epsilon \inf (P \epsilon (X))$.

(a) $\mathcal{P}_\mathcal{A} = \emptyset$.

(b) There is a $(\epsilon)$-minimal element $M \epsilon \mathcal{P}_\mathcal{A}$.

(c) $M$ is a positive linear functional on $E$, $M(1) = 1$ and $x$ is the barycenter of $M$.

(d) If $d_1, d_2, \ldots \epsilon C$ and $d_1 \epsilon d_2 \epsilon \ldots$ then there exists $x \epsilon X$ such that $\lim \inf d_i(x) = \lim \inf d_i(x)$.

(e) If $g_1, g_2, \ldots \epsilon E$, $g_1 \epsilon g_2 \epsilon \ldots$ and, for all $x \epsilon X$, $\lim \inf g_i(x) = 0$ then $\lim \inf M(g_i) = 0$.

Proofs. (a) is immediate since $S_g \epsilon \mathcal{P}_\mathcal{A}$ and (b) follows from Zorn's Lemma.

(c) If $d \epsilon C$ and $\epsilon \epsilon E$ we write $N = \epsilon \inf (M(\epsilon) - d \epsilon E)$ and $\epsilon \epsilon C$. Then $N \epsilon \mathcal{P}_\mathcal{A}$ and $\epsilon \epsilon C$. Since $M$ is minimal, $M(\epsilon) - d \epsilon E$ and so $M$ is linear on $\epsilon \epsilon C$. $M$ is $\epsilon$-continuous on $E$ and, from the Stone-Weierstrass Theorem, $C \epsilon E$ is $\epsilon$-dense in $E$ hence $M$ is linear on $E$. If $g \epsilon E$, $g \epsilon g_0$ then $M(g) \epsilon S_g(g) \epsilon 0$ hence $M$ is positive. If $d \epsilon C$ then $M(d) \epsilon S_g(d)$, $d \epsilon C$, i.e., $M$ is a balayage of $g_0$. This implies that $M(1) = 1$ and $x$ is the barycenter of $M$.

(d) For all $n \epsilon N$ there exists $x_\epsilon \mathcal{P}_\mathcal{A}$ such that $d_i(x_\epsilon) \epsilon \lim \inf d_i(x_\epsilon) + \epsilon / n$.

We let $y$ be a cluster point of $(x_\epsilon)$, $x \epsilon N$ then $d_i(x_\epsilon) \epsilon \lim \inf d_i(x_\epsilon)$ and so $\lim \inf d_i(y) \epsilon \lim \inf d_i(x_\epsilon)$. On the other hand, for all $x \epsilon X$, $\lim \inf d_i(x) \epsilon \lim \inf d_i(y)$. It follows that

$\{y : y \epsilon X, \lim \inf d_i(y) = \lim \inf d_i(x_\epsilon)\}$

is a closed face of $X$ and, by the usual argument, contains an extreme point of $X$.

(e) Let $\epsilon > 0$. We choose $d_1, d_2, \ldots \epsilon C$ such that $d_1 \epsilon d_2$ and $M(d_1) \epsilon M(d_2) + \epsilon / 2$.
and, for \( n = 2, 3, \ldots \)
\[
d_n \geq d_{n-1} - g_{n-1} + g_n \quad \text{and} \quad M(d_n) \leq M(d_{n-1} - g_{n-1} + g_n) + \varepsilon/2^n,
\]
from which it follows that \( d_1 \leq d_2 \leq \ldots \) and, for \( n = 1, 2, \ldots \)
\[
(14) \quad d_n \geq g_n \quad \text{and} \quad M(d_n) - M(g_n) \leq \varepsilon \left( \frac{1}{2} + \ldots + \frac{1}{2^n} \right) \leq \varepsilon.
\]

For all \( x \in X \), \( \lim_n d_n(x) \geq \lim_n g_n(x) \geq 0 \) hence, from (d), \( \lim_n \inf d_n(X) \geq 0 \) and so \( \lim_n M(d_n) \geq 0 \). Combining this with (14), \( \lim_n M(g_n) = -\varepsilon \) and the required result follows since \( \varepsilon \) is arbitrary.

References


On the function \( g_t \) and the heat equation

by C. Segovia (Princeton, N. J.) and R. L. Wheeden (New Brunswick, N. J.)

INTRODUCTION AND NOTATIONS

In the present paper, a function analogous to the \( g_t \) function of Littlewood, Paley, Zygmund, Stein (see [13] and [10]) is introduced for functions \( u(x, t, y) \) which are solutions of the boundary problem

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial u}{\partial y^2}, \quad y > 0
\]

and

\[
\lim_{x \to e} u(x, t, y) = f(x, t).
\]

The definition of \( g_t \) is given in section 3, (2.1), and its properties concerning the preservation of \( I^p \) classes are discussed in theorems (2.2), (2.3), and (2.4). The method used here is an adaptation to the parabolic case of the one found in C. L. Pefferman’s doctoral dissertation [2]. In section 3, theorem (3.1), the function \( g_t \) is applied to obtain a characterization of the \( L^2 \) spaces introduced by B. F. Jones in [4] and [5]. This characterization is suggested by those given by Hirschman [3] and Stein [9]. Also, a generalization of the \( g \)-function of Littlewood-Paley involving fractional derivatives is considered (theorem (2.22)).

For an analogue in the case of analytic and harmonic functions, see [3] and [8].

We shall denote by \( E_{n+1} \) the set of all \((n+1)\)-tuples \( (x_1, \ldots, x_n, t) = (x, t) \) of real numbers, with the explicit intention of distinguishing the last variable. \( E_{n+1} \) denotes the set of all \((n+3)\)-tuples \( (x_1, \ldots, x_n, t, y) \) of real numbers with \( y > 0 \). By \( |x| \) we denote the absolute value of \( (x_1, \ldots, x_n) \), which is given by \( \left( \sum x_i^2 \right)^{1/2} \). The complement of a set \( A \) is denoted by \( A' \) and its Lebesgue measure by \( |A| \). The definition of Fourier