In this paper we prove a theorem on the composition of the \( p \)-absolutely summing and \((s,t)\)-absolutely summing operators which is a generalization of a theorem proved by Pietsch (see [7]) concerning the composition of \( p \)-absolutely summing operators. The proof of the theorem follows Pietsch's proof.

As an application of this theorem we prove that for some class of spaces the ideals of \((s,t)\)-absolutely summing operators have properties quite analogous to those of ideals of \((s,t)\)-absolutely summing operators in a Hilbert space provided \(1/t - 1/s = 1/p\) and \(t \leq 2\). The proof is quite analogous to that of the theorem stating that \(A_{11}(b, X) \ast A_{11}(b, X)\) if \(r \leq 2\) (see [5]).

Definition. Let \( X \) and \( Y \) be Banach spaces, let \( T \in B(X, Y) \) and let \( 1 \leq q \leq p < \infty \). Put

\[
a_{p,q}(T) := \inf \{ C : \left[ \sum_{i} \| T x_i \|^q \right]^{1/q} \leq C \sup_{x \in X} \left[ \sum_{i} \| x_i \|^p \right]^{1/p} \}
\]

for \( x_i \in X \), \( i = 1, \ldots, n \) and \( n = 1, 2, \ldots \). \[\]

An operator \( T \) is said to be \((p,q)\)-absolutely summing \((T \in A_{p,q}(X, Y)\)) if \( a_{p,q}(T) < \infty \).

It turns out that \( A_{p,q}(X, Y) \) with the norm \( a_{p,q}(\cdot) \) is the Banach ideal.

Proposition. Let \( X, Y \) and \( Z \) be Banach spaces, \( T \in A_{p,q}(X, Y) \) and \( S \in A_{r,s}(Y, Z) \). Then the operator \( ST \in B(X, Z) \) is \((r,q)\)-absolutely summing, where

\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{s} \leq 1, \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{t} \leq 1
\]

and \( a_{p,q}(ST) \leq a_{p,q}(S)a_{p,q}(T) \).

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Proof. Since the operator \( T \) is \( p \)-absolutely summing, there is a regular positive Borel measure \( \mu \) on the unit ball \( K^p \) of \( X^* \) such that for \( x \in X \)

\[
\|T x\| \leq a_{p,\mu}(T) (\sum_{n} \|x^n(x^n)f(x^n)\|\|T x^n\|\|\mu\|)\|x^n(x^n)\|^{1/p}.
\]

(see [6] and [7]).

Let \( (a_n)_{n=1}^\infty \) be an arbitrary finite sequence. Put

\[
a^n_n = \left( \int_{K^n} \|x^n(x^n)f(x^n)\|\|T x^n\|\|\mu\| \right)^{1/p} a_n \quad \text{for } n = 1, \ldots, N.
\]

Applying the Hölder inequality and the fact that \( S \) is \((a, t)\)-absolutely summing, we obtain

\[
\left( \sum_n \|S T x^n\|\|x^n(x^n)f(x^n)\|\|\mu\| \right)^{1/p} \leq \left( \sum_n \|S T x^n\|^{1/p} (\sum_{n} \|x^n(x^n)f(x^n)\|^p\|\mu\|)\|x^n(x^n)\|^{1/p} \right)^{1/p}.
\]

Since \( T \) is \( p \)-absolutely summing, the diagram

\[
\begin{array}{ccc}
     & C(K^p) & \rightarrow & L_p(K^p, \mu) & \rightarrow & Z \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow \\
X & \rightarrow & Y & \rightarrow & \rightarrow & \rightarrow
\end{array}
\]

is commutative where \( Z \) is a Banach space, \( i : X \rightarrow C(K^p) \) is the canonical isometry \( x \rightarrow x^n(x^n) \), and \( I : C(K^p) \rightarrow L_p(K^p, \mu) \) is the identity map \( f \rightarrow f \).

Let \( E \) denote the closure of \( i(X) \). Consider an arbitrary functional \( g^* \in Y^* \).

