STUDIA MATHEMATICA, T. XXXV. (1970)

On an equation with reflection of order n

by

BARBARA MAZBIC-KULMA (Warsawa)

If a differential equation contains together with the unknown function \( x(t) \) the function \( x(-t) \), then it is called a differential equation with reflection.

D. Przeworska– ROLEWICZ gives in [1] the general solution of an equation with reflection of order 1, i.e. of the equation

\[
a_0 x(t) + b_0 x(-t) + a_1 x'(t) + b_1 x'(-t) = y(t),
\]

where \( a_0, b_0, a_1 \) and \( b_1 \) are scalars.

In the present paper we consider the differential equation with reflection of order \( n \),

\[
(1) \quad a_n x(t) + b_n x(-t) + \cdots + a_1 x^{(n-1)}(t) + b_1 x^{(n-1)}(-t) = y(t),
\]

where the coefficients \( a_0, \ldots, a_n, b_0, \ldots, b_n \) are constants. We give a general form of the solution of (1) under the following assumptions:

1° \( a_n^2 - b_n^2 \neq 0 \);

2° \( a_{j-k}a_k - b_{j-k}b_k \neq 0 \) \((k = 0, 1, \ldots, n \text{ and } j = k+1, \ldots, k+n)\);

3° the polynomial \( \sum_{j=0}^{n} \lambda_j t^j \) has single roots only for \( k = 0, 1, \ldots, n \),

where

\[
\lambda_j = \begin{cases} 
\sum_{k=0}^{j} \sigma_k & \text{for } 0 \leq j \leq n, \\
\sum_{k=j+1}^{n} \sigma_k & \text{for } n < j \leq 2n,
\end{cases}
\]

\[
\sigma_k = (-1)^{k-1} (a_{j-k}a_k - b_{j-k}b_k) (a_n^2 - b_n^2)^{-1}.
\]

1. Let \( S \) be a reflection: \( S x(t) = x(-t) \). Since \( S^2 = I \), where \( I \) is the identity operator, \( S \) is an involution. We write

\[
(2) \quad D x(t) = x'(t).
\]
It can be proved that the operator \( S \) satisfies the following conditions:

1° \( S \) is commuting with the operator \( D^n \):

\[
SD^n - D^n S = 0;
\]

(3)

2° \( S \) is anticommuting with the operator \( D^{n+1} \):

\[
SD^{n+1} = -D^{n+1} S = 0.
\]

(4)

2. Let \( X \) be a linear space over the field of complex scalars. We consider a linear equation of the form

\[
(a_1 I + b_1 S) + (a_2 I + b_2 S)D + \ldots + (a_n I + b_n S)D^n = y,
\]

where \( S \) is an involution on \( X \) and \( D \) is a linear operator transforming \( X \) into itself and anticommuting with \( S \); \( a_1, \ldots, a_n, b_1, \ldots, b_n \) are scalars.

Let us write

\[
A = \sum_{m=0}^{n} (a_m I + b_m S)D^m.
\]

We prove for the operator \( A \) the following

**Theorem 1.** Let

\[
B = \sum_{m=0}^{n} \left( (-1)^m a_m I - b_m S \right) D^m
\]

and \( R_A = (a_n^2 - b_n^2) \). Then

\[
AR_A = R_A A = \sum_{j=1}^{n} \lambda_j D^j,
\]

where \( \lambda_j \) and \( \xi_j \) are defined by (i) and (ii) respectively.

**Proof.** We have

\[
BA = \sum_{m=0}^{n} \sum_{m=0}^{n} \left( (-1)^m a_m I - b_m S \right) \left( D^m (a_k I + b_k S) D^k \right) = \sum_{m=0}^{n} \sum_{m=0}^{n} \left( (-1)^m a_m I - b_m S \right) \left( D^m + (-1)^n b_k S D^n \right) D^k
\]

\[
= \sum_{m=0}^{n} \sum_{m=0}^{n} \left[ (-1)^m a_m I - b_m S \right] \left[ a_k + (-1)^n b_k S D^n \right] D^k
\]

\[
= \sum_{m=0}^{n} \sum_{m=0}^{n} \left[ (-1)^m a_m I - b_m S \right] \left[ a_k + (-1)^n b_k S D^n D^k \right]
\]

\[
= \sum_{m=0}^{n} \sum_{m=0}^{n} \left[ (-1)^m a_m I - b_m S \right] \left[ a_k + (-1)^n b_k S D^n \right] D^{n+k}
\]

\[
= \sum_{m=0}^{n} \sum_{m=0}^{n} \left[ (-1)^m a_m I - b_m S \right] \left[ a_k + (-1)^n b_k S D^n \right] D^{n+k}.
\]

Let us remark that

\[
(a_m b_k - a_k b_m) D^{n+k} = - (a_k b_m - a_m b_k) S D^{n+k},
\]

hence

\[
\sum_{m=0}^{n} \sum_{m=0}^{n} (a_m b_k - a_k b_m) S D^{n+k} = 0.
\]

This implies

\[
BA = \sum_{m=0}^{n} \sum_{m=0}^{n} (-1)^m (a_m I - b_m S) D^{n+k}.
\]

(7)

Similarly, we can show that \( BA = AB \). Putting \( m = j - k \) in (7), we have

\[
BA = \sum_{j=1}^{n} \sum_{j=1}^{n} (-1)^{j-k} (a_{j-k} I - b_{j-k} S) D^j.
\]

Now we write

\[
\lambda_j = \left\{ \begin{array}{ll}
\sum_{k=0}^{j} (-1)^{j-k} (a_{j-k} I - b_{j-k} S) (a_n^2 - b_n^2)^{-1} & \text{for } 0 < j < n,
\sum_{k=0}^{n-j} (-1)^{j-k} (a_{j-k} I - b_{j-k} S) (a_n^2 - b_n^2)^{-1} & \text{for } n < j < 2n
\end{array} \right.
\]

and

\[
R_A = (a_n^2 - b_n^2)^{-1} B.
\]

It is easy to check that

\[
AR_A = R_A A = \sum_{j=1}^{n} \lambda_j D^j
\]

and that \( AR \) contains only even powers of \( D \). Finally, we obtain

\[
AR_A = \sum_{j=1}^{n} \lambda_j D^j.
\]

Let now \( D_T \) denote the domain of the operator \( T \) and \( Z_T \) the kernel of \( T \):

\[
Z_T = \{(x \in D_T : Tx = 0)\}.
\]

**Theorem 2.** 1° \( Z_T \subset Z_{xT} \) and 2° \( Z_{xT} \subset Z_T \), where \( T = \sum_{j=1}^{n} \lambda_j D^j \).

Indeed, if \( x \in Z_T \), then \( Ax = 0 \) and

\[
\left\| \sum_{j=0}^{n} \lambda_j D^j \right\| x = E_A (Ax) = 0,
\]

hence \( x \in Z_T \). This implies that \( Z_A = Z_T \). The proof of 2° is analogous.

In the following we make use of assumption 3° (p. 69) in view of which the polynomial \( \sum_{j=1}^{n} \lambda_j D^j \), considered as a polynomial with respect
to the variable $D^j$, has only single roots. In [1] for $n = 1$ the roots are single because the corresponding polynomial is of the form $D^j - \lambda$. For $n > 2$ this polynomial may have multiple roots. Since we assume that the polynomial \( \sum_{j=0}^{n} \lambda_j D^j \) has single roots only, we can write that

\[
T = \prod_{j=1}^{n} (D^j - u_j I),
\]

where $u_j$ denotes the $j$-th root.

**Theorem 3.** We have

\[
Z_T = \{ z : z = \sum_{q=0}^{n} (a_q + S \zeta^q), \text{ for } z_q, \zeta^q \in Z_{D - \sqrt[n]{u_q}} \},
\]

where $T = \prod_{j=1}^{n} (D^j - u_j I)$.

**Proof.** Let us suppose that $z$ is of the form (10). Then

\[
| \prod_{j=1}^{n} (D^j - u_j I) | z = | \prod_{j=1}^{n} (D^j - u_j I) | \sum_{q=0}^{n} (a_q + S \zeta^q) = \sum_{q=0}^{n} \zeta^q = 0.
\]

Therefore $z \in Z_T$.

Conversely, let us suppose that $z \in Z_T$. We can decompose the space $Z_T$ into a direct sum,

\[
Z_T = \bigoplus_{q=0}^{n} [Z_{D - \sqrt[n]{u_q}} \oplus Z_{D - \sqrt[n]{u_q}}],
\]

because $D$ is an algebraic operator on the space $Z_T$ with single characteristic roots (cf. [2], p. 31–32). Hence

\[
z = \sum_{q=0}^{n} (a_q + S \zeta^q),
\]

where $z_q \in Z_{D - \sqrt[n]{u_q}}$ and $\zeta^q \in Z_{D - \sqrt[n]{u_q}}$ for $q = 1, 2, \ldots, n$.

We have to prove that $z_q^q = S \zeta_q^q$ for $q = 1, 2, \ldots, n$.

But $z_q^q \in Z_{D - \sqrt[n]{u_q}}$, hence $Dz_q^q = -V_u z_q^q$, and $V_u S \zeta_q^q = S(V_u \zeta_q^q) = -S z_q^q = Dz_q^q$. Therefore

\[
(D - \sqrt[n]{u_q} I) S \zeta_q^q = 0 \quad \text{and} \quad z_q^q = S \zeta_q^q \in Z_{D - \sqrt[n]{u_q}}
\]

for $q = 1, 2, \ldots, n$. But $z_q^q = \bar{S} z_q^q = S(z_q^q) = S \zeta_q^q$, which gives the required form of $z$.

**Theorem 4.** We have

\[
Z_T = \{ z : z = \sum_{q=0}^{n} \sum_{r=0}^{n} \left( \left[ \sum_{a_{01} - a_{11} + V_u a_{01} \theta_{01}^q} \right] I - \left[ \sum_{a_{11} + a_{21} + V_u a_{01} \theta_{01}^q} \right] S \theta_{01}^q \right) z_q \},
\]

where $\zeta_q \in Z_{D - \sqrt[n]{u_q}}$, $\theta_{01}^q$ being a scalar, $q = 1, 2, \ldots, n$.

**Proof.** Theorem 2 implies $Z_T = Z_T$. From Theorem 3 we infer that every $z \in Z_T$ is of the form $\sum_{q=0}^{n} (a_q + S \zeta_q)$, where $z_q \in Z_{D - \sqrt[n]{u_q}}$.

Similarly as in the proof of Theorem 2.4 in [1] we have for $q = 1, 2, \ldots, n$

\[
A z_q = \sum_{r=0}^{n} \left( \left[ \sum_{a_{01} - a_{11} + V_u a_{01} \theta_{01}^q} \right] I - \left[ \sum_{a_{11} + a_{21} + V_u a_{01} \theta_{01}^q} \right] S \theta_{01}^q \right) z_q
\]

But

\[
Dz_q = V_u z_q
\]

because $z_q \in Z_{D - \sqrt[n]{u_q}}$ and $S \zeta_q = z_q$ for $q = 1, 2, \ldots, n$. Hence

\[
Dz_q = V_u \zeta_q^q,
\]

\[
Dz_q = (-1)^q \zeta_q^q
\]

for $q = 1, 2, \ldots, n$. Then

\[
Dz_q = \zeta_q^q,
\]

\[
Dz_q = (-1)^q \zeta_q^q
\]

for $q = 1, 2, \ldots, n$. Thus

\[
A z_q = \sum_{r=0}^{n} \left( \left[ \sum_{a_{01} - a_{11} + V_u a_{01} \theta_{01}^q} \right] I - \left[ \sum_{a_{11} + a_{21} + V_u a_{01} \theta_{01}^q} \right] S \theta_{01}^q \right) z_q
\]

Similarly, we can show that

\[
A S z_q = \sum_{r=0}^{n} \left( \left[ \sum_{a_{01} - a_{11} + V_u a_{01} \theta_{01}^q} \right] I - \left[ \sum_{a_{11} + a_{21} + V_u a_{01} \theta_{01}^q} \right] S \theta_{01}^q \right) S z_q.
\]

Hence

\[
A z = \sum_{q=0}^{n} \sum_{r=0}^{n} \left( \left[ \sum_{a_{01} - a_{11} + V_u a_{01} \theta_{01}^q} \right] I - \left[ \sum_{a_{11} + a_{21} + V_u a_{01} \theta_{01}^q} \right] S \theta_{01}^q \right) z_q + \sum_{q=0}^{n} \sum_{r=0}^{n} \left( \left[ \sum_{a_{11} + a_{21} + V_u a_{01} \theta_{01}^q} \right] I - \left[ \sum_{a_{01} - a_{11} + V_u a_{01} \theta_{01}^q} \right] S \theta_{01}^q \right) S z_q + \sum_{q=0}^{n} \sum_{r=0}^{n} \left( \left[ \sum_{a_{01} - a_{11} + V_u a_{01} \theta_{01}^q} \right] I - \left[ \sum_{a_{11} + a_{21} + V_u a_{01} \theta_{01}^q} \right] S \theta_{01}^q \right) S z_q
\]

for $q = 1, 2, \ldots, n$. But $z_q = S z_q = z_q = z_q = \zeta_q$, which gives the required form of $z$.
(see the proof of Theorem 3), and \( z_q \in Z_{D, y} \), \( S_{y_q} \in Z_{D, y} \), and \( T = \prod_{q=1}^{n} (D^q - u_q I) \), where \( q = 1, 2, \ldots, n \). Thus the equality \( Ax = 0 \) holds if and only if

\[
\sum_{q=1}^{n} \sum_{q'=1}^{n} [(a_{q'q} + a_{qq}) V u_{q'}] u_q u_{q'} = 0, \tag{11}
\]

\[
\sum_{q=1}^{n} \sum_{q'=1}^{n} [(a_{q'q} - a_{qq}) V u_{q'}] u_q u_{q'} = 0.
\]

Acting with \( S \) on both sides of the second equation of (11) and applying the property \( S^2 = I \), we obtain the following system of equations:

\[
\sum_{q=1}^{n} \sum_{q'=1}^{n} [(a_{q'q} + a_{qq}) V u_{q'}] u_q u_{q'} + [(b_{q'q} - b_{qq}) V u_{q'}] u_q u_{q'} = 0, \tag{12}
\]

\[
\sum_{q=1}^{n} \sum_{q'=1}^{n} [(a_{q'q} - a_{qq}) V u_{q'}] u_q u_{q'} + [(b_{q'q} + b_{qq}) V u_{q'}] u_q u_{q'} = 0.
\]

From these equations it follows that \( z_q \) and \( z_q' \) are linearly dependent for \( q = 1, 2, \ldots, n \). Indeed, the space \( X \) is a direct sum, which implies that (13) holds if and only if

\[
\sum_{q=1}^{n} \sum_{q'=1}^{n} [(a_{q'q} + a_{qq}) V u_{q'}] u_q u_{q'} + \sum_{q=1}^{n} [(b_{q'q} - b_{qq}) V u_{q'}] u_q u_{q'} = 0, \tag{13}
\]

\[
\sum_{q=1}^{n} \sum_{q'=1}^{n} [(a_{q'q} - a_{qq}) V u_{q'}] u_q u_{q'} + \sum_{q=1}^{n} [(b_{q'q} + b_{qq}) V u_{q'}] u_q u_{q'} = 0.
\]

This shows the linear dependence of \( z_q \) and \( z_q' \).

We can show that the determinant of the system (13) is

\[
V = \sum_{q=1}^{n} \sum_{q'=1}^{n} (-1)^{n} (a_{q'q} - a_{qq}) V u_{q'} u_q = \sum_{q=1}^{n} \lambda_q u_q^2.
\]

Since \( u_q \) \((q = 1, 2, \ldots, n)\) are roots of the polynomial \( \sum_{q=1}^{n} \lambda_q D^q \) considered as a polynomial with respect to the variable \( D^q \), we have

\[
V = \sum_{q=1}^{n} \lambda_q u_q^2 = 0.
\]

It follows that (13) has non-zero solutions for \( z_q \) and \( z_q' \).

If we write

\[
\xi_q = \sum_{q=1}^{n} (b_{q'q} + b_{qq}) V u_{q'}, \quad \xi_q' = \sum_{q=1}^{n} (a_{q'q} - a_{qq}) V u_{q'},
\]

we obtain from the second equation of (13) that \( \xi_q z_q + \xi_q' z_q' = 0 \) for \( q = 1, 2, \ldots, n \). Hence

\[
z = \sum_{q=1}^{n} \sum_{q'=1}^{n} [(a_{q'q} - a_{qq}) V u_{q'}] I - [(b_{q'q} + b_{qq}) V u_{q'}] S z_q,
\]

which was to be proved.

