On convergence of sequences of periodic distributions

by

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1. In this paper we are concerned with distributions in the $q$-dimensional Euclidean space $\mathbb{R}^q$. The points of $\mathbb{R}^q$ are denoted by $x = (x_1, \ldots, x_q)$. If nothing else is said explicitly, the values of distributions are in a fixed Banach space.

A distribution $f$ in $\mathbb{R}^q$ will be called periodic, iff (iff = if and only if) $f(x+p) = f(x)$ for every integral point $p \in \mathbb{R}^q$, i.e., for every point $p = (\pi_1, \ldots, \pi_q)$ whose the coordinates $\pi_i$ are integers. In the space of periodic distributions, we consider six kinds of convergence:

I. Tempered strong convergence.
II. Strong convergence.
III. Weak convergence.
IV. Tempered weak convergence.
V. Weak convergence in Fourier coefficients.
VI. Strong convergence in Fourier coefficients.

We prove cyclically the implications

$$
\begin{array}{c}
\text{II} \rightarrow \text{III} \\
\text{I} \quad \text{IV} \\
\text{VI} \leftarrow \text{V}
\end{array}
$$

so that it is seen that all the kinds of convergence are equivalent. In fact, the diagram of implications, given in section 10, is some more complicated, because it is needed to consider every kind of convergence in two different ways: as a convergence to a limit and as a convergence which says nothing about the limit. So we have to consider, in the whole, 12 kinds of convergence.

All proofs in this paper have a classical character, i.e., do not use any tools from topology or functional analysis.

We shall first establish, in section 2, the notation and recollect some more or less known facts about periodic distributions.
2. It is known that if $f$ and $g$ are distributions in $\mathcal{R}'$, one of them being of bounded carrier and one of them (the same or the other one) real-valued, then the convolution $f \ast g$ does exist (see, e.g., [1]). Using the general notation $g_0(x) = y(x + l)$, we have for every $\lambda \in \mathcal{R}'$

$$f \ast (g \circ \lambda) = f \ast g \circ \lambda$$

for every $\lambda \in \mathcal{R}$.

On the other hand, if $f$ is any distribution and $g$ a smooth function, we have for the ordinary product,

$$(g \circ \lambda) \ast 0 = f \ast (g \circ \lambda) \ast 0.$$

In the sequel, we shall use the following notation:

$$f \ast (gh) = f \ast (gh) \quad \text{and} \quad f \ast h = (f \circ \lambda) \ast h.$$

This means that if we have to perform the multiplication and the convolution, we first perform the multiplication.

If $x = (\xi_1, \ldots, \xi_d) \in \mathcal{R}'$ and $\lambda \in \mathcal{R}$, then by

$$x < l, \quad \lambda < x, \quad x < l, \quad \lambda < l,$$

we understand, respectively,

$$\xi_j < l, \quad \lambda < \xi_j, \quad \lambda < \xi_j, \quad \lambda < \xi_j,$$

where $j = 1, \ldots, g$. Thus, if $x \in \mathcal{R}'$, the interval $0 < x < 1$ denotes the set of points $x$ with $0 < \xi_j < 1, j = 1, \ldots, g$.

Let $n$ denote the characteristic function of the interval $0 < x < 1$.

Then the convolution $f \ast n$ exists for every distribution $f$ in $\mathcal{R}'$. In particular

$$1 \ast n = 1.$$

Let $P$ denote the set of all integral points of $\mathcal{R}'$. Then evidently

$$\sum_{x \in P} n_0(x) = 1,$$

where $n_0(x) = n(x + p)$.

Let $g$ be a real-valued smooth function of bounded carrier such that

$$1 \ast l = 1 \quad \text{and} \quad n \ast l = 1 \ast l = 1, \quad x \in P,$$

By (1), it is easy to verify that

$$\sum_{x \in P} n_0(x) = 1.$$

A distribution $f$ is periodic, iff

$$f_0(x) = f$$

for $p \in P$.

Sequences of periodic distributions

For periodic distributions $f$ we have

$$f \ast 0 = (f \ast 0) \ast 0 = (f \ast 0) \ast (f \ast 0) \ast 0 = f \ast (f \ast 0) \ast 0.$$

Hence, by (2) and (1),

$$f \ast n = \sum_{x \in P} (f \ast 0) \ast n = f \ast \frac{1}{\lambda} \sum_{x \in P} n_0(x) \ast n = f \ast 1,$$

thus

$$f \ast n = f \ast 1.$$

Given any $\lambda \in \mathcal{R}'$, we have

$$(f \ast 1) \ast 0 = f \ast 1 \ast 0 = f \ast 1.$$

Thus we see that the convolution $f \ast n$, where the distribution $f$ is periodic, is a constant function.