Then the formula

\[
\tilde{\beta}_p(a^n) = g^*(T x^n)
\]

determines a functional \( \tilde{\beta}_p \) on \( E \). It follows from the Hahn-Banach theorem and from the fact that \( \{L_p(K^p, \mu)\}^* \) is isometrically isomorphic to \( L_p(K^p, \mu) \) that there is an element \( f \in L_p(K^p, \mu) \) such that

\[
g^*(T x^n) = \int_{K^n} x^n(x^n)f(x^n)\|T x^n\|\|\mu\|
\]

and

\[
\left( \int_{K^n} \|x^n(x^n)f(x^n)\|\|T x^n\|\|\mu\| \right)^{1/p} \leq a_{p,\mu}(T) \|g^*\|.
\]

By Hölder's inequality, we obtain

\[
\|g^*(T x^n)\| \leq \int_{K^n} \|x^n(x^n)f(x^n)\|\|T x^n\|\|\mu\|
\]

\[
= \int_{K^n} \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\|^p \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\|^p
\]

\[
\leq \int_{K^n} \|x^n(x^n)f(x^n)\|\|T x^n\|\|\mu\| \left( \int_{K^n} \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\|^p \right)^{1/p} \left( \int_{K^n} \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\|^p \right)^{1/p}.
\]

Hence for arbitrary \( g^* \in Y^* \), \( \|g^*\| \leq 1 \) and for \( n = 1, \ldots, N \)

\[
\|g^*(T x^n)\| \leq \int_{K^n} \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\| \left( \int_{K^n} \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\|^p \right)^{1/p}.
\]

Finally, we get

\[
\left( \sum_n \|g^*(T x^n)\|^p \right)^{1/p} \leq \left( \sum_n \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\| \right)^{1/p} \left( \sum_n \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\|^p \right)^{1/p}.
\]

Consequently,

\[
\left( \sum_n \|S T x^n\|\|x^n(x^n)f(x^n)\|\|\mu\| \right)^{1/p} \leq \left( \sum_n \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\| \right)^{1/p} \left( \sum_n \|x^n(x^n)f(x^n)\|^p\|T x^n\|\|\mu\|^p \right)^{1/p}.
\]

Thus, by the definition of the norm \( a_{p,\mu}(S; T) \), we have

\[
a_{p,\mu}(S; T) \leq a_{p,\mu}(S) a_{p,\mu}(T).
\]

This completes the proof.

**Theorem.** Let \( X \) be a Banach space isomorphic to a subspace of an \( L_p(\mu) \)-space for some measure \( \mu \), and let \( Y \) be an arbitrary Banach space. Then for \( 1 \leq r \leq 2 \)

\[
A_{r,2}(X, Y) = A_{r,2}(X, Y), \quad \text{where } 1/r_1 = 1/r - 1/2.
\]

Proof. First, observe that \( A_{r,2}(X, Y) \subset A_{r,2}(X, Y) \) since \( 1 - 1/r = 1 - 1/r_1 \) (see [4], 0.7).

The inclusion \( A_{r,2}(X, Y) \subset A_{r,2}(X, Y) \) results from the Proposition and from the following facts:

(a) If \( X \) is isomorphic to a subspace of an \( L_p(\mu) \)-space, then every operator \( S \in B(l_2, X) \) is \( 2 \)-absolutely summing (see [2] and [6]).

(b) Let \( T : X \rightarrow Y \) be a linear operator from a Banach space \( X \) into a Banach space \( Y \). Then \( T \in A_{r,2}(X, Y) \) if and only if \( TS \in A_{r,2}(l_2, Y) \) for every \( S \in B(l_2, X) \).

To prove (b), assume that \( T \in A_{r,2}(X, Y) \). Then there is a sequence \( (x_n) \subset X \) such that the series \( \sum x_n \) is unconditionally convergent, but \( \sum \|T x_n\| = \infty \).

Put \( S(x_n) := \sum a_n x_n \) for \( (a_n) \in l_\infty \). Since the series \( \sum a_n \) is unconditionally convergent, \( S \in B(l_\infty, X) \) (see [1]).

