On the form of pointwise continuous positive functionals and isomorphisms of function spaces

by

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In §1 we are concerned with the form of pointwise continuous positive functionals defined on a linear space of continuous real-valued functions. In §2 we give a sufficient condition under which the existence of an isomorphism of function spaces implies the existence of a homeomorphism of spaces of arguments. §3 contains a generalization of these results to the case of linear spaces satisfying less restricted conditions than those assumed in §§1 and 2.

§1.

Given a topological space $X$, we denote by $C(X)$ ($C^*(X)$) the set of all continuous (all bounded continuous) real-valued functions defined on $X$. It is known that

(A) A space $X$ is a $Q$-space if and only if each non-trivial linear multiplicative functional $\varphi$ defined in $C(X)$ can be written in the form

$$
\varphi(f) = f(p_0)
$$

where $p_0$ is a fixed point of $X$.

Now let $R$ be any linear ring contained in $C(X)$ and satisfying the following conditions:

(a) all constant functions belong to $R$;

(b) if $f, g \in R$, $0 \leq f(p) \leq g(p) \leq 1$ for $n - 1$, then there exists a sequence $a_n$ of positive numbers such that $\sum a_n < +\infty$ and $\sum a_n f_n \in R$;

(c) if $f \in R$ and $f(p) \neq 0$ for each $p$ in $X$, then $1/f \in R$.

(1) All topological spaces under consideration are supposed to be Hausdorff completely regular.

(2) A functional is called non-trivial if it does not vanish identically.

(3) The original Hewitt definition of $Q$-spaces [2] is as follows: a maximal ideal $R$ in the ring $C(X)$ is said to be hyper-real provided that the quotient field $C(X)/R$ contains (within isomorphism) the field of real numbers as a proper subset; $R$ is called free if for each $p$ in $X$ there is an $f$ in $R$ with $f(p) = 0$. Then a space $X$ is said to be a $Q$-space if each maximal proper free ideal in $C(X)$ is hyper-real. Restating this definition in terms of functionals, we obtain (A).
In [6] the author proves following theorem (a):

(B) Each non-trivial multiplicative linear functional \( \varphi \) defined on any linear ring \( E \subseteq C(X) \) which satisfies conditions (a), (b) and (c) can be written in the form (**) if and only if \( E \) is a Lindelöf space (i).

An analogous situation may be observed in connection with pointwise continuous positive linear functionals. Namely, in [5] the author proves that (i):

(C) Each pointwise continuous positive linear functional \( \varphi \) defined on \( C^0(X) \) can be written in the form (**)

\[
\varphi(f) = a_1 f(p_1) + \cdots + a_n f(p_n),
\]

where \( p_1, \ldots, p_n \) are fixed points of \( X \) and \( a_1, \ldots, a_n \) are fixed real numbers if and only if \( X \) is a \( Q \)-space.

Now let \( E \subseteq C^0(X) \) be a linear space satisfying conditions (a) and (b) and the following one:

(D) If \( f \in E \), then \( \|f\| \leq r \).

Then the following analogue of statement (B) can be proved:

(D) Each pointwise continuous positive linear functional \( \varphi \) defined on any linear space \( E \subseteq C^0(X) \) which satisfies conditions (a), (b) and (d) can be written in the form (**) if and only if \( X \) is a Lindelöf space (i).

We shall deduce statement (D) from a more general statement which will be given in the next section. Now we explain the role of condition (d).

Notice that condition (d) is satisfied if \( E \) is a linear subring of \( C^0(X) \), which is closed with respect to uniform convergence and satisfies condition (a). Indeed, in this case, the function \( \|f\| \) can be written as the sum of a uniformly convergent series of members of \( E \).

Condition (d) is essential. We shall show that, even in the case when \( X \) is the unit interval \([0,1]\), there exists a linear space \( E \subseteq C^0(X) \) which satisfies conditions (a) and (b), and a pointwise continuous positive linear functional \( \varphi \) defined on \( E \) which cannot be written in the form (**). We shall use the following unpublished result of S. Mazur:

Suppose that \( p \) is an increasing sequence of positive integers with \( \sum 1/p_k < \infty \) and \( f_k(a) = a^{p_k} + \sum a^{j+k} \), where the series \( \sum a^{j+k} \) are convergent in the whole interval \([0,1]\). If \( f_k(a) \to f(a) \) for each \( 0 < a < 1 \), then \( f(a) \) is also of the form \( f(a) = a^{\sum a^{j+k}} \), where the series \( \sum a^{j+k} \) is convergent in the whole interval \([0,1]\); moreover, \( f_k \) converges to \( f \) almost uniformly in the interval \([0,1]\).