If $E_p(x) = e^{2\pi i p \cdot x}$, then the products $fE_p$ are also periodic distributions for $f$ periodic. The elements

$$c_p = fE_p \ast n \ast 1$$

are called Fourier coefficients of the periodic distribution $f$.

We shall prove generally that, if $g$ is any distribution and $\varphi$ is a real-valued smooth function of bounded carrier, then for every order $k$,

$$\hat{g}^{(k)} \ast 1 = (-1)^k \hat{g}^{(k)} \ast 1,$$

where $(-1)^k = (-1)^{k_1 + \cdots + k_g}$. In fact, this formula holds, in particular, if $g$ is a smooth function. If $g$ is an arbitrary distribution and $g_n$ is a sequence of smooth functions, convergent to $g$, then the sequence $\hat{g}_n^{(k)} \ast 1$ converges to $\hat{g}^{(k)} \ast 1$ and the sequence $g_n \hat{g}^{(k)} \ast 1$ converges to $g \hat{g}^{(k)} \ast 1$ (see e.g., [2], Theorem 5).

This implies (4).

Applying formula (4), we can easily estimate the coefficients $c_p$.

In fact, we may write

$$c_p = fE_{-p} \ast 1.$$

Let $I$ be the carrier of $x$. There is an order $k$ and a continuous function $F$ such that $F^{(k)} = f$ in $I$. Thus

$$c_p = fE_p \ast 1 = (-1)^k F(E_{-p} \ast 0) \ast 1 = (-1)^p \frac{1}{\lambda} \sum_{x \in P} n_0(x) \ast n \ast 1.$$

But

$$E_{-p} \ast 0 = \sum_{n \in \mathbb{Z}^d} \left( \frac{1}{\lambda} \sum_{x \in P} n_0(x) \ast n \ast 0 \right) \ast (m - l).$$

If $m = (\mu_1, \ldots, \mu_d)$ and $p = (a_1, \ldots, a_d)$, then generally by $a^m$ we understand the product $a^{\mu_1} a^{\mu_2}$. If in this product there occurs any-
where $\theta_i$, we read it as 1. Using this general notation, we can easily verify that $E_{-p}^{(\omega)} = (1 - 2\pi\frac{\omega}{p})^k E_{-p}$. Hence we obtain

$$|c_p| \leq M \sum_{\nu \in \mathbb{Z}^k} \left(2\pi\frac{\nu}{p}\right)^n M \left(1 + 2\pi\frac{\nu}{p}\right)^k,$$

where $\nu = (\nu_1, \ldots, \nu_k)$ for $p = (\nu_1, \ldots, \nu_k)$ and $1 + 2\pi\frac{\nu}{p}$ denotes the point of $\mathbb{R}^k$ whose $i$th coordinate is $1 + 2\pi\nu_i$; $i = 1, \ldots, k$, whereas the constant $M$ in (5) is equal to the product of the maximum of $|\nu|$ on $I$, of the maximum of $|\nu^n|$ on $I$ for all $0 \leq n \leq k$, and of the volume of $I$.

Let $\varphi$ be a real-valued smooth function of bounded carrier. If $f$ is any distribution, then

$$(\varphi * f) E_p = \varphi E_p * f E_p.$$

In fact, this equality holds for smooth functions $f$. If $f$ is an arbitrary distribution, then there is a sequence of smooth functions $f_n$ convergent to $f$. Since $(\varphi * f_n) E_p = \varphi E_p * f_n E_p$, we obtain (6) on letting $n \to \infty$ (see [1], Theorem 5).

If $f$ is a periodic distribution, then $(\varphi * f) \rho_i = \varphi * f \rho_i = \varphi \ast f$, thus $g = \varphi \ast f$ is a periodic smooth function. Applying (6), we can calculate the Fourier coefficients of $g$, namely:

$$g E_{-p} \ast \nu = (\varphi \ast f) E_{-p} \ast \nu = (\varphi E_{-p} * f E_{-p}) \ast \nu = \varphi E_{-p} * (f E_{-p} \ast \nu) = \varphi E_{-p} \ast \nu = f E_{-p} \ast \nu.$$

Letting $a_p = f E_{-p} \ast \nu$, we see that the Fourier coefficients of $\varphi \ast f$ are $a_p \varphi_p$.

It follows that if all Fourier coefficients $\varphi_p$ of $f$ vanish, then so do the Fourier coefficients of $\varphi \ast f$. This implies $\varphi \ast f = 0$, because $\varphi \ast f$ is a smooth function. If $\delta_0$ is a delta sequence of smooth functions, then $\delta_0 \ast f = f$. Since $\delta_0 \ast f = 0$, this implies $f = 0$. Thus we have proved that if all the Fourier coefficients of a periodic distribution vanish, then $f = 0$.

This implies that periodic distributions are defined uniquely by their Fourier coefficients. In fact, if $f E_{-p} \ast \nu = g E_{-p} \ast \nu$ for $p \neq 0$, then $(f - g) E_{-p} \ast \nu = 0$. Thus the Fourier coefficients of $f - g$ vanish. Hence $f - g = 0$.

Let, as before, $c_p$ be Fourier coefficients of $f$ and let

$$g = \sum_{p \neq 0} c_p E_p;$$

this series converges distributionally, which is ensured by inequality (5). Hence

$$g E_{-p} \ast \nu = \sum_{p \neq 0} (c_p E_{-p} \ast \nu)$$

(see e.g., [1], Theorem 5). But $E_{p-\nu} \ast \nu = 0$ except for the case $p = \nu$, in which we have $E_{\nu} \ast \nu = 1 \ast \nu = 1$. Thus $g E_{-p} \ast \nu = c_p$. This means that the Fourier coefficients of $g$ are $c_p$. Hence $f = g$, and by (7),

$$f = \sum_{p \neq 0} c_p E_p.$$

Thus, we have proved that every periodic distribution equals its Fourier expansion. This is the well known theorem of Schwartz.

We shall prove that every periodic distribution is a tempered distribution, i.e., is the derivative of some order of a continuous function of polynomial growth.

We say that a periodic distribution $f$ belongs to class $(r)$, where $r$ is an order (i.e., an integer point satisfying $r > 0$), iff, for (8),

$$\sum_{p \neq 0} |c_p| < (1 + 2\pi|\nu|)^r < \infty.$$

By (5), every periodic distribution belongs to some class $(r)$. Evidently, distributions of class $(0)$ are continuous functions, thus they are tempered distributions. Assume that all distributions of some class $(r)$ are tempered and let $f$ be a distribution of class $(r + \epsilon)$, where $\epsilon$ is fixed. Letting (8), the distribution

$$g = \sum_{p \neq 0} d_p E_p$$

with

$$d_p = \frac{c_p}{(1 + 2\pi|\nu|)^r},$$

belongs to $(r)$ and therefore is tempered. We have

$$f = \sum_{p \neq 0} (1 + 2\pi|\nu|) d_p E_p = g - i\theta^0 h,$$

where

$$h = \sum_{p \neq 0} \text{sgn}(\nu) d_p E_p.$$

Evidently, $h \in (r)$; thus $h$ is tempered, and so is $h^0$. Thus $f$ is tempered. Since $\epsilon$ may be chosen arbitrarily from $1, \ldots, q$, it follows by induction that distributions of all classes $(r)$ are tempered. Thus, every periodic distribution is tempered.

3. We say that a sequence of distributions $f_n$ in $R^k$ converges tempered strongly to $f$, iff there are continuous functions $F_n, F$ and $\alpha$, the last being positive and of polynomial growth in $R^k$, such that $F_n = f_n$, $F = f$ for some order $\alpha$, and the sequence $F_n^{\alpha}$ converges to $F^{\alpha}$ uniformly in $R^k$. 
We say that a sequence of distributions \( f_n \) in \( \mathbb{R}^d \) converges strongly to \( f \), if, given any open bounded interval \( I \subset \mathbb{R}^d \), there is an order \( k \) and there are continuous functions \( F_n, F \) such that \( F_n^{(k)} = f_n \) and \( F^{(k)} = f \) in \( I \), and \( F_n \) converges to \( F \) uniformly in \( I \). Such a convergence is called just convergence, in (3), or distributional convergence.

The following theorem is trivially true:

\[ I \rightarrow II. \quad \text{If a sequence of distributions converges tempered strongly to } f_n, \text{ then it converges strongly to } f. \]

4. We say that a sequence of distributions \( f_n \) in \( \mathbb{R}^d \) converges weakly to \( f \), if the sequence of integrals \( \int f_n \varphi \) converges to \( \int f \varphi \) for every real-valued smooth function of bounded carrier \( \varphi \). For such a function \( \varphi \), the integral \( \int f \varphi \) is a regular operation, it is therefore defined uniquely for every distribution \( f \) as the limit of \( \int f_n \varphi \), where \( f_n \) is a fundamental sequence for \( f \) (see [3]).

\[ II \rightarrow III. \quad \text{If a sequence of distributions converges strongly to } f_n, \text{ then it converges weakly to } f. \]