**Theorem 5.** If \( \tilde{x} \) is a solution of the equation

\[
\prod_{q=1}^{n} (D^q - u_q I) x = y,
\]

then \( x = R_d \tilde{x} \) is a solution of the equation \( Ax = y \).

**Proof.** Let \( \tilde{x} \) satisfy equation (13). Then

\[
Ax = AR_d \tilde{x} = \prod_{q=1}^{n} (D^q - u_q I) \tilde{x} = y.
\]

Similarly, \( t = A \tilde{x} \) is a solution of the equation \( R_d t = y \).

Finally, we obtain the main theorem on the general form of the solution of the equations \( Ax = y \) and \( R_d t = y \):

**Theorem 6.** Let

\[
A = \sum_{q=1}^{n} (a_q + b_q S) D^q,
\]

where \( S \) is an involution acting in a linear space \( X \), let \( D \) be an operator transforming \( X \) into itself and anticommuting with \( S \) and \( L \), finally, \( a_q, b_q, \ldots, b_q \), be scalars. We assume that assumptions 1°-3° (p. 69) are satisfied.

If \( \tilde{x} \) is a solution of equation (13), then every solution of the equation

\[
x = R_d \tilde{x} + \sum_{q=1}^{n} \sum_{q'=1}^{n} [(a_{q'q} - a_{qq}) V u_{q'}] I - [(b_{q'q} + b_{qq}) V u_{q'}] S z_q,
\]

where \( a_q \in Z_{D, y} \), \( R_d = (a_q - b_q)^{-1} B \) and

\[
B = \sum_{q=1}^{n} (-1)^n a_{q'q} - b_{qq} S) D^q, \quad AR_d A = \prod_{q=1}^{n} (D^q - u_q I).
\]

Similarly, any solution of the equation \( R_d t = y \) is of the form

\[
t = Ax + \sum_{q=1}^{n} \sum_{q'=1}^{n} [(a_{q'q} + a_{qq}) V u_{q'}] I - [(b_{q'q} + b_{qq}) V u_{q'}] S z_q.
\]
On conditional bases in non-nuclear Fréchet spaces

by

W. WOJTYŃSKI (Warszawa)

In the present paper we give some criteria for the nuclearity of Fréchet spaces with bases. Our main result is the following:

A. Let $X$ be a Fréchet space with a basis. Then $X$ is nuclear if and only if every basis of $X$ is absolute (the basis $(e_n)$ is absolute if $\sum_n \|e_n e_n\| < \infty$ for each $x = \sum_n e_n e_n$ and each pseudonorm $\|\cdot\|$ on $X$).

For countably Hilbert spaces this result is strengthened as follows:

B. A Hilbertian Fréchet space $X$ with a basis is nuclear if and only if every basis $(e_n)$ of $X$ is unconditional (i.e. $\sum_n |\langle x, e_n \rangle| < \infty$ for each $x = \sum_n e_n e_n X$, and each linear functional $\langle x^*, e_n \rangle$).

Observe that the part “only if” of our results is a consequence of the Dynin–Mitjagin theorem [9] which asserts that in a nuclear space each basis is unconditional. We do not know whether the converse is true, however, we believe the following holds:

CONJECTURE (see [9]). A Fréchet space $X$ with a basis is nuclear provided each basis in $X$ is unconditional.

The conjecture is already established for Banach spaces, because the class of nuclear Banach spaces coincides with the class of finite-dimensional spaces, and, by result of Pełczyński and Singer [9], in every infinite-dimensional Banach space with a basis there exists a conditional basis.

Statement B can be regarded as a generalization of a result due to Babenko asserting that in a Hilbert space there exists a conditional basis; [1], cf. also [4], [5] and [7].

Statement A is a generalization of an unpublished result of professor J. Rutherford (presented on the conference on functional analysis in Sopot 1968) that a Fréchet space satisfying the assumption of A is a Schwartz space.