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Singular invariant measures on the line

by

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**INTRODUCTION**

In this paper we give a method of obtaining a large class of $\sigma$-finite measures on the real line which are invariant under translation by every real number. These measures are defined on sub-$\sigma$-rings of rings of Borel subsets of $\mathbb{R}$, the real line. Most of these measures are non-atomic and attach positive finite mass to some Lebesgue null set. The object of this paper is to study such measures with special reference to some problems in Harmonic Analysis. We feel that the study of such measures is not only interesting in itself but it has also strong bearing on some unresolved problems raised by Helson and Lowdenslager [7].

In Section 1 we recall the definition of sets of translation and obtain some of their properties. Large part of this section derives from Kahane and Salem [9], Chapter 1. In Section 2 we get a method of obtaining invariant measures as described above and obtain a result regarding the ergodicity of such measures. In Section 3 we define functions called cocycles and coboundaries introduced by Helson and Lowdenslager in [8] and give an example of a cocycle taking values $+1$ or $-1$ and which is not a coboundary. In Section 4 we derive certain elementary consequences of Beurling’s well known description of invariant subspaces of $H^2$ [1, 8] and in Section 5 we apply these results to obtain the multiplicity of the spectral measure associated with a unitary group of translations on the $L^2$ of an invariant measure. In Section 6 we give some results about “dual” measures. The results of Sections 5 and 6 are rather special but we feel that they are interesting.

Ergodicity plays a special role in our discussions. The work has connections with the previous work of de Leeuw and Glicksberg [2] on...

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(1) First such example was given by [3] for compact groups with ordered duals.
analytic measures on compact locally abelian groups and its generalization by Fournier [4] to arbitrary locally compact spaces where ergodicity was first explicitly considered in connection with analytic measures.

1. SETS OF TRANSLATION

In this section we shall describe the construction and properties of sets of translation (or translation sets). This description will follow Kakane and Salem [9] with only slight modifications which shall result in some technical advantage for our purpose.

1.1. Construction. Let \( n \) be an integer \( \geq 2 \) and let \( \eta_1, \eta_2, \ldots, \eta_n \) be \( n \) distinct numbers satisfying \( 0 = \eta_1 < \eta_2 < \ldots < \eta_n < 1 \). Let \( \xi > 0 \) satisfy \( \xi < \eta_1 - \eta_n, \xi < \eta_2 - \eta_n, \ldots, \xi < 1 - \eta_n \). Let \([a, b]\) denote the closed interval \( a \leq x \leq b \) of length \( \lambda \). By dissection of \([a, b]\) of type \((\eta_1, \eta_2, \ldots, \eta_n, \xi)\) we mean the subset of \([a, b]\) consisting of union of \( n \) intervals \([a+\lambda \eta_1, a+\lambda \eta_2+\xi], \ldots, [a+\lambda \eta_n+\xi, b]\), \( \xi = 1, 2, \ldots, n \). Each of these intervals is called a white interval and each component interval of the complement (with respect to \([a, b]\)) of this union is called a black interval. A dissection of type \((\eta_1, \eta_2, \ldots, \eta_n, \xi)\) is called equally spaced if \( \eta_1 = 0 \) and \( \eta_k = (k-1)\eta_n, \) \( k = 1, 2, 3, \ldots, n \).

Let \( \mathcal{E} \) denote the interval \( 0 < x < 1 \). Let \( \mathcal{E}_1 \) be the dissection of \( \mathcal{E} \) of type \((\eta_1, \eta_2, \ldots, \eta_n, \xi_1)\). Let \( \mathcal{E}_2 \subseteq \mathcal{E}_1 \) be obtained from \( \mathcal{E}_1 \) by performing on each white interval of \( \mathcal{E}_1 \) a dissection of type \((\eta_1, \eta_2, \ldots, \eta_n, \xi_2)\); thus \( \mathcal{E}_2 \subseteq \mathcal{E}_1 \) and consists of \( n \) white intervals. Generally, let \( \mathcal{E}_n \) denote the set obtained from \( \mathcal{E}_{n-1} \) by performing on each white subinterval of \( \mathcal{E}_{n-1} \), a dissection of type \((\eta_1, \eta_2, \ldots, \eta_n, \xi_n)\).

Definition 1.1. Let \( \mathcal{E} = \bigcup_{k=1}^{\infty} \mathcal{E}_k \), where \( \mathcal{E}_1 \supset \mathcal{E}_2 \supset \ldots \supset \mathcal{E}_k \) are sets as described above. Then \( \mathcal{E} \) is called a set of translation. A translation set \( \mathcal{E} \) is called equally spaced if each dissection \((\eta_1, \eta_2, \ldots, \eta_n, \xi_k)\) is equally spaced.

Each left end-point of the white interval in \( \mathcal{E}_k \) is of the type

\[
\eta(j_1, 1) + \xi \eta(j_2, 2) + + \xi \eta(j_n, k),
\]

where we have written \( \eta(j, x) \) for \( \eta_j + x \eta_n \) and where \( 1 \leq j_k \leq n \). These left end-points and group and semi-group generated by them will play a special role in our discussions. Since every point of \( \mathcal{E} \) is a limit point of each of these end-points, every \( x \in \mathcal{E} \) has a representation of the type

\[
x = \eta(j_1, 1) + \xi \eta(j_2, 2) + + \xi \eta(j_n, k) + \ldots
\]

1.2. Lebesgue function. Let \( \lambda \) denote the Lebesgue measure on \( \mathcal{E}_k \) normalised to \( \lambda = 1 \) and regard \( \lambda \) as defined on \( \mathcal{E} \) by making \( \mu_0 \) zero for sets outside \( \lambda \). Let \( \lambda_k \) denote the distribution function of \( \lambda_k \), i.e., \( \lambda_k(x) = \lambda((0, k)) \). It is easy to see that the value of \( \lambda_k \) on the left end-point \( (1, 1) \) is

\[
\lambda_k((0, k)) + \xi \lambda_k((0, k)) + \ldots + \xi \lambda_k((0, k)) = \lambda + \xi \lambda + \ldots + \xi \lambda^{k-1}
\]

\( \lambda_k \) increases linearly on each white interval of \( \mathcal{E}_k \) and \( \lambda_k \) is constant on each subinterval of the complement of \( \mathcal{E}_k \). The functions \( \lambda_k \) converge uniformly to a continuous function \( \lambda \) which has all its points of increase in \( \mathcal{E} \) and is constant on subintervals of complement of \( \mathcal{E} \).

For any \( x \in \mathcal{E} \) with representation \( x = \eta(j_1, 1) + \xi \eta(j_2, 2) + + \xi \eta(j_n, k) + \ldots \),

\[
L(x) = \frac{1}{\eta_1} + \frac{1}{\eta_2} + \ldots + \frac{1}{\eta_n} + \ldots
\]

For \( n = 1, 2, \ldots \) and call the new set \( \mathcal{E}_n \), then the restriction of the function \( L \) to \( \mathcal{E} \) is one-one and increasing on \( \mathcal{E} \). It maps \( \mathcal{E} \) onto \( 0 < x < 1 \). Henceforth we shall regard \( L \) as a function from \( \mathcal{E} \) onto \( I = (x: 0 < x < 1) \) and write \( \mathcal{E} \) for \( \mathcal{E} \).

1.3. Measure induced by \( L \). Let \( g \) denote the measure on \( \mathcal{E} \) induced by the function \( L \). If \( W = \mathcal{E} \) is the \( \lambda^{th} \) white interval of \( \mathcal{E} \) counting from left, then

\[
L(W \cap \mathcal{E}) = \frac{\lambda}{\eta_1 + \ldots + \eta_n} \quad \text{and} \quad g(W \cap \mathcal{E}) = \frac{1}{\eta_1 + \ldots + \eta_n}
\]

Since any function in \( L' \) or \( L'' \) of \( I \) with Lebesgue measure is approximable by linear combinations of indicator functions of the interval

\[
\left( \frac{1}{\eta_1 + \ldots + \eta_n}, \frac{1}{\eta_1 + \ldots + \eta_n} \right)
\]

we conclude that any function of \( L'(\mathcal{E}, g) \) or \( L''(\mathcal{E}, g) \) is approximable in the respective norms by linear combinations of the indicator functions of the sets of the type \( W \cap \mathcal{E} \), where \( W \) is a white interval in \( \mathcal{E} \) for some \( k \).

Another property of the measure \( g \) which we shall use is the following [19, p. 19]:

(*) If \( A \) is a measurable subset of \( \mathcal{E} \) and \( t \) is a real number such that \( A + t \subseteq \mathcal{E} \), then \( g(A) = g(A + t) \), i.e., measurable subsets of \( \mathcal{E} \) congruent by translation have the same measure \( g \).
Let $Q$ be the group of real numbers generated by the left endpoints of the intervals in $E_0$, $k = 1, 2, \ldots$ Let $K$ be the group of real numbers in $0 \leq x < 1$ of type

$$k = \frac{j_1 - 1}{v_1} + \frac{j_2 - 1}{v_2} + \ldots + \frac{j_n - 1}{v_n}, \quad 0 < j_n \leq v_n,$$

where the addition in $K$ is defined modulo 1. It is clear that $K$ is a dense subgroup of the group $Q$ of real numbers (modulo 1) and hence any measurable subset of $Q$ invariant under translation by $K$ has Haar measure zero or one $(?)$. The next lemma transfers this fact to $E$.

**Lemma 1.1.** Let $A \subset E$ be a measurable set such that $(A \pm q) \cap E \subset A$ for every $q \in Q$. Then either $g(A) = 0$ or $g(E - A) = 0$.

**Proof.** It is enough to show that $L(A)$ is invariant under translation by $K$. Now let $y \in L(A)$ and let it have the representation

$$y = \sum_{i=1}^{m} \frac{j_i - 1}{v_i} e_i, \quad 1 \leq j_i \leq v_i.$$

Let

$$x = \sum_{i=1}^{m} \frac{p_i - 1}{v_i} e_i, \quad 1 \leq p_i \leq v_i.$$

We shall show that $y + x \in L(A)$. Now $y + x$ has a representation of type

$$y + x = \sum_{i=1}^{m} \frac{q_i - 1}{v_i} + \sum_{i=1}^{m} \frac{j_i - 1}{v_i},$$

so that the terms in the representation of $y + x$ and $y$ agree from $(n + 1)$th term on. Hence we have

$$L^{-1}(y + x) = \sum_{i=1}^{m} \xi_i \xi_l q_i (j_0, l) + \sum_{i=1}^{m} \xi_i \xi_l j_0 (j_0, l),$$

$$L^{-1}(y) = \sum_{i=1}^{m} \xi_i \xi_l q_i (j_0, l) + \sum_{i=1}^{m} \xi_i \xi_l j_0 (j_0, l).$$

Consequently, by (1.3),

$$L^{-1}(y + x) = L^{-1}(y) = \left( \sum_{i=1}^{m} \xi_i \xi_l q_i (j_0, l) - \sum_{i=1}^{m} \xi_i \xi_l j_0 (j_0, l) \right) \in Q.$$

Let $g$ denote this element in $Q$; then $L^{-1}(y + x) = L^{-1}(y) + g$ belongs to $(A + g) \cap E \subset A$. Hence $L^{-1}(y + x) \subset A$, so that $y + x \in L(A)$. Thus $L(A)$ is invariant under $K$, q.e.d.

**Remark.** Lemma 1.1 permits us to classify the sets of translation in the following manner. Let $E_1$ and $E_2$ be two sets of translation with $g_1$ and $g_2$ the associated measures supported on $E_1$ and $E_2$ respectively. Then either $g_1(E_1 \cap (E_1 + t)) = 0$ for all $t$ or there exist $t_0, t_1, \ldots$ such that $g_1(E_1 - \bigcup_{n=1}^{\infty} (E_1 + t_n)) = 0$.

**2. INVARIANT MEASURES**

A $\omega$-finite measure $\nu$ defined on the Borel $\sigma$-field $\mathcal{B}$ of $B$ is called locally invariant if there exists a support $B$ of $\nu$ such that any two measurable subsets of $B$ congruent by translation have the same measure $\nu$. $B$ is then called an admissible support of $\nu$. If $\nu$ is locally invariant with admissible support $B$, then $\nu$ is locally invariant with admissible support $B + t$, where $\nu$ is defined by $\nu(A) = \nu(A - t), \ A \in \mathcal{B}$. Further $\nu$ and $\nu_1$ agree on Borel subsets of $B \cap (B + t)$. Let $G$ be a subgroup of $B$. For each $t \in G$, let $S_t$ be the $\sigma$-ring of Borel measurable subsets of $B - t$ and let $B$ be the $\sigma$-ring generated by $\bigcup_{t \in G} S_t$. $S$ consists of measurable subsets of $B$ which can be covered by countably many translates of $B$ by members of $G$.

**Theorem 2.1.** Let $\nu$, $B$ and $S$ be as above. Then there exists a unique measure $\mu$ on the $\sigma$-ring $\mathcal{B}$ such that

(i) $\mu(A) = \mu(A + t)$ for all $A \in \mathcal{B}$ and $t \in G$;

(ii) restriction $\mu|_S$ of $\mu$ to $S$ is $\nu$.

**Proof.** Let $A = \bigcup_{t \in G} A_t \subset A = \emptyset$ if $t \neq 0$ and $A_t$ is a Borel subset of $B - t$ for some $t$. Define $\mu$ for this $A$ as

$$\mu(A) = \sum_{t \in G} \nu(A_t).$$

Since $\nu$ and $\nu_1$ agree on Borel subsets of $(B + t) \cap B$, $\mu$ is unambiguously defined. It is easy to prove that $\mu$ is invariant under $G$ and $\mu|_S = \nu$ and that $\mu$ is unique, q.e.d.

Now if in Theorem 2.1 we take $G = B$, we get a measure invariant under translation by every real number. We can take $\mu$ to be the measure $g$ associated with a translation set $E$, $E$ being an admissible support of $\nu$. We can also take $\mu$ to be any non-atomic finite measure supported on an independent set $(?) E$ and $E$ then becomes an admissible support of $\mu$.

$(?)$ Observe that an independent set and its non-zero translate can intersect in at most one point.
We thus see that there are many measures on $R$, other than the Lebesgue measure and the cardinality measure which are invariant under translation. A $\sigma$-finite measure space $(X, S, \mu)$ is called totally $\sigma$-finite if there exists an $A \in S$ such that $\mu(B) = 0$ for every set $B \in S$ disjoint from $A$.

2.3. Ergodicity. Henceforth we shall deal with totally $\sigma$-finite measures on Borel subsets of $E$ which are invariant under a countable dense subgroup of $E$. A totally $\sigma$-finite measure $\mu$ defined on a sub-$\sigma$-ring of $\mathcal{B}$ will be regarded as a measure on $\mathcal{B}$ simply by setting $\mu(A) = 0$ for those subsets of $\mathcal{B}$ which do not intersect a measurable support of $\mu$.

**Definition 2.1.** Let $\mu$ be a $\sigma$-finite measure on $\mathcal{B}$ which is invariant under translation by a countable dense subgroup $Q$. We say that $\mu$ is **ergodic with respect to $Q$** if for every measurable set $A$ such that $A + g = A$ for all $g \in Q$, either $\mu(A) = 0$ or $\mu(E - A) = 0$.

Now let $\mathcal{B} = E$ be a set of translation and $Q$ the group generated by the left end-points as in Lemma 1.1. Let $g$ be the locally invariant measure on $E$ given by the Lebesgue function on $E$. Let $\mu$ be the measure obtained by setting in Theorem 3.1 $\nu = g$, $B = E$ and $G = Q$.

**Theorem 2.2.** $\mu$ is ergodic under $Q$.

Proof. Let $A$ be a measurable set which is invariant under $Q$, i.e., $A + g = A$ for all $g \in Q$. Let $A = E \cap A$. Then $(A + g) \cap E = A$ for all $g \in Q$. Hence by Lemma 1.1 either $\nu(A) = 0$ or $\nu(E - A) = 0$. Now $\mu$ is invariant under $Q$ and supported on $\bigcup_{g \in Q} (E + g)$, with $\nu$ the restriction of $\mu$ to the Borel subsets of $E$. It follows that either $\mu(A) = 0$ or $\mu(E - A) = 0$. This proves the ergodicity of $\mu$, q.e.d.

On the other hand, if a non-atomic $\sigma$-finite measure $\mu$ is invariant under a subgroup $Q$ and attaches positive mass to a perfect independent set $E$, then $\mu$ can never be ergodic under any subgroup. Thus in terms of their ergodic behaviour perfect independent sets are quite opposite of sets of translation.

3. COCYCLES AND COBOUNDARIES

3.1. Let $\mu$ be a measure defined on $\mathcal{B}$ which is invariant under a countable semi-group $P$ of $E$.

**Definition 3.1.** A non-vanishing complex-valued function $A$ on $P \times E$ is called a cocycle if $\Phi(x, y)$ is $\mathcal{B}$ measurable in $x$ for every $q$ and $A$ satisfies, for all $q_1$ and $q_2$,

$$\Phi(q_1 + q_2, x) = \Phi(q_1, x) \Phi(q_2, x + q_1) \quad \text{a.e.} \ [\mu].$$

(3.1)

It is called a unitary cocycle if $|\Phi(q, x)| = 1$ a.e. $[\mu]$ for all $q \in P$.

Let $B$ be a non-vanishing $\mathcal{B}$-measurable function and write

$$\Phi(q, x) = \frac{B(q + x)}{B(x)}.

Then $A$ is easily verified to be a cocycle. A cocycle of this type is called a coboundary. A cocycle defined on $P \times E$ can be uniquely extended to $Q \times E$, where $Q$ is the group generated by $P$, so that (3.1) holds on $Q \times E$. For this we need only write $\Phi(-p, x) = \{\Phi(p, x + p)\}^\ast$. In view of this we shall assume henceforth that a cocycle is defined with respect to $Q$.

3.2. We now give an example of a cocycle which is not a coboundary. It will also serve to show that some results of later sections are not vacuous. First example of a cocycle which is not a coboundary was constructed by Helson and Lowdenslager [8] in connection with invariant subspaces in the space of square integrable functions on the Bohr group, where the terminology of cocycles and coboundaries was introduced. Subsequent examples and new results were given by Gamelin [5]. Our example differs from those of above authors in that we are dealing on the real line rather than finite- or infinite-dimensional tori.

**Example.** Let $G$ be a dense subgroup of $E$ and let $G_n$ be the group $G \cap a_n$ where $a_n$ is a cyclic group. Let $\lambda_n > 0$ be the generator of $G_n$ and let $\lambda_n/a_n + 1 = a_0$. Define

$$B_n(x) = (\lambda_n)^x, \quad k_{a_n} \leq x < (k + 1)_{a_n}, \quad 0 < k < \infty,$$

$$B_1(x) = (\lambda_n)^x, \quad k_{a_n} \leq x < (k + 1)_{a_n}, \quad 0, \ldots, a_n, a_{n - 1},$$

and extend $B_1$ outside $[0, \lambda_n)$ by making it periodic with period $\lambda_n$. Generally, define

$$B_n(x) = (\lambda_n)^x, \quad k_{a_n} \leq x < (k + 1)_{a_n}, \quad 0 < k < a_n - 1,$$

and extend $B_n$ outside $[0, \lambda_n - 1]$. By making it periodic with period $\lambda_n - 1$. Let $\Phi_n = B_1 \cdots B_n$ and define $A(q, x) = \Phi_n(q + x)\Phi_n(x)g(x)$. Suppose $m > n$; then

$$\Phi_m(q + x)\Phi_m^{-1}(x) = \frac{\prod_{i=1}^n B_i(x + g)}{\prod_{i=1}^m B_i(x + g)}.$$

Now $B_{n+1}, \ldots, B_n$ are periodic with period $\lambda_n$ and so if $g \in G_n$, then $\Phi_m(q + x)\Phi_n(x) = \Phi_n(x + g)/\Phi_n(x)$ which shows that $A$ is unambiguously defined.
Singular invariant measures

4. SOME CONSEQUENCES OF BEURLING'S DISCRIPION
OF INVARIANT SUBSPACES OF $H^2$

4.1. $L_a(R^+, \mathcal{B}_1, \mu)$ and shifts on it. Let $R^+$ denote the set of non-negative real numbers. Let $0 < \lambda < \infty$, and let $\mathcal{B}_1$ denote the $\sigma$-ring of sets generated by the intervals $n\lambda \leq x < (n + 1)\lambda$, $n = 0, 1, 2, \ldots$