Proof. If \( f_n \) converges strongly to \( f \) and the carrier of \( \varphi \) is in a bounded interval \( I \), then there are an order \( k \) and continuous functions \( F_n, F \) such that \( F_n^{(k)} = f_n \) and \( F^{(k)} = f \) in \( I \), and \( F_n \) converges to \( F \) uniformly in \( I \). We have therefore

\[ \int f_n \varphi = \int F_n^{(k)} \varphi = \int \underbrace{(-1)^k F_n \varphi}_{= \psi_n} = \int \underbrace{(-1)^k F \varphi}_{= \psi} = \int f \varphi, \]

where \( (-1)^k = (-1)^{\sum_{\alpha \in \mathbb{Z}^d} \alpha_\alpha} \) with \( k = (\alpha_1, \ldots, \alpha_d) \). This proves the assertion.

5. The class of all real-valued smooth functions of bounded carrier is denoted by \( \mathcal{S} \). The class of all rapidly decreasing real-valued functions is denoted by \( \mathcal{S} \).

In the sequel, we shall need the following

**Theorem 1.** Given any function \( \varphi \in \mathcal{S} \), if there is a function \( \varphi \in \mathcal{S} \) such that \( f \varphi = \varphi \) holds for every distribution \( f \).

Proof. Let \( x \in \mathcal{S}, \sum |x_\alpha| = 1 \). We shall show that the function

\[ \varphi = \sum |x^{(k)}| \]

has the required properties. Let

\[ J = \sum |x^{(k)}| \quad \text{and} \quad \varphi = \sum |x^{(k)}|. \]

Evidently, \( \varphi \) is a continuous function of bounded carrier, and \( \varphi \) is a continuous function whose product with any polynomial is bounded.

We have

\[ \sum_{k \in \mathbb{N}} |x^{(k)}| < \sum_{k \in \mathbb{N}} |x^{(k-1)}| < \sum_{k \in \mathbb{N}} |x^{(k-1)}| + |x^{(k-1)}| < \sum_{k \in \mathbb{N}} \varphi < \sum_{k \in \mathbb{N}} \varphi < 2^k. \]

It is easily seen that the series \( \sum_{k \in \mathbb{N}} |x^{(k)}| < 2^k \cdot M \cdot \varphi \).

\[ \sum_{k \in \mathbb{N}} |x^{(k)}| < 2^k \cdot M \cdot \varphi. \]

Now, let \( F \) be a continuous function of polynomial growth such that \( F^{(k)} = f \). Then

\[ f \varphi = f \sum \varphi = F \sum (x - y)^{k-1} \varphi, \]

where (and everywhere in the remaining part of this proof) the sum is stretched on all \( \sigma \in F \). In the last convolution, inequality (9) allows to interchange the signs \( \sum \) and \( f \) so that we obtain

\[ f \varphi = \sum F \varphi = \sum f \varphi = \sum f \varphi \varphi \]

But the carriers of \( \varphi \varphi \) are included in the carrier of \( \varphi \) thus we may interchange again the signs \( \sum \) and \( f \) in the last convolution, provided the series \( \sum \varphi \varphi \) is distributionally convergent (see [1], Theorem 5). But this series converges even almost uniformly which follows from the fact that \( \varphi \in \mathcal{S} \). Similarly, the series \( \sum \varphi \varphi \) converges uniformly for every fixed order \( k \), which implies that \( \sum \varphi \varphi \) is a smooth function. Subsequently, \( \varphi = \sum \varphi \varphi \) and

\[ f \varphi = f \sum \varphi \varphi = f \varphi. \]

6. If \( f \) is any tempered distribution and \( \varphi \in \mathcal{S} \), then the convolution \( g = f \varphi \) is a smooth function. In fact, there is a continuous function \( F \) such that \( F^{(k)} = f \) and \( |F(x)| < M + |x|^p \) for some order \( k \) and positive numbers \( M \) and \( p \). Hence, for \( h \in \mathbb{R}^d, |h| < 1 \), we obtain

\[ |F(x + h)| < M + |x|^p \]

and subsequently

\[ |F(x + h)| < 2M + |x|^p + (1 + |x|^p) = G(x), \]

where \( G \) is, evidently, a continuous function of polynomial growth. Now,

\[ |g(x + h) - g(x)| \leq |F(x + h - t) - F(x - t)| \cdot |\varphi(t)| dt \quad \text{for } |h| < 1. \]

If \( x \) is fixed and \( h \to 0 \), then the integrand converges to 0 at every point \( t \) and is bounded by the product \( G(x - t) |\varphi(t)| \) which is an integrable function of \( t \). Hence \( g(x + h) - g(x) \to 0 \), as \( h \to 0 \). Since \( x \) may be chosen
formulate similar definitions of convergence, but without speaking explicitly of the limit of the sequence. The corresponding kinds of convergence will be denoted by \( I^*, II^*, III^* \) and \( IV^* \), namely:

- **I** A sequence \( f_n \) converges tempered strongly, iff there are continuous functions \( F_\alpha \) and \( \phi_\alpha \) of polynomial growth, such that \( F_\alpha = f_n \) for some fixed order \( \alpha \), and \( F_\alpha \) converges uniformly in \( R^d \).
- **II** A sequence \( f_n \) converges strongly, iff, given any open bounded interval \( I \subseteq R^d \), there is an order \( \alpha \) and there are continuous functions \( F_\alpha \) such that \( F_\alpha = f_n \) in \( I \), and \( F_\alpha \) converges uniformly in \( I \).
- **III** A sequence \( f_n \) converges weakly, iff \( \int f_n \psi \) converges for every \( \psi \in C \).
- **IV** A sequence \( f_n \) converges tempered weakly, iff \( \int f_n \psi \) converges for every \( \psi \in C \).

The following implications are trivial: \( I \to I^*, II \to II^*, III \to III^* \) and \( IV \to IV^* \). It is also evident that \( I \to I^* \) and \( II \to II^* \), i.e., if a sequence converges, then it converges to some limit. We can also state the implication \( III \to IV^* \), which immediately follows from Corollary in section 6. So we have already proved the following implications:

\[
\begin{align*}
I \to I^* & \quad II \to II^* & \quad III \to III^* & \to IV^* \\
\uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
I \to II \to III \to IV
\end{align*}
\]

8. We shall say that a set \( \{a_n\} \) of real numbers \( a_n \) with \( p \in \mathbb{P} \) is rapidly decreasing, iff, for every positive number \( \mu \), we have \([p]^{\mu} |a_n| \to 0\), as \( |p| \to \infty \).

Given any \( \mu > 0 \), we have \([p]^{\mu} \leq (1 + 2\mu |p|)^{\mu} \), where the vector \( \mu \) has all its coordinates equal to \( 2 |p| \). Similarly, given any order \( \kappa \), we have \((1 + 2\kappa |p|)^{\mu} \leq |p|^{\mu} \) with \( \mu = 2 |p| \), provided \( |p| \) is large enough (\( \geq 7 \)). From these inequalities it follows that \( \{a_n\} \) is rapidly decreasing, iff, for every order \( \kappa \), we have \((1 + 2\kappa |p|)^{\mu} |a_n| \to 0\), as \( |p| \to \infty \).

If \( a_n \) is rapidly decreasing, then given any order \( \kappa \), we have,

\[
(1 + 2\kappa |p|)^{\mu} |a_n| \leq (1 + 2\kappa |p|)^{\mu} |a_n| \to 0,
\]

for some positive number \( M \). This implies

\[
(1 + 2\kappa |p|)^{\mu} |a_n| \to 0,
\]

Conversely, from the last inequality it follows that

\[
(1 + 2\kappa |p|)^{\mu} |a_n| \to 0,
\]

as \( |p| \to \infty \).

Thus, a set \( \{a_n\} \) is rapidly decreasing, iff inequality (11) holds for every order \( \kappa \).
If \( c_\mu \) are Fourier coefficients of a periodic function, then from inequality (11) it follows that

\[
\sum_{\mu \in \mathbb{Z}^d} |c_\mu|^2 < \infty.
\]

If a set \((c_\mu)\) of elements of the Banach space satisfies, for some order \( k \), the inequality \(|c_\mu| \leq (1 + 2\pi \mu)^k\), then we shall say that \((c_\mu)\) is tempered. Thus, the set of Fourier coefficients of a periodic distribution is tempered.

Inequality (12) holds for any rapidly decreasing set of real numbers \((a_\mu)\) and any tempered set \((c_\mu)\) of elements of the given Banach space.

We shall say that a sequence of periodic distributions

\[
f_n = \sum_{\mu \in \mathbb{Z}^d} c_{n\mu} E_\mu
\]

converges weakly in Fourier coefficients, iff the sequence

\[
b_n = \sum_{\mu \in \mathbb{Z}^d} a_\mu c_{n\mu}
\]

is convergent for every rapidly decreasing set \((a_\mu)\). We shall say that \(f_n\) converges weakly in Fourier coefficients to the limit

\[
f = \sum_{\mu \in \mathbb{Z}^d} c_\mu E_\mu,
\]

if \(b_n \to b = \sum_{\mu \in \mathbb{Z}^d} a_\mu c_\mu\). These kinds of convergence will be denoted by \(\mathcal{V}^o\) and \(\mathcal{V}\) respectively. Evidently \(\mathcal{V} \subset \mathcal{V}^o\). We shall prove the following implication:

\(\mathcal{V}^o \Rightarrow \mathcal{V}^o\). If a sequence of periodic distributions converges tempered weakly, then it converges weakly in Fourier coefficients.

**Proof.** Let \((a_\mu)\) be any given rapidly decreasing set and let \(\psi\) be a smooth function such that \(\psi(0) = 1\) and \(\psi(x) = 0\) for every \(x\) not belonging to the interval \(-1 < x < 1\). Then

\[\sigma = \sum_{\mu \in \mathbb{Z}^d} a_\mu \psi(x) e^{2\pi i \mu x},\]

Let \(\psi\) be the Fourier transform of \(\sigma\), i.e.,

\[\psi(x) = \int \sigma(t) e^{2\pi i \mu x} dt.\]

Then \(\psi \ast \sigma\) and

\[\sigma(x) = \int \psi(t) e^{2\pi i \mu x} dt.\]

Hence

\[\int \psi E_\mu = \sigma(p) = a_\mu.
\]

Let

\[f_n = \sum_{\mu \in \mathbb{Z}^d} c_{n\mu} E_\mu
\]

be a sequence of periodic distributions which converges tempered weakly. Then, by Theorem 2, we have

\[(f_n \ast \psi)(x) = \sum_{\mu \in \mathbb{Z}^d} a_\mu c_{n\mu} E_\mu(x)
\]

and hence, letting \(x = 0\),

\[\int f_n \psi = \sum_{\mu \in \mathbb{Z}^d} a_\mu c_{n\mu},\]

where \(\psi(x) = \psi(-x)\). But \(\psi \ast \sigma\), thus the sequence \(\int f_n \psi\) is convergent, by assumption that \(f_n\) converges tempered weakly. Therefore, the sequence \(\sum_{\mu \in \mathbb{Z}^d} a_\mu c_{n\mu}\) is convergent for \(n \to \infty\). Since \((a_\mu)\) is an arbitrary rapidly decreasing set, this proves that \(f_n\) converges weakly in Fourier coefficients.

9. We shall say that a sequence of periodic distributions

\[
f_n = \sum_{\mu \in \mathbb{Z}^d} c_{n\mu} E_\mu
\]

converges strongly in Fourier coefficients, iff there is an integer \(i \geq 0\) such that \(c_{n\mu}(1 + |\mu|)^{-i}\) converges uniformly in \(p \ast P\), as \(n \to \infty\). We say that \(f_n\) converges strongly in Fourier coefficients to

\[
f = \sum_{\mu \in \mathbb{Z}^d} c_\mu E_\mu,
\]

iff \(c_{n\mu}(1 + |\mu|)^{-i}\) converges to \(c_\mu(1 + |\mu|)^{-i}\) uniformly in \(p \ast P\), as \(n \to \infty\). These kinds of convergence will be denoted by \(\mathcal{VI}^o\) and \(\mathcal{VI}\), respectively. The implication \(\mathcal{VI} \Rightarrow \mathcal{VI}^o\) is obvious. But also from \(\mathcal{VI}^o\) there follows \(\mathcal{VI}\), provided we take

\[f = \sum_{\mu \in \mathbb{Z}^d} c_\mu E_\mu\]

with \(c_\mu = \lim_{n \to \infty} c_{n\mu}\).

We shall still prove the following implications:

\(\mathcal{V}^o \Rightarrow \mathcal{VI}^o\). If a sequence of periodic distributions converges weakly in Fourier coefficients, then it converges strongly in Fourier coefficients.

\(\mathcal{VI} \Rightarrow \mathcal{V}\) and \(\mathcal{VI} \Rightarrow \mathcal{I}\). If a sequence of periodic distributions converges strongly in Fourier coefficients to \(f\), then it converges to \(f\) weakly in Fourier coefficients and converges tempered strongly.

**Proof of \(\mathcal{V}^o \Rightarrow \mathcal{VI}^o\).** We order all \(p \ast P\) into a sequence \(p_1, p_2, \ldots\). Let

\[t_{ij} = i + |j|\]

\((i, j = 1, 2, \ldots)\).

The matrix \(T = (t_{ij})\) has the following properties: (i) \(t_{ij} \geq i\), (ii) for every \(i\), there is an \(i_s\) with \(\sum_{j=1}^{i_s} t_{ij} \to \infty\). We say that a sequence \(b_\mu\) of elements of a Banach space is \(T\)-tempered, iff there is an index \(i\) such that the sequence \(t_{i_1}^{-1} b_1, t_{i_2}^{-1} b_2, \ldots\) is bounded.
If \( c_\alpha \) are Fourier coefficients of a periodic distribution, then the sequence \( b_j = c_\alpha \) (\( j = 1, 2, \ldots \)) is tempered, which follows from inequality (5).

Assume that a sequence of distributions

\[
f_n = \sum_{\alpha \in \mathbb{Z}} c_{\alpha, n} E_\alpha
\]

converges weakly in Fourier coefficients. Let \( B_n = (c_{\alpha, 1}, c_{\alpha, 2}, \ldots) \). We shall show that the sequence of scalar products

\[
RB_n = \sum_{i=1}^{\infty} R_i c_{\alpha, i}
\]

is convergent for every sequence of real numbers \( R = (r_1, r_2, \ldots) \) such that

\[
\sum_{i=1}^{\infty} |r_i| |r_j| < \infty.
\]

Let \( a_{\alpha,i} = r_i \) (\( j = 1, 2, \ldots \)). Then

\[
EB_n = \sum_{\alpha \in \mathbb{Z}} a_{\alpha,n} c_{\alpha,n}
\]

and it suffices to show that the set \( (a_{\alpha,n}) \) is rapidly decreasing. In fact, we have for every index \( i = 1, 2, \ldots \),

\[
\sum_{\alpha \in \mathbb{Z}} |p|^i |a_{\alpha,i}| = \sum_{\alpha \in \mathbb{Z}} |p|^i |r_i| \leq \sum_{n=1}^{\infty} |r_i| |r_j| < \infty,
\]

which implies that \( |p|^i |a_{\alpha,i}| \to 0 \), as \( |p| \to \infty \). Thus the set \( (a_{\alpha,n}) \) is rapidly decreasing.

Now we apply a known theorem which says that, if for tempered sequences \( B_n = (b_{\alpha,1}, b_{\alpha,2}, \ldots) \) the sequence of scalar products \( RB_n \) converges for every \( R = (r_1, r_2, \ldots) \) satisfying (13), then there is an index \( i \) such that the sequence \( A_n = (\sum_{\alpha} b_{\alpha,i} A_\alpha) \) converges uniformly, as \( n \to \infty \). Thus, in our case, there is an index \( i \) such that \( t_{\alpha,i} c_{\alpha,i} \) converges uniformly in \( j \), as \( n \to \infty \). I.e., if \( t_{\alpha,i} c_{\alpha,i} \) denotes the limit of \( t_{\alpha,i} c_{\alpha,i} \), there is a sequence of positive numbers \( \varepsilon_{\alpha,i} \) tending to 0, such that

\[
|c_{\alpha,i} - c_{\alpha,0}| < \varepsilon_{\alpha,i} \quad \text{for} \quad j = 1, 2, \ldots
\]

Hence

\[
|c_{\alpha,i} - c_{\alpha,0}| < \varepsilon_{\alpha,i} \quad \text{for} \quad p \in P, \quad n = 1, 2, \ldots
\]

and subsequently

\[
|c_{\alpha,i} - c_{\alpha,0}| \leq (i+1) \varepsilon_{\alpha,i} \quad \text{for} \quad p \in P, \quad n = 1, 2, \ldots
\]

This proves that \( c_{\alpha,i} (1 + |p|)^{-i} \) converges uniformly in \( p \in P, \) as \( n \to \infty \).

Proof of VI \( \rightarrow \) V. Let \( f_n = \sum_{\alpha \in \mathbb{Z}} c_{\alpha, n} E_\alpha \) be a sequence of periodic distributions which is VI-convergent to \( f = \sum_{\alpha \in \mathbb{Z}} c_\alpha E_\alpha \); the signs of \( \sum \) are stretched, here and everywhere in the remaining part of this paper, over all \( p \in P \). Thus, there is an integer \( \mu \geq 0 \) such that \( c_{\alpha,i} (1 + |p|)^{i-\mu} \) converges to \( c_\alpha (1 + |p|)^{-\mu} \) uniformly in \( p \in P, \) as \( n \to \infty \). Let \( (a_\alpha) \) be any rapidly decreasing set of real numbers. Then

\[
\sum |c_{\alpha,i} - c_\alpha| a_\alpha = \sum |c_{\alpha,i} - c_\alpha| (1 + |p|)^{-\mu} (1 + |p|)^{\mu} a_\alpha
\]

\[
\leq c_\alpha \sum (1 + |p|)^{\mu} a_\alpha,
\]

where \( c_\alpha \) is a sequence of positive numbers tending to 0. This implies that \( f_n \) is V-convergent.

Proof of VI \( \rightarrow \) I. As in the preceding proof, there is a positive integer \( \mu \) such that \( c_{\alpha,i} (1 + |p|)^{-\mu} \) converges to \( c_\alpha (1 + |p|)^{-\mu} \) uniformly in \( p \in P, \) as \( n \to \infty \). If \( r \) denotes the vector in \( \mathbb{R}^2 \) whose all the coordinates are \( \mu + 2 \), then the sequence

\[
\sum_{\alpha \in \mathbb{Z}} c_{\alpha,i} (1 + 2\pi p r) E_\alpha
\]

converges uniformly in \( \mathbb{R}^2 \) to a continuous periodic function, as \( n \to \infty \). The scheme of the following proof is like by the end of section 2, where we proved that every periodic distribution is tempered. We say that a sequence of periodic distributions \( f_n \) belongs to the class \( (r) \), iff the corresponding sequence \( (14) \) is uniformly convergent. If a sequence \( f_n \) belongs to \((0)\), then, evidently, it is I-convergent. Assume that all sequences of periodic distributions which belong to some class \( (r) \) are I-convergent, and let \( f_n \) be a sequence which belongs to \((r+\varepsilon)\), where \( \varepsilon \) is fixed arbitrary. If \( f_n = \sum_{\alpha \in \mathbb{Z}} c_{\alpha,n} E_\alpha \), then the sequence \( g_n = \sum_{\alpha \in \mathbb{Z}} d_{\alpha,n} E_\alpha \) with

\[
d_{\alpha,n} = \frac{c_{\alpha,n}}{1 + 2\pi r |\varepsilon|}
\]

belongs to \((r)\) and, therefore, is I-convergent. We have

\[
f_n = \sum (1 + 2\pi |\varepsilon|) d_{\alpha,n} E_\alpha = g_n - \varepsilon \theta(\varepsilon),
\]

where \( i \) is the imaginary unit and \( h_n = \sum \text{sgn } \varepsilon d_{\alpha,n} E_\alpha \). Evidently, the sequence \( h_n \) belongs to \((r)\), thus is I-convergent and so is its derivative
Thus $f_n$ is I-convergent. Since $j$ may be chosen arbitrary, it follows by induction that all sequences belonging to some $(r)$ are I-convergent. Thus, every VI-convergent sequence is I-convergent.

10. The following diagram shows which implications between the considered kinds of convergence have been stated, so far:

From this diagram we can immediately read that all possible implications hold between the 12 kinds of convergence, i.e., that all the 12 kinds of convergence are, for sequences of periodic distributions, equivalent.

References


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**On symplectic mappings of contraction operators**

by

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Dedicated to

Stanisław Mazur and Władysław Orlicz

One of the more familiar theorems in function theory states that every conformal mapping of the unit disk onto itself is a fractional linear transformation. In 1943, Siegel [3] proved that this result holds as well for symmetric complex matrices. Our purpose is to generalize this theorem still further and show that it holds both for contraction operators and for symmetric (as distinguished from Hermitian symmetric) contraction operators.

More precisely, let $\mathcal{F}_1$ denote the set of all strictly contractive linear operators on a Hilbert space $H$, $\mathcal{F}_1 = \{J; |J| < 1\}$, and let $\mathcal{F}_2$ denote the set of all strictly contractive symmetric linear operators on $H$, $\mathcal{F}_2 = \{Z; |Z| < 1 \text{ and } Z = \overline{Z}\}$, where for a given conjugation $\mathcal{C}$, $Z = \mathcal{C}Z^*\mathcal{C}$.

We shall consider the group $\mathcal{G}([\mathcal{F}_1]$ of one-to-one bi-analytic mappings $\psi$ of $\mathcal{F}_1 ([\mathcal{F}_2])$ onto itself with the metric $|\psi_1 - \psi_2| = \sup |\psi_1(J) - \psi_2(J)|$ over $\mathcal{F}_1$ (or $\mathcal{F}_2$).

Let $\mathcal{G}_1 ([\mathcal{F}_2]$) denote the principal component of $\mathcal{G}([\mathcal{F}_1])$. It will turn out that $\mathcal{G}_1 = \mathcal{G}$. The analogous assertion does not hold for $\mathcal{F}$ even in the case of matrices; for example $\varphi(J) = J'$ belongs to $\mathcal{F}$ but not to $\mathcal{G}$.

The transformation

$$J \rightarrow (A + B)(DJ + D)^{-1}$$

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