Since \( \sum \|T x_n\| = \infty \), there exists a sequence of real numbers \( \eta_n \) such that \( \lim \eta_n = 0 \) and \( \sum \|T x_n\| = \infty \).

Since \( \sum \|T (x_n(a_n))\| = \sum \|T x_n\| = \infty \), where \( a_0, \ldots, 0, 1, 0, \ldots \), \( TS \in A_{r,2}(l_\infty, Y) \), and this completes the proof of (b).
COROLLARY 1. Let $X$ be an $L_p$-space (see [6]). Let $1 \leq r \leq 2$, $1 \leq p \leq 2$. Then for every Banach space $Y$ we have

$$A_{r,1}(X, Y) = A_{r,1}(X, Y),$$

where $1/r_1 = 1 - 1/2$.

This corollary is a special case of the Theorem, since $L_p$ is a subspace of $L_1$ ($\mu$) for some measure $\mu$ (see [6], Section 7).

COROLLARY 2. Let $1 < r < 2$ and $1 < p < 2$. Then for every Banach space $Y$ we have

$$A_{r,1}(L_0, Y) = A_{r,1}(L_0, Y),$$

where $1/r_1 = 1 - 1/2$.

Definition. We denote by $H_1$ the space of Lebesgue-integrable functions on the circle such that

$$\int_0^{2\pi} f(t) dt = 0$$

for $n = 1, 2, \ldots$ (see [3]).

COROLLARY 3. Let $1 < r < 2$ and let $Y$ be an arbitrary Banach space. Then

$$A_{r,1}(H_1, Y) = A_{r,1}(H_1, Y),$$

where $1/r_1 = 1 - 1/2$.

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References


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The estimation of an integral arising in multiplier transformations

by

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The aim of this note is to prove the following general estimate:

Theorem. Let $a_1 < a_2 < \ldots < a_n$ be fixed non-negative real numbers and let $b_1, \ldots, b_n$ be real numbers. Then

$$\int_0^\infty \exp \left( i \left( b_1 [x] \right)^a + b_2 [x]^a \ldots + b_n [x]^a \right) \frac{dx}{x} \leq K(a_1, a_2, \ldots, a_n),$$

where $K$ does not depend on $b_1, b_2, \ldots, b_n$.

(The integral is defined by integrating over $\varepsilon \leq |x| \leq R$ and then letting $R \to \infty$ and $\varepsilon \to 0$.)

For fixed real $a$ the symbol $[x]^a$ may stand for either $|x|^a$ or $\text{sgn} x |x|^a$.

The proof of the Theorem is based on the following Lemma of Van der Corput:

Lemma 1. Let $f(t)$ be a real-valued differentiable function on $u \leq t \leq v$. Suppose $f'(t)$ is monotonic and that $\left| f'(t) \right| > \frac{1}{2} > 0$. Then

$$\int_u^v \exp \left( i f(t) \right) dt < \frac{1}{2}.$$

For the proof of Lemma 1, see [3], p. 197.

To apply Van Der Corput's Lemma, it is necessary to obtain estimates on the measure of the set on which an expression of the form

$$(1.1) g(x) = d_1 x^j + d_2 x^j + \ldots + d_{m-1} x^{j-1} + x^m$$

is small.

Lemma 2. Let $g(x)$ be defined by (1.1) with $d_i$ real and $c_i > 0$. Assume further that $c_i > c_{i-1} + 1$, $2 \leq j \leq m$, and that $c_1 = 1$. Then the graph of $g(x)$ for $1 \leq x \leq \infty$ consists of $\nu$ intervals $(I_k)$ on each side of which $g(x)$ is monotonic. On each of the intervals $(I_k)$, $k = 1, \ldots, \nu$, $|g(x)| \geq 1$ except on a subinterval of length at most $\mu_1$, and what is most important $\nu$ and the numbers $\mu_k$ may be chosen so as not to depend on the numbers $d_1, d_2, \ldots, d_{m-1}$.