Let $\mu$ be a measure defined on $\mathcal{B}_1$ which is invariant under translation by $\lambda$. We shall assume that $0 < \mu(I_1) < \infty$, where $I_1$ is the interval $0 \leq x < \lambda$. It is easy to see that $L_a(R^+, \mathcal{B}_1, \mu)$ is not much different from $l_1$, the space of square summable sequences. Every function in $L_a(R^+, \mathcal{B}_1, \mu)$ is constant on $n\lambda < x < (n + 1)\lambda$ and if $C_n$ be this value, then

$$\sum_{n=1}^{\infty} |C_n|^2 < \infty.$$

By shift on $L_a(R^+, \mathcal{B}_1, \mu)$ we mean the operator $S$ defined by

$$(Sf)(x) = \begin{cases} 0 & \text{if } 0 \leq x < \lambda, \\ f(x-\lambda) & \text{if } x \geq \lambda. \end{cases}$$

$S$ is an isometry from $L_a(R^+, \mathcal{B}_1, \mu)$ into $L_a(R^+, \mathcal{B}_1, \mu)$. Let $a$ be a function which is $\mathcal{B}_1$-measurable and of absolute value 1. Let $U$ be the unitary operator $Uf = af$, where $f \in L_a(R^+, \mathcal{B}_1, \mu)$. Write $T$ for the isometric operator $U^{-1}SU$. Now since $U$ is unitary, the subspace spanned by $f, Uf, U^2f, \ldots$ is whole of $L_a(R^+, \mathcal{B}_1, \mu)$ if and only if the subspace spanned by $Uf, SUf, S^2Uf, \ldots$ is the whole of $L_a(R^+, \mathcal{B}_1, \mu)$. Let $a_n$ and $f_n$ stand for values of $a$ and $f$ on the interval $n\lambda < x < (n + 1)\lambda$.

The next theorem will be useful in Section 5:

**Theorem 4.1.** (i) $Uf, SUf, S^2Uf, \ldots$ spans $L_a(R^+, \mathcal{B}_1, \mu)$ if and only if \( \sum_{n=1}^{\infty} a_n f_n x \) is an outer function in $H^1$.

(ii) Suppose $f_i = 0$ for $i > n$. Then $Uf_i, S^2Uf_i, \ldots$ spans $L_a(R^+, \mathcal{B}_1, \mu)$ if and only if \( \sum_{i=1}^{n} a_i f_i x \) has no zeros in $|x| < 1$.

(iii) Suppose that $a_i f_i = (-1)^i$ for $i = 0, 1, 2, \ldots$ and $f_i = 0$ otherwise. Then $Uf_i, SUf_i, S^2Uf_i, \ldots$ spans $L_a(R^+, \mathcal{B}_1, \mu)$.

**Proof.** (i) follows from Beurling's well-known discription of invariant subspaces of $H^2$.

(ii) follows because a polynomial in $x$ is outer if and only if it has no zeros inside the unit disc.

(iii) follows from (ii) because \( \sum_{i=1}^{n} (-1)^i f_i x \) has no zeros in $|x| < 1$. 
5. UNITARY GROUP OF TRANSLATIONS

5.1. Unitary group of translations (*)

Let \( E = \bigcup_{k=0}^{\infty} E_k \) be an equally spaced set of translation, where \( E_k \) is a dissection of \([0, 1]\) of the type \([v_0 + k, v_0 + k, \ldots, v_n + k]\) and \( E_0 \) is obtained from \( E_{-1} \) by performing on each subinterval of \( E_{-1} \) a dissection of the type \([v_0 + 1, v_0 + 1, \ldots, v_n + 1]_k\). Let \( g \) be the locally invariant measure on \( E \) induced by the Lebesgue function \( L \), \( \eta \) being an admissible support of \( g \). Let \( Q \) denote the group generated by the left end-points of subintervals of \( E_0 \), \( k = 1, 2, \ldots, i.e., by the real numbers of the type \( \lambda v_1 \xi_1 + \lambda v_2 \xi_2 + \ldots + \lambda v_{n-1} \xi_{n-1} + \lambda v_n \xi_n, \quad \lambda \leq \lambda \leq \eta \).

\( Q^+ \) shall denote the semi-group of non-negative real numbers of \( Q \). We shall denote by \( W_0 \) the \( L^2 \) subinterval of \( E_0 \) counting from left. Let \( \mu \) be the measure obtained as in theorem 2.1 by taking \( G = Q, r = g \) and \( B = E \). Let \( A \) be a unitary cocycle on \( Q \times R \) satisfying

\[
A(q, x + q) = A(q, x) A(q, x + q)
\]

n.a. \([\mu]\). Consider the group of unitary operators on \( L_q(R, \mathcal{A}, \mu) \) defined by

\[
(U_qf)(x) = A(q, x) f(x + q), \quad q \in Q.
\]

It can be verified using the functional equation of the cocycle that \( U_q \) is a group of unitary operators.

**Theorem 5.1.** Assume that

(i) for all non-negative integers \( n \) and \( k, A(w \xi z \xi_2 \ldots \xi_{n-1} \xi_n, w) \) is constant on \( W_k \cap E \) and let \( c_k \) be this constant value;

(ii) \( \sum_{2^k} c_k^2 \eta^2 \) has no zeros in \([x] < 1 \) for all \( k \).

Then

(a) \( U_q L_q, q \in Q^+ \), spans the closed subspace of functions in \( L_q(R, \mathcal{A}, \mu) \) which vanish for negative real numbers;

(b) \( U_q L_q, q \in Q \), spans \( L_q(R, \mathcal{A}, \mu) \). Here \( L_q \) denotes the indicator function of \( E \).

**Proof.** Let \( \mathcal{A} \) be the \( \sigma \)-algebra generated by \( W_k \cap E + k \eta \) on \( 0 \leq x < \infty \). Since for all \( n, A(w \eta, x) \) is constant on \( W_k \cap E \),

\[
A(w \eta, x) = \frac{B_k(x + w \eta)}{B_k(x)}
\]

for function \( B_k \) defined by \( B_k(x + w \eta) = A(w \eta, x), 0 \leq x < \eta \). We see now that in \( L_q(R^+; \mathcal{A}, \mu) \), \( U_{\eta w}, n = 0, 1, 2, \ldots \), is \( B_k^* \) \( B_k \) for \( B_k \) also stands for the operator consisting of multiplication by \( B_k \). Now \( B(x + w \eta) = A(w \eta, x) \) for \( 0 \leq x < \eta \). Because of (ii) therefore

\[
\sum_{2^k} c_k^2 \eta^2 = \sum_{2^k} B(k \eta, x)
\]

has no zeros in \([x] < 1 \). By theorem 4.1 (b) it follows that \( U_{\eta w} L_q, k \geq 0 \), spans \( L_q(R^+; \mathcal{A}, \mu) \) and hence the indicator functions of \( W_k \cap E + k \eta \), \( k \geq 0 \), all belong to the span of \( U_{\eta w} L_q \). Again let \( \mathcal{A} \) be the \( \sigma \)-algebra generated by \( W_k \cap E + k \eta \xi_k, k \geq 0 \). Since \( \sum_{2^k} c_k^2 \eta^2 \) has no zeros in \([x] < 1 \), by argument same as before we conclude that \( U_{\eta w} L_q \), \( k \geq 0 \), spans \( L_q(R^+; \mathcal{A}, \mu) \) and hence the indicator functions of \( W_k \cap E + k \eta \xi_k, k \geq 0 \), all belong to the span of \( U_{\eta w} L_q \). If we note that \( A(k \eta, x) \) is constant on each \( W_k \cap E, k \geq 0 \), \( \xi_k \) it can be seen that \( U_{\eta w} L_q \), \( k \geq 0 \), has a span which contains the indicator functions of each of the sets \( W_k \cap E + k \eta \xi_k, k \geq 0 \). Proceeding thus we see that the closed linear subspace spanned by \( U_{\eta w} L_q \), \( q \in Q \), contains the indicator functions of each of the sets

\[
W_k \cap E + k \eta \xi_k, k \geq 0 \quad \xi_k \geq 0
\]

But every function in \( L_q(R^+; \mathcal{A}, \mu) \) is approximable by linear combinations of indicator functions of such sets. Consequently, \( U_{\eta w} L_q \), \( q \in Q^+ \), spans the subspace of functions in \( L_q(R^+; \mathcal{A}, \mu) \) which vanish for negative real numbers.

(b) this is now obvious, q.e.d.

**Corollary.** Translates of \( 1_q \) by members of \( Q^+ \) span \( L_q(R^+; \mathcal{A}, \mu) \).

**Proof.** In this case each \( c_k^2 \) is 1, so that \( \sum_{2^k} c_k^2 \eta^2 \) has no zero in \([x] < 1 \).

Hence theorem 5.1 is applicable, q.e.d.

The cocycle constructed in 5.2 together with theorem 4.1 (iii) can be used to show that our theorem is non-vacuous in a non-trivial way, i.e., \( A \) really chosen to be a cocycle which is not a coboundary and for which the hypothesis of theorem 5.1 are satisfied. We leave the details of this verification to the reader.

Remark. In contrast with theorem 5.1 there exist measures \( \mu \) invariant under a group \( Q \) such that for every group \( U_q \), \( q \in Q \), of unitary operators on \( L_q(R, \mathcal{A}, \mu) \) of type \( (5.1) \) no function \( f \) exists in \( L_q(R, \mathcal{A}, \mu) \) so that \( U_q f, q \in Q \), spans \( L_q(R, \mathcal{A}, \mu) \) [3].

(*) Actually our unitary groups are translations times a cocycle as given by formula (5.1).
6. DUAL MEASURES AND THEIR PROPERTIES

6.1. Dual measures. Let $\mu$ be a $\sigma$-finite measure on $\mathcal{A}$ which is invariant under a countable dense subgroup $G$ and let $A(\cdot, \cdot)$ be a unitary cocycle on $Q \times R$, which satisfies functional equation (3.1) a.e. with respect to $\mu$. Let $U_{t}$ $(q \in G)$ and $V_{t}$ $(t \in R)$ be groups of unitary operators on $L_{2}(E, \mathcal{F}, \mu)$ defined by

$$U_{t}f(q \cdot x, \cdot) = A(q, x)f(x),$$

$$V_{t}f(q \cdot x, \cdot) = e^{it}f(q \cdot x).$$

It is easy to check that $U_{t}$ and $V_{t}$ together satisfy the following important equation:

$$(6.1) \quad V_{t}U_{t} = e^{it}U_{t}V_{t}.$$  

Now consider $\mathcal{K}$ as an abelian group with discrete topology and let $B$ denote its compact dual. There is a continuous isomorphism $\Phi$ of the real line with usual topology into $B$ given by $(\Phi(f), q) = \Phi(f)(q) = e^{iq}$. It can further be shown that $\Phi(E)$ is dense in $B$. Now by Stone's theorem for groups we can write

$$U_{q} = \int_{\mathcal{K}} q \cdot \beta(d\nu),$$

where $\beta$ is a spectral measure on the Borel subsets of $B$, whose values are projections in $L_{1}(E, \mathcal{F}, \mu)$. It follows as a consequence of (6.1) that

$$(6.2) \quad V_{t}\beta(A) V_{t}^{-1} = \beta(A + \Phi(t)),$$

where $A$ is a measurable subset of $B$. See [9].

We assume now that $L_{2}(E, \mathcal{F}, \mu)$ has a single generator $f$ with respect to $U_{q}$, $q \in Q$. Then $\beta$ is of multiplicity 1 and $T_{t}U_{q}f(q \cdot \cdot)x = A(q, \cdot)\cdot \cdot f(x)$ is an invertible isometry from $L_{2}(E, \mathcal{F}, \mu)$ into $L_{2}(E, \mu)$, where $v$ is the measure $v(A) = \beta(A)f$. Because of (6.2) the measure $v$ is quasi-invariant under $\Phi(E)$, i.e., $v(A) = 0$ if and only if $v(A + \Phi(t)) = 0$, $t \in R$. Further one can prove

**Theorem 6.1.** If $\mu$ is ergodic under $Q$, then $v$ is ergodic under $\Phi(E)$.

**Remark.** Our method shows a way of getting on the dual $B$ of a subgroup of $E$ a measure which is quasi-invariant and ergodic under $\Phi(E)$ but which is neither equivalent to the Haar measure on $B$ nor equivalent to the linear measure on a coset of $\Phi(E)$. Here equivalent measures means measures mutually absolutely continuous.

References


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*Singular invariant measures 13*