Consider the following infinite matrix of positive integers:

\[
\begin{array}{cccccc}
2 & 1 & 2 & 1 & 2 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 \\
\end{array}
\]

Of course, the sum of the inverses of numbers standing in the \( k \)-th row of the matrix does not exceed \( 2k/2^k \). Let \( p_k \) be the increasing sequence consisting of all terms of the matrix. If follows from the preceding remark that the series \( \sum 1/p_k \) is convergent.

Let \( E \) be the set of all functions \( f \) on \( X \) which can be represented in the form \( f(x) = a_0 + \sum a_k x^{p_k} \), where the series \( \sum a_k x^{p_k} \) is convergent in the whole interval \([0,1]\). By Mazur's result quoted above, \( E \) is a linear subspace of \( C^0(X) \) satisfying conditions (a) and (b).

Now let

\[
\varphi(f) = \int f(x) \, dx \quad \text{for} \quad f \in E.
\]

Of course, \( \varphi \) is a positive linear functional and it follows from Mazur's result quoted above that \( \varphi \) is pointwise continuous. But \( \varphi \) cannot be written in the form (**). Indeed, let \( a_1, \ldots, a_n \) be any points of the interval \([0,1]\) and let

\[
f(x) = (x^{a_2} - x^{a_1}) \cdots (x^{a_n} - x^{a_1}).
\]

Then \( f \in E \) (indeed, exponents of the variable \( x \) occur in the \( k \)-th row of the matrix), \( f(a_j) = \delta (j = 1, \ldots, k) \), \( \varphi(f) \neq 0 \).
1. The main result. Before formulating the main result of the present paragraph we recall some definitions.

A subset \( P \) of a topological space \( S \) is said to be \( Q \)-closed in \( S \) provided that for each \( p \in S \setminus P \) there exists a \( G \)-set \( G \subset S \) which contains \( p \) and is disjoint from \( P \).

Of course we have

(i) A set \( P \subset S \) is \( Q \)-closed in \( S \) if and only if for each \( p \in S \setminus P \) there exists a continuous real-valued function \( f \) on \( S \) which is positive on \( P \) and zero at \( p \).

Moreover, we have (see [7])

(ii) A space \( A \) is \( Q \)-closed in \( \beta X \) if and only if \( X \) is a \( Q \)-space.

(iii) A space \( X \) is \( Q \)-closed in each of its compactifications \( (*) \) if and only if \( X \) is a Lindelöf space.

Suppose \( \mathcal{F} \) is a family of bounded continuous functions, each defined on a space \( X \). Let \( I \) be the interval of values of a function \( f \in \mathcal{F} \) (i.e. \( I \) is the interval \([\inf \{ f(p) \}, \sup \{ f(p) \}]\)) and denote by \( I^0 \) the Cartesian product \( \prod_{p \in X} I_p \). Let \( F \) be the mapping of \( X \) into \( I^0 \) which carries a point \( x \in X \) into the point \( (x)^0 \) whose \( f \)-th coordinate, is equal to \( f(p) \). Clearly, \( F \) is a continuous mapping.

**Theorem 1.** Let \( E \in C^*(X) \) be a linear space satisfying conditions \( (\alpha) \), \( (\beta) \) and \( (\gamma) \). Then each pointwise continuous positive linear functional \( \varphi \) defined on \( E \) can be written in the form \((**)\) if and only if \( F_E(X) \) is \( Q \)-closed in \( F_E(X) \) (the bar indicates the closure with respect to \( I^0 \)).

The proof of the above theorem will be given in the next sections. Now we shall show that statements \((C)\) and \((D)\) are immediate consequences of this theorem.

To begin with, if \( E = C^*(X) \), then \( F_E \) is a homeomorphism and \( F_{E^*}(X) = BF_E(X) = \beta X \) (\( \ast \)). Hence, each pointwise continuous positive linear functional \( \varphi \) defined on \( C^*(X) \) can be written in the form \((**)\) if and only if \( X \) is \( Q \)-closed in \( \beta X \), i.e. if \( X \) is a \( Q \)-space. Thus \((C)\) is proved. Now suppose that \( X \) is a Lindelöf space. If \( E \subset C^*(X) \) is any linear space satisfying conditions \( (\alpha) \), \( (\beta) \) and \( (\gamma) \), then \( F_E(X) \), being the continuous image of \( X \), is also a Lindelöf space, whence \( F_E(X) \) is \( Q \)-closed in \( F_E(X) \). Conversely, suppose that \( X \) is not a Lindelöf space. Then, by \((\text{iii})\), there exists a space \( Y \) which is a compactification of \( X \) and is such that \( X \) is not \( Q \)-closed in \( Y \). Denote by \( E \) the set of all functions on \( X \) which can be continuously extended over \( Y \). Then \( E \) is a linear subspace of \( C^*(X) \) which satisfies conditions \( (\alpha) \), \( (\beta) \) and \( (\gamma) \). Moreover, \( F_E \) is a homeomorphism and \( F_E \) can be extended to a homeomorphism which maps \( Y \) onto \( F_E(X) \). It follows that \( F_E(X) \) is not \( Q \)-closed in \( F_E(X) \) and \((D)\) is proved.

II. Auxiliary theorems. We say that a family \( F \) of functions, each defined on a space \( S \), distinguishes points of \( S \) if for every \( p, p' \in S \), \( p \neq p' \), there is an \( f \in F \) with \( f(p) \neq f(p') \); we say that \( F \) distinguishes points and closed sets of \( S \) if for each \( A \subset S \) and \( p \in S \) there is an \( f \in F \) with \( f(p) = 0 \) for \( p \in A \) and \( f(p) = 1 \) for \( p \notin A \).

**Lemma 1.** Suppose that \( S \) is a compact space and that \( E \) is a linear subspace of \( C^*(S) \) satisfying conditions \( (\alpha) \), \( (\beta) \) and \( (\gamma) \) and distinguishing points and closed sets of \( S \). Then a set \( P \subset S \) is \( Q \)-closed in \( S \) if and only if for each \( p_0 \in S \) \( P \) there is a function \( f \in E \) and a sequence \( f_1, f_2, \ldots \) \((f_1 \in E)\) such that \( f_n(p) \to f(p) \) for each \( p \in P \) and \( f(p_0) = 1 \).

**Proof.** Let \( P \subset S \) be a \( Q \)-closed set and let \( p_0 \) be any point in \( S \setminus P \). Then there exists a sequence \( G_1, G_2, \ldots \) of open sets such that \( p_0 \in G_n \quad (n = 1, 2, \ldots) \) and \( F \cap \bigcap G_n = \emptyset \). Let \( F_{\alpha} \) be any point in \( S \setminus P \). There is a function \( h_0 \in F \) such that \( h_0(p_0) = 0 \) and \( h_0(p) = 1 \) for \( p \in F_{\alpha} \). Setting \( g_n(p) = \min \{ 1, \max \{ 0, h_0(p) \} \} \) we have \( g_n(p_0) = 0 \), \( g_n(p) = 1 \) for \( p \in F_{\alpha} \) and \( 0 < g_n(p) < 1 \) for \( p \in S \setminus F_{\alpha} \). By condition \((\beta)\), there exists a sequence \( a_n \) of positive numbers such that \( \sum_{n=0}^{\infty} a_n = \infty \) and \( g \in \sum_{n=0}^{\infty} g_n \cdot E \).

Of course, \( g(p_0) = 0 \) and \( g(p) > 0 \) for \( p \in S \). Setting \( f_n = \min \{ 1, a_n \} \) we have \( f_n \in E \), \( f_n(p) \to 1 \) for \( p \in S \) and \( f_n(p_0) \to 1 \).

Conversely, suppose that \( P \) is not \( Q \)-closed in \( S \). Then, by \((i)\), there exist a point \( p_0 \in S \setminus P \) such that for each continuous function \( h \) on \( S \) the condition \( h(p_0) > 0 \) for each \( p \in P \) implies \( h(p_0) > 0 \). Now let \( f \) be any function in \( E \) and let \( f_1, f_2, \ldots \) \((f_1 \in E)\) be any sequence such that \( f_n(p) \) \( \to f(p) \) for each \( p \in P \). Let

\[
\begin{align*}
\eta(p) &= |f(p) - f_0(p)| \quad (0 < \eta(p_i) < 1) \quad (i = 1, 2, \ldots)\\
\eta &\to 0 \quad (p \to p_0)\\
\eta &\to 0 \quad (p \to p_0)
\end{align*}
\]

Then \( \eta \) is a continuous function on \( S \) and \( \eta(p_0) = 0 \) and it follows that \( |f(p_i) - f(p_j)| < \eta(p_i) \) for \( p_i \in P \). Hence \( \eta(p_i) = 0 \) implies \( |f(p_i) - f(p_0)| \) and \( f_n(p_i) \to f(p_0) \) for \( n = 1, 2, \ldots \) Since \( f_n(p_i) \to f(p_1), f_n(p_0) \to f(p_0) \) and the lemma follows.

**Lemma 2.** Let \( E \) be a compact space and \( E \) a linear subspace of \( C^*(S) \) which satisfies conditions \( (\alpha) \), \( (\beta) \) and \( (\gamma) \). Then if \( E \) distinguishes points, then \( E \) distinguishes points and closed sets.

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(*) By a compactification of a space \( X \) we understand any compact space which contains \( X \) as a dense subset.

(**) Theorem of E. Cheh: see [1].
III. The case of a compact space. In this section we shall prove the following

**Theorem 1a.** Suppose that $X$ is a compact space and that $E$ is a linear subspace of $C^*(X)$ which satisfies conditions (a), (b) and (c) and distinguishes points and closed sets. Then each pointwise continuous positive linear functional $\varphi$ defined on $E$ can be written in the form $\varphi(f) = \varphi_0(f) = \int_X f(x) \, \mu(dx)$ where $\mu$ is a positive measure on $X$.

Suppose that $\varphi$ is any pointwise continuous positive non-trivial linear functional defined on $E$. For any $f$ in $E$ we denote by $Z(f)$ the set $\{ x \in X : f(x) = 0 \}$ and let $Z$ be the intersection of all sets of the form $Z(f)$ where $f$ is any non-negative function in $E$ with $\varphi(f) = 0$. The proof of Theorem 1a is based on the following lemmas:

**Lemma 3.** If $f$ is a non-negative function in $E$ and if $\varphi(f) = 0$, then $Z(f) \neq \emptyset$.

**Proof.** Suppose that $Z(f) = \emptyset$. Then there is $g \in E$ with $\varphi(g) = 0$. Let $h = |g|$, then $h \in E$ and $\varphi(h) \geq \varphi(g) = 0$. Let $f_n = \min\{h, n \cdot f\}$. Then $f_n \geq f$ and $f_n \rightarrow h$ in $E$, whence $\varphi(f_n) \rightarrow \varphi(h)$. It follows that $\varphi(f) = \varphi(h) = 0$, which leads to a contradiction.

**Lemma 4.** The set $Z$ is non-empty.

**Proof.** If $f$ and $g$ are non-negative functions in $E$, then $Z(f) \cap Z(g) = Z(h)$ where $h = f + g$. It follows by Lemma 3 that $Z$ is the intersection of a closed system of closed non-empty subsets of $X$. By the compactness of $X$, $Z$ is non-empty.

**Lemma 5.** If $f$ is any non-negative function in $E$, then $\varphi(f) = 0$ if and only if $Z(f) \subset Z(f)$.

**Proof.** Suppose that $f$ is any non-negative function in $E$, then $f$ can be written in the form $f(x) = \int_X f(x) \, \mu(dx)$ where $\mu$ is a positive measure on $X$. The proof is by contradiction. Suppose that $Z(f) \subset Z(f)$. Then $\varphi(\chi_{Z(f)}) = 0$ for every function $\chi_{Z(f)}$ with $\varphi(\chi_{Z(f)}) = 0$. But $\varphi(\chi_{Z(f)}) = f(f)$ and $\varphi(\chi_{Z(f)}) = f(f)$ if and only if $Z(f) \subset Z(f)$. Hence, the proof is by contradiction.

**IV. Proof of Theorem 1.** Let $E$ be any linear subspace of $C(X)$ satisfying conditions (a), (b) and (c). We denote by $E_i$ the set of all continuous functions $h$ defined on $Y = F_p(X)$ for which there is a function $f$ in $E$ such that

$$f(x) = h(f_p(x))$$

for each $x$ in $X$.

Of course, for each $f$ in $E$ there exists exactly one function $h$ in $E_i$ satisfying (1), namely the $f$-th coordinate of a point $y \in Y$. It follows that $E_i$ is a linear subspace if $O(X)$ satisfying conditions (a), (b) and (c). Moreover, for each $f$ in $E$, the function $f$ belongs to $E_i$ (we recall that the coordinates of points of $Y$ are enumerated by means of members of $E$ and $p(y)$ denotes the $f$-th coordinate of a point $y \in Y$). It follows
that $E_1$ distinguishes points of $Y$. Since $Y$ is compact, $E_1$ distinguishes points and closed sets of $Y$.

Now assume that $F_E(X)$ is $Q$-closed in $Y$ and let $\varphi$ be any pointwise continuous positive linear functional defined on $E$. Let $\varphi_1$ be the functional defined on $E_1$ by the equality

$$\varphi_1(h) = \varphi(f),$$

where $f$ is a member of $E$ satisfying (1). Of course, $\varphi_1$ is a pointwise continuous positive linear functional. By Theorem 1a, we have

$$\varphi_1(h) = a_1 h(y_1) + \ldots + a_n h(y_n) \quad \text{for each } h \in E_1,$$

where $a_1, \ldots, a_n$ are fixed real numbers and $y_1, \ldots, y_n$ are fixed points of $Y$.

We shall show that $y_i \in F_E(X)$. Assume that $y_i \in Y \setminus F_E(X)$ and $a_i \neq 0.$ Since $F_E(X)$ is $Q$-closed in $Y$ by Lemma 1, there is a function $h \in E_1$ and a sequence $h_1, h_2, \ldots, (h_n) \in E_1$ such that $h_n(y) \to h(y)$ for each $y$ in $F_E(X)$ and $h_n(y_i) \to h(y_i)$. Let $h_0 = h_0 - h$. Then $h_0 \in E_1$, $h_0(y) \to 0$ for each $y$ in $F_E(X)$ and $h_0(y_i) \to 0$. Since $E_1$ distinguishes points and closed sets of $Y$, there exists a function $g$ in $E_1$ such that $g(y_0) = 1$ and $g(y) = 0$ for $i \neq 0$. Let $g_0 = \max[0, \min\{g, h_0\}]$. Then $g_0 \in E_1$, $g_0(y) \to 0$ for each $y$ in $F_E(X)$ and from (3) it follows that $\varphi_1(g_0) \to 0$. But from (1) and (2) it follows that $g_0(y) \to 0$ for each $y$ in $F_E(X)$, then $\varphi_1(g_0) \to 0$, which leads to a contradiction. Thus $y_i \in F_E(X)$ for $i = 1, \ldots, n$.

Now let $p_i$ be any point in $X$ with $F_E(p_i) = y_i$ (i = 1, \ldots, h). It follows from (1) and (2) that

$$\varphi(f) = a_1 f(p_1) + \ldots + a_n f(p_n) \quad \text{for each } f \in F_E.$$

Conversely, suppose that $F_E(X)$ is not $Q$-closed in $Y$. Then, by Lemma 1, there is a point $y_i \in X \setminus F_E(X)$ such that for any function $h \in E_1$ and each sequence $h_1, h_2, \ldots, (h_n) \in E_1$ the condition $h_n(y_i) \to h(y_i)$ for each $y$ in $F_E(X)$ implies the condition $h_n(y_i) \to h(y_i)$. Let $\varphi$ be the functional defined on $E$ by the equality $\varphi(f) = h(y)$ where $h$ is a function in $E_1$ satisfying (1). Then $\varphi$ is a pointwise continuous positive linear functional which cannot be written in the form ($\star$). Thus the proof of the theorem is complete.

§ 2.

In this paragraph we shall show that if $E \subseteq C^*(X)$ is a linear space which satisfies conditions (a), (b), (3) and distinguishes points and closed sets of $X$ and if $F_E(X)$ is $Q$-closed in $F_E(X)$, then the topology of $X$ is, in a certain sense, determined by $E$. Before an exact formulation of the theorem we give a definition.

Let $E_1$ and $E_2$ be linear spaces consisting of functions. A linear one-to-one mapping $\xi$ of $E_1$ onto $E_2$ will be called an isomorphism if $\xi$ satisfies the following conditions

1. if $f \in E_1, \ f = \xi(f)$, then $f \geq 0$ if and only if $f \geq 0$;
2. if $f(x) = \xi(f), \ f \geq 0$, then $f_n \to 0$ if and only if $f_n \to 0$.

We shall prove the following theorem:

Theorem 2. Suppose $E_1 \subseteq C^*(X), E_2 \subseteq C^*(X)$, are linear spaces satisfying conditions (a), (3) and (3) and such that $E_1$ distinguishes points and closed sets of $X_1$ and $F_E(X_1)$ is $Q$-closed in $F_E(X_1)$ ($i = 1, 2$). If the spaces $E_1$ and $E_2$ are isomorphic, then the spaces $X_1$ and $X_2$ are homeomorphic.

The proof of Theorem 2 will be given in section III. Now we shall show some elementary properties of isomorphisms ($\Xi$ denotes the function which is identically equal to 1 on $X$):

(iv) $\xi(f) = \xi(f)$ for each $f \in E_1$.

We have $-\xi(f) \leq \xi(f) \leq \xi(f)$.

Finally, $\xi(f) = \xi(f)$.

(v) If $f \in F_E(X), \ f \geq 0, \ g = \xi(f)$, then $Z(f) = 0$ if and only if $Z(g) = 0$.

Suppose that $Z(f) = 0$. Let $g_0 = \xi(f) = \xi(g)$.

Since $Z(f) = 0$, we have $Z(f) = Z(g)$.

(vi) $\xi(f)$ is a strictly positive function on $X$.

Remark 1. It can be shown that condition (vi) actually characterizes isomorphisms among linear one-to-one mappings satisfying condition (I). In fact, suppose that $\xi$ is such a mapping of $E_1 \subseteq C^*(X)$ onto $E_2 \subseteq C^*(X)$ and let $f_0 \in E_1, \ f_0 \geq 0, \ g_0 = \xi(f_0)$, $g_0 = \xi(f_0)$. Suppose that $g_0 \geq 0$ and let $p_0$ be any point of $X_1$. Let $f_0 = \min\{f_0, \xi(p_0)\} = \xi(f_0)$. By condition (3) there exists a sequence $\alpha_n$ of positive numbers such that $\alpha_n \geq +\infty$ and $\xi = \sum \alpha_n f_0 + E_1$. Let $g = \xi(f_0)$.

Of course, $f_0 = \xi(f_0)$ and $g_0 = \xi(g_0)$, and it follows from condition (vi) that $g_0 = 0$ for some $g_0 \subseteq X$. On the other hand, let $f_0 = \sum \alpha_n f_0$. Of course, $0 \leq f_0 \leq \sum \alpha_n f_0$, where $\alpha_n \leq \xi(f_0)$, $\alpha_n \leq \xi(f_0)$, and thus $\xi(f_0) = 0$; consequently $g = \xi(f) = \xi(f_0)$, and it follows that $g_0 = 0$ for
n = 1, 2, ... But by condition (iv) (its proof does not depend on the condition (I)) \( \theta_n = \min \{ g_n, |g_n - f_n(p)| \} \). Since \( g_n(p_0) > 0 \) (here we use again condition (v)), \( g_n(p_0) = f_n(p_0). \) Since \( g_n(p_0) \to 0 \), \( f_n(p) \to 0 \). Since \( p_0 \) is an arbitrary point of \( X_1, f_n \to 0 \). In an analogous manner one can show that the assumption \( f_n \to 0 \) implies \( g_n \to 0 \).

II. Given a linear space \( E \subset C^*(X) \), we denote by \( \Phi(E) \) the class of all non-trivial positive linear functionals \( \varphi \) on \( E \) which satisfy the following conditions:

\[
\varphi(f) = |\varphi(f)| \quad \text{for each } f \in E;
\]

if \( f \neq 0 \), \( f_n \to 0 \) then \( \varphi(f_n) \to 0 \).

LEMMA 1. If \( E \) is a linear space satisfying the conditions (a), (b) and (c) and \( \varphi \in \Phi(E) \), then \( \varphi \) is a pointwise continuous functional.

Proof. Suppose that \( \varphi \in \Phi(E) \). We can assume, without loss of generality, that \( \varphi(e) = 1 \). First let us notice that if \( f \neq 0 \), \( f \geq 0 \) and \( \varphi(f) = 0 \), then \( \varphi(f) = 0 \) (the proof of Lemma 3 of § 1 applies to this case). Now let \( f_n \to 0 \) (\( f_n \in E \)). Let us set \( g_n = \min \{ [1, \varphi(f_n) \in C^*(X)] \} \). By condition (b) there exists a sequence \( a_1, a_2, \ldots \) of positive numbers such that \( \sum a_n < +\infty \) and \( g = \sum g_n \in E \). Let \( r_n = \sum a_n g_n \). Of course, \( r_n \in E \) and \( r_n \to 0 \), whence \( \varphi(r_n) \to 0 \). It follows that \( \varphi(g) = \sum \varphi(a_n g_n) \). But \( \varphi(g_n) \) is a positive number which depends only upon the point \( p \). We shall show that \( \varphi \) is a one-to-one mapping. If \( g \neq 0 \), \( \varphi(g) = 0 \), then there is a function \( f \in E \) with \( f(p) = 1, f(q) = 0 \). Let \( g = \varphi(f) \). Then, by (1), \( \varphi(g) = 1 \). If \( \varphi(g) = 0 \), then it follows that \( \varphi(p) = 0 \).

We shall show that \( \varphi(p) \neq 0 \).

LEMMA 2. If \( E \subset C^*(X) \) is a linear space which satisfies conditions (a), (b) and (c) and distinguishes points and closed sets of \( X \), and if \( F_E(X) \) is \( \theta \)-closed in \( F_E(X) \), then each functional \( \varphi \in \Phi(E) \) is of the form \( \varphi(f) = \alpha \cdot f(p_0) \), where \( \alpha > 0 \) and \( p_0 \) is a fixed point of \( X \) which is uniquely determined by the functional \( \varphi \) (10).

Proof. By the preceding lemma and Theorem 1, \( \varphi \) is of the form

\[
\varphi(f) = a_1 f(p_1) + \cdots + a_k f(p_k),
\]

where \( a_1, a_2, \ldots, a_k \) are mutually distinct. We shall show that at most one \( a_i \) is different from 0. Assume for instance that \( a_1, a_2 \neq 0 \). Since \( E \) distinguishes points and closed sets of \( X \), there is a function \( f_n \in E \) such that \( f_n(p_1) = 1, f_n(p_2) = -1, \) and \( f_n(p_i) = 0 \) for \( i = 3, \ldots, k \). Of course, \( f_n(f_n) = 0 \) and \( \varphi(f_n) = 0 \), which leads to a contradiction. Thus we have \( \varphi(f) = \alpha f(p_0) \) for some \( p_0 \in X \) and for each \( f \in E \). Since \( \varphi \) is non-trivial, \( \alpha > 0 \). Since \( E \) distinguishes points and closed sets of \( X \), the point \( p_0 \) is uniquely determined by the functional \( \varphi \).

III. Proof of Theorem 2. Let \( \xi \) be an isomorphism of \( E_1 \) onto \( E_2 \). For each \( p \in X_1 \), we denote by \( \varphi(p) \) the functional defined by the equality \( \varphi(p) = \xi(p) \). Of course, \( \varphi \in \Phi(E_1) \) and it follows by (iv) that the functional \( \varphi \) belongs to \( \Phi(E_2) \); denote by \( \varphi \) the point which corresponds to the functional \( \varphi(p) \). We see that the following relation is satisfied:

\[
\varphi \cdot (\xi^{-1}(g)) = a \cdot \varphi \cdot (h(p)) \quad \text{for each } g \in E_1,
\]

where \( a \) is a positive number which depends only upon the point \( p \).

We shall show that \( \varphi \) is a one-to-one mapping. If \( g \neq 0 \), \( \varphi(g) = 0 \), then there is a function \( f \in E_1 \) with \( f(p) = 1, f(q) = 0 \). Let \( g = \xi(f) \). Then, by (1), \( a \cdot \varphi \cdot (h(p)) = 1 \). If \( \varphi \cdot (h(p)) = 0 \), then it follows that \( \varphi(p) = 0 \).

We shall show that \( \varphi(p) \neq 0 \).

\[
\varphi(p) \cdot (\xi^{-1}(g)) = 1 \quad \text{for each } g \in E_1,
\]

whence, by (1), \( \varphi = h(p) \).

It remains to show that \( h \) and \( \varphi \) are continuous. Let \( V \) be any neighbourhood of a point \( h(p) \in X_2 \). Since \( E_2 \) distinguishes the points and closed sets of \( X_2 \), there is a function \( \xi(f) \in E_2 \) which is 1 at \( h(p) \) and 0 on \( X_2 \) \( \setminus V \). Let \( f_1 = \xi^{-1}(g) \). By (1), \( f_1(p) = 1 \), hence \( U = \{ q \in X_1 : f_1(q) > 0 \} