Table des matières du tome XXVIII, fascicule 1

Page
J. MIKUŚIŃSKI, Convolution approximation and shift approximation. 1-8
W. ŻUŁAWEK, On generalized topological divisors of zero in m-convex locally convex algebras 9-16
C. SWARTZ, A generalized function calculus based on the Laplace transform 17-30
R. COPPEN and G. WEISS, A kernel associated with certain multiply connected domains and its applications to factorization theorems 31-68
R. L. WHEEDEN, Derivatives of Fourier series and integrals 69-80
E. B. FABER, Singular integrals and partial differential equations of parabolic type 81-131
N. M. RIVIERE and Y. SAGHER, The converse of Wiener-Lévy-Marcinkiewicz theorem 133-138

CONVOLUTION APPROXIMATION AND SHIFTS APPROXIMATIONS

Part I. Convolution approximation

1. In this paper, the convolution

\[ \int_0^t g(t-s) k(s) ds \]

will be denoted by \( g * k \) (instead of the usual notation \( g \ast k \)). The ordinary product of two functions will be denoted, on writing explicitly the arguments, e.g. \( g(t)k(t) \). Such a notation is also used in my book [3].

Let \( C[0, T] \) \( (0 < T < \infty) \) be the class of real continuous functions on the interval \( [0, T] \), \( C^1[0, T] \) the subclass of \( C[0, T] \) of functions which vanish at \( 0 \), and \( C^0[0, T] \) the class of infinitely derivable functions in \( [0, T] \) which vanish at 0 together with all their derivatives.

Theorem 1. For any fixed \( g \in C[0, T] \) which does not vanish identically in the right neighbourhood of 0, the set of convolutions \( gk \) with \( k \in C^0[0, T] \) is dense in \( C^0[0, T] \).

This theorem is, in fact, due to Foiaş [1], who needed it to prove that the set of continuous functions is dense in the space of operators. However, Foiaş formulated it in a slightly different form: For any fixed \( g \in L^2[0, T] \), not vanishing almost everywhere at the neighbourhood of 0, the set of convolutions \( gk \) with \( k \in C^0[0, T] \) is dense in \( L^2[0, T] \). Another formulation of Theorem 1 is given in the paper [3]. There it is proved that, for any fixed \( g \in C[0, T] \), non-vanishing in the neighborhood of 0, the set of convolutions \( gk \) with \( k \) absolutely continuous functions is dense in \( C[0, T] \). The proof given in [3] can be used for Theorem 1; the only needed modifications are the following: one assumes that \( g \in C[0, T] \) (instead of \( g \in C[0, T] \)), \( k \), \( k \in C^0[0, T] \) (instead of \( k \), \( k \in AC \) (absolutely continuous functions)) and one has \( k \in C^0[0, T] \) (instead of \( k \in L^2[0, T] \)).

Evidently, Theorem 1 is a little stronger than my earlier formulation in [3]. It is easy to see that it is also stronger than the formulation
in Poincaré's paper [1]. In fact, let \( g \in L([0, T]) \) and let
\[
L = \int_0^T g(t) \, dt.
\]

By Theorem I, the set of convolutions \( L \), where \( k \in C_0^\infty([0, T]) \) is dense in \( C([0, T]) \). Since \( L \) is dense, we may also say that the set of convolutions \( k \) where \( k \in C_0^\infty([0, T]) \) is dense in \( C([0, T]) \). A fortiori, the set of convolutions \( g \) where \( g \in C([0, T]) \) is dense in \( C([0, T]) \). Since \( C([0, T]) \) is dense in \( L([0, T]) \), this set is dense in \( L([0, T]) \).

In the Part I of this paper we are going to strengthen Theorem I in three steps. In this way we shall obtain Theorems II, III and IV, each of them being stronger than the preceding one.

**Theorem II.** Given any fixed \( g \in C([0, T]) \) which does not vanish identically in the right neighbourhood of 0, the set of convolutions \( k \) with \( k \in C_0^\infty([0, T]) \) is dense in \( C([0, T]) \).

**Proof.** Let \( f \in C([0, T]) \) and let \( \epsilon_n \) be a sequence of positive numbers, tending to 0 as \( n \to \infty \). By Theorem I, there exists, for every positive integer \( n \), a function \( k_\epsilon \in C([0, T]) \) such that
\[
|g(k_\epsilon f - f)| \leq T^{-\epsilon_n}.
\]

This implies that
\[
|g(k_\epsilon f - f)| < T^{-\epsilon_n} \quad \text{for} \quad i = 1, \ldots, n.
\]

Thus, for any fixed \( i \), the sequence \( g(k_\epsilon f) \) converges uniformly to \( f \) as \( \epsilon \to 0 \). This means that \( g \to f \) in the topology of \( C([0, T]) \), which proves Theorem II.

The fact that Theorem II is stronger than Theorem I follows from the remark that \( C([0, T]) \) is dense in \( C([0, T]) \).

Let \( C([0, \infty)) \) denote the class of continuous functions in the interval \([0, \infty)\) and \( C([0, \infty)) \) the class of indefinitely derivable functions in that interval. We say that a sequence of functions from the class \( C([0, \infty)) \) or from the class \( C([0, \infty)) \) is convergent in \( C([0, \infty)) \) or in \( C([0, \infty)) \) respectively, if the corresponding sequence of functions restricted to any bounded interval \([0, T]\) is convergent in the proper topology of \( C([0, T]) \) or \( C([0, T]) \).

**Theorem III.** For any fixed \( g \in C([0, \infty)) \) which does not vanish identically in the right neighbourhood of 0, the set of convolutions \( k \) with \( k \in C([0, \infty)) \) is dense in \( C([0, \infty)) \).

**Proof.** Let \( f \in C([0, \infty)) \). By Theorem II, there is, for any positive integer \( p \), a sequence of functions \( k_\epsilon \in C([0, \infty)) \) such that \( |g(k_\epsilon f - f)| < \epsilon \) in \([0, \infty)\), where \( 0 < \epsilon_n \to 0 \) as \( n \to \infty \). This implies that the diagonal sequence \( g(k_\epsilon f) \) converges to \( f \) uniformly in every interval \([0, T]\) \((0 < T < \infty)\), which proves Theorem III.

Evidently, Theorem III reduces to Theorem II, when restricting the considered functions from \([0, \infty)\) to \([0, T]\).

2. Let \( M_p \) denote the space of all operators \( a = p/q \) (see [2]), where \( p, q \in C([0, \infty)) \) and \( a \) does not vanish identically in any right neighbourhood of 0. We say that a function \( g \in C([0, \infty)) \) non-vanishing identically in any right neighbourhood of 0, such that all the operational products \( a \cdot g \) are functions of class \( C([0, \infty)) \) and \( a \cdot g \) converges almost uniformly in \( C([0, \infty)) \) (i.e., uniformly in every bounded interval \([0, T]\)). We say that an operator \( a \in M_p \) does not vanish in the right neighbourhood of 0, if it is of the form \( p/q \), where both \( p \) and \( q \) are functions which do not vanish identically in the right neighbourhood of 0 (see [4]).

**Theorem IV.** For any fixed operator \( g \) which does not vanish in the right neighbourhood of 0, the set of elements \( k \), where \( k \in C([0, \infty)) \), is dense in \( C([0, \infty)) \).

**Proof.** Let \( f \in C([0, \infty)) \). There is a function \( g \in C([0, \infty)) \), non-vanishing identically in the right neighbourhood of 0, such that \( g \cdot f \in C([0, \infty)) \); evidently the function \( g \) does not vanish either identically in the right neighbourhood of 0. Thus, by Theorem III, there are functions \( k_\epsilon \in C([0, \infty)) \) such that \( g(k_\epsilon f) \) converges to \( f \) in the topology of \( C([0, \infty)) \). Since \( g(k_\epsilon f) \in C([0, \infty)) \), Theorem IV is proved.

In order to see that Theorem IV is stronger than Theorem III, it suffices to observe that \( C([0, \infty)) \) is a subset of \( M_p \).

**Part II. Shift approximation**

3. Let \( S \) be a linear subspace of \( M \), containing \( C([0, \infty)) \), with the following properties:

1. \( S \) is a locally convex topological space such that every sequence \( f_k \to f \) converges in \( C([0, \infty)) \) converges also in \( S \) to the same limit; moreover, every sequence \( f_k \to f \) converges in \( S \) converges also in \( M \) to the same limit; finally, we assume that \( C([0, \infty)) \) is dense in \( S \);

2. If \( f \to S \), then \( h \cdot f \to S \) (h shift-operator) for every \( \lambda \geq 0 \). In the topology of \( S \), \( h \cdot f \) is a continuous function of \( \lambda \) in the interval \( 0 \leq \lambda < \infty \);

3. There is a family of semi-norms \( ||f||_n \) with \( a \cdot A \) such that for any \( a \in A \) there is a number \( \lambda_n > 0 \) such that \( \lambda > \lambda_n \) implies \( ||h \cdot f||_n = 0 \) for every \( f \in S \).
We are going to give a few examples of the space $S$.

(i) **Space $C^\infty_0[0, \infty)$.** Here, we have

\[
h_f(t) = \begin{cases} f(t - \lambda) & \text{for } t \geq \lambda, \\ 0 & \text{for } 0 \leq t < \lambda. \end{cases}
\]

For $A$, we can take the set of pairs $a = (p, r)$ of integers $p, r$ ($p \geq 0, r \geq 1$) and then let

\[ ||f||_{\alpha,r} = \max_{t \in \mathbb{R}} |f^{(r)}(t)|. \]

(ii) **Space $C^p[0, \infty)$.** The elements of this space are functions in $[0, \infty)$, derivable up to the order $p$ in that interval, and vanishing together with these derivatives at 0. A sequence $f_n \in C^p[0, \infty)$ is said to converge in $C^p[0, \infty)$, if for any $i = 0, \ldots, p$, the sequence $f_n^{(i)}$ converges uniformly in every interval $[0, T]$ ($0 < T < \infty$). Formula (1) holds also in the actual case. For $A$, we can take $a = (i, r)$ of integers $i, r (0 \leq i \leq p, r \geq 1)$ and then let

\[ ||f||_{i,r} = \max_{t \in \mathbb{R}} |f^{(r)}(t)|. \]

(iii) **Space $C_0[0, \infty)$.** Its elements are continuous functions in $[0, \infty)$, vanishing at 0. This is a particular case of the preceding example (with $p = 0$). Actually, for $A$ we can take the set of all positive integers $a = r$ and let

\[ ||f||_r = \max_{t \in \mathbb{R}} |f(t)|. \]

(iv) **Space $L^p[0, \infty)$, $p \geq 1$.** The elements of this space are functions which are locally $p$-integrable in $[0, \infty)$, i.e., $p$-integrable on every bounded interval $[0, T]$. Formula (1) holds also in the present case. For $A$, we can take the set of all positive integers and let

\[ ||f||_p = \left( \int_0^\infty |f(t)|^p \, dt \right)^{1/p}. \]

(v) **Space $D_\infty$.** As elements of this space we take the distributions whose support lies on $[0, \infty)$. It turns out that to say that these elements are distributions defined on the whole line $(-\infty, \infty)$ and vanish in $(-\infty, 0)$, In particular, continuous functions in $(-\infty, \infty)$, vanishing in $(-\infty, 0)$, are distributions. In order to imbed $C^\infty_0[0, \infty)$ into $D_\infty$, we extend the definition of $f \in C^\infty_0[0, \infty)$ into the negative part of the real axis, assuming that $f$ vanishes on that part. Evidently, formula (1) makes sense in the case of $D_\infty$. For $A$, we can take the set of all smooth (infinitely derivable) functions $\alpha$ of bounded support (vanishing outside a bounded interval). Then we let for $f \in D_\infty$,

\[ ||f||_\alpha = \left( \int_0^\infty |f(t)| \, dt \right)^{1/\alpha}. \]

(vi) **Space $M_\infty$.** The elements of this space are operators (elements of $M_\infty$) which can be represented in the form $f = p/q$, where $p, q \in C_0[0, \infty)$ and $q$ does not vanish identically in the right neighbourhood of 0. The space $M_\infty$ is thus determined by the function $q$. For $A$, we can take the set of positive integers and let

\[ ||f||_\alpha = \max_{t \in \mathbb{R}} |g(t)| \]

($g$ is a continuous function).

We have evidently $C^\infty_0[0, \infty) \subseteq C_0^\infty[0, \infty) \subseteq C_0[0, \infty) \subseteq L^p[0, \infty) \subseteq D_\infty$. We also have $D_\infty \subseteq M_\infty$, provided we take for $g$ a function of class $C^\infty_0[0, \infty)$; then $D_\infty$ is a proper subset of $M_\infty$.

**Theorem V.** For every operator $g : S$ which does not vanish in the right neighbourhood of 0, the set of elements of the form

\[ \lambda_1 h^1 g + \cdots + \lambda_r h^r g, \]

where $\lambda_i$ and $\tau_i$ are real numbers, $\tau_i > 0$, and $h$ is the shift-operator, is dense in $S$.

**Proof.** Let us consider the integral

\[ \int_\mathbb{R} h^\tau g(t) \, \lambda \, dt \]

where $k \in C^\infty_0[0, \infty)$. Remark that, in the interpretations (i)-(v), this integral can be written in the form

\[ \int_\mathbb{R} g(t + \tau) h^\tau \, dt, \]

because of formula (1). Thus it equals the convolution $gh$. We shall show that it equals $gh$ also in the general case. In fact, the integral

\[ \int_{-\infty}^\infty h^\tau g(t) \, dt \]

is defined for every finite $b > 0$, since its integrand is continuous. The value of (3) is to be considered as the limit of (4), as $b \to \infty$. The existence of that limit follows from the inequality

\[ \int_{\tau_1}^{\tau_2} h^\tau g(t) \, dt \leq \int_{\tau_1}^{\tau_2} |h^\tau| \, dt \leq \int_{\tau_1}^{\tau_2} |h^\tau| \, dt (\tau_1 \leq \tau_2) \]
and from the fact that \( \|k\|_u = 0 \) for \( \lambda_n < r_1 \leq \tau \leq \tau_n \). On the other hand, by \( \tau_n \), the limit (2) can be also considered in the operational sense, and we see that it equals
\[
g \int_0^\infty k \cdot k(t) \, dt = g \ast S
\]
(see e.g. formula (9.1), p. 387, of [2]).

Let
\[
w_n(k^{1/n}) = \lambda_1 k^{1/n} + \lambda_2 k^{1/n} + \ldots + \lambda_n k^{1/n},
\]
where
\[
\lambda_i = \int_{(i-1)/n}^i k(t) \, dt \quad (n = 1, 2, \ldots, \lambda_n).
\]

We have
\[
\|g - w_n(k^{1/n})\|_u \leq \|k \tau - g \tau\|_u \leq \|k \tau\|_u + \|\lambda g\|_u.
\]

For \( n > \lambda_n \), the last integral vanishes, so we can write
\[
\|g - w_n(k^{1/n})\|_u \leq \sum_{i=1}^n \int_{(i-1)/n}^i \|k \tau - g \tau\|_u \leq \|\lambda g\|_u + \|\lambda g\|_u.
\]

For \( \tau > (i-1)/n > \lambda_n \), we have \( \|k \tau - g \tau\|_u \leq \|\lambda g\|_u + \|\lambda g\|_u = 0 \), it therefore suffices to consider, in (5), only expressions
\[
\|\lambda g - k \tau\|_u.
\]
with \( \tau \) and \( \lambda \) belonging to the bounded interval \( 0 \leq \tau \leq \lambda + 1 \). Since the function \( k \tau \) is supposed continuous, expression (6) becomes less than any given \( \varepsilon > 0 \) if \( \lambda_n < i/n \leq \tau < i/n < \lambda_n + 1 \) and \( n \) is sufficiently large, say \( n > \lambda_n \). Thus we obtain, for \( n > \lambda_n \),
\[
\|g - w_n(k^{1/n})\|_u \leq \varepsilon \int_{i/n}^{i/n+1} |k(t)| \, dt.
\]

This proves that \( w_n(k^{1/n})g \to g \) in the topology of \( S \). We can also say that the set of elements \( w(k^{1/n})g \), where \( k \) are polynomials with real coefficients, is dense in the set of convolutions \( g \) with respect to the topology of \( S \). Since the set of convolutions \( g \) is dense in \( C_0^\infty(0, \infty) \), by Theorem IV, and the last set is dense in \( S \), by hypothesis, the set of elements \( w(k^{1/n})g \) is dense in \( S \), which proves Theorem V.

4. We are now going to discuss some particular cases of Theorem V.

If \( S \) is one of the spaces \( C_0^\infty(0, \infty) \), \( C_0^\infty[0, 1] \), or \( C_0^\infty[0, \infty) \), then the hypothesis that the operator \( g \) does not vanish in the right neighbourhood of 0 means that the function \( g \) (actually \( g \) is a function) does not vanish identically in this neighbourhood. Theorem V says that, for any fixed \( g \) with that property, the set of elements
\[
\lambda_1 g(t-\tau_1) + \ldots + \lambda_n g(t-\tau_n)
\]
with \( \tau_1 > 0 \) is dense in the considered space.

This implies, in particular, that every function \( f \in C_0^\infty(0, \infty) \) (not necessarily vanishing at 0) can be approximated almost uniformly in \( [0, \infty) \) by sums (7) with \( g \in C_0^\infty[0, \infty) \), where at most the number \( r_1 \) is negative, all others \( r_{i+1} \) \( (i > 1) \) being positive.

Indeed, there is a point \( t_i > 0 \) such that \( g(t_i) \neq 0 \). The function
\[
f(t-\tau_1) + \ldots + \lambda_n g(t-\tau_n)
\]
belongs evidently to \( C_0^\infty(0, \infty) \) and can be therefore approximated almost uniformly by sums
\[
\lambda_1 g(t-\tau_1) + \ldots + \lambda_n g(t-\tau_n).
\]

Hence, our assertion follows, on taking \( \lambda_1 = f(t_i)/g(t_i) \) and \( \tau_1 = -t_i \).

If we restrict the functions to a bounded interval \( [0, T], \) then we obtain a theorem proved in [3].

In a similar way we can show that every function \( f \in C_0^\infty(0, \infty) \) can be approximated almost uniformly together with their derivatives up to the order \( p \) by sums (7), where at most \( p+1 \) numbers \( r_1, \ldots, r_{p+1} \) are negative.

If \( S = D'(0, \infty) \) \( (p > 1) \), then the assumption that \( g \) does not vanish in the right neighbourhood of 0 means that there is no right neighbourhood of 0 in which the function \( g \) vanishes almost everywhere. Theorem V says that the set of elements (7) is dense in \( D'(0, \infty) \). If we restrict the functions to a bounded interval \( [0, T], \) then we obtain a theorem proved by Skorokhod in [3].

If \( S = D'_0 \), then Theorem V says that every distribution from \( D'_0 \) can be approximated by sums (7) with any other distribution \( g \) from \( D'_0 \) which does not vanish in the right neighbourhood of 0, and positive numbers \( r_1 \).

In particular, it can be approximated by sums with the delta distribution: \( \lambda_1 \delta(t-\tau_1) + \ldots + \lambda_n \delta(t-\tau_n) \). It might seem to seem, at first, more paradoxical that the delta distribution \( \delta(t) \) can be approximated by sums (7) with positive \( r_i \) and with an arbitrary given function \( g \in C_0^\infty(0, \infty) \).

Finally, it follows from Theorem V that every operator \( f \ast \mathcal{M}_s \), can be approximated operationally by sums (2) with any other operator \( g \ast \mathcal{M}_s \) which does not vanish in the right neighbourhood of 0. In fact, there exists
a function $q \ast C[0, \infty)$ which does not vanish identically in the right neighbourhood of 0 such that $f = p/q$ and $g = p_{1}/q$, where $p_{1}, p_{2}, q \in C[0, \infty)$.

Let $M_{q}$ be the set of all operators which can be represented in the form $p/q$ ($p \ast C[0, \infty)$). By Theorem V, $f$ can be approximated by sums (2) in the topology of $M_{q}$. But every sequence which converges in the topology of $M_{q}$ also converges operationally, which proves our assertion. Taking in particular $q = 1$, we see that every operator from $M_{q}$ can be approximated by polynomials $\lambda_{1}h^{1} + \ldots + \lambda_{n}h^{n}$ of the shift-operator with positive $\tau$.

5. We have considered, so far, functions, distributions and operators defined on the one-dimensional real space $R$. However, all the theorems can also be interpreted in the Euclidean space $R^{m}$ of any number of dimensions. Then by an interval $[0, T]$ we understand the set of points $t = (t_{1}, \ldots, t_{m})$ whose coordinates satisfy the inequalities $0 \leq t_{i} \leq T_{i}$, where $T = (T_{1}, \ldots, T_{m})$. Similarly, the interval $[0, \infty)$ means the set of points $t$ with $t_{i} \geq 0$. By the convolution

$$\int_{t_{i}}^{T_{i}} g(t - \tau) h(\tau) d\tau$$

we understand an integral stretched on the set $0 \leq t_{i} \leq t_{i}$ ($i = 1, \ldots, m$).

The proof of Theorem I is based on the Titchmarsh theorem, which holds for any number of dimensions (see [5]). This theorem permits to introduce the class of $m$-dimensional operators $a = p/q$, where $q$ does not vanish identically in the $m$-dimensional right neighbourhood of 0. Then all the preceding considerations remain true in the new, more general, interpretation.

References


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On generalized topological divisors of zero in $m$-convex locally convex algebras

by

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By a topological algebra we mean in this paper a topological linear space together with an associative jointly continuous multiplication. An element $x$ of a topological algebra $A$, $x \neq 0$, will be called a left (right) topological divisor of zero if there exists a non-void subset $P \subset A$ such that zero is not in the closure $\overline{P}$ of $P$ but $0 \not\in \overline{P} \cap (0 \times V)$. Here, as usual, $UV = \{xy : x \in U, y \in V\}$. An element $x \in A$ is called a topological divisor of zero in $A$ if it is both a right and a left topological divisor of zero. It is a classical fact of the theory of Banach algebras, due to Šilov [3] (for algebras without a unit, see [5]) that a complex Banach algebra either possesses topological divisors of zero or is isomorphically homeomorphic to the field of complex numbers. The same holds for locally bounded algebras — a class more general than the class of Banach algebras [5]. Here we investigate the problem for another generalization of Banach algebras, namely for the class of locally convex multiplicatively convex topological algebras (shortly, we shall call them $m$-convex algebras throughout this paper). An $m$-convex algebra is a topological algebra (over complexes) with a basis for neighbourhoods of the origin consisting of sets $U$ which are convex, symmetric and idempotent, i.e. such that $UU \subset U$. Or, which is equivalent, it is a locally convex algebra with the topology given by means of family $\mathcal{P}$ of submultiplicative pseudonorms:

$$\|xy\| \leq \|x\|\|y\|$$

and, in the case where the algebra in question possesses a unit $e$,

$$\|e\| = 1$$

for each $\|\cdot\| \in \mathcal{P}$. We may assume that $\mathcal{P}$ consists of all continuous pseudonorms satisfying (1) and (2) in the case where there is a unit element.

The theory of these algebras was created by Arens [1] and Michael [2].

The statement that an $m$-convex algebra either possesses topological divisors of zero or is isomorphically homeomorphic to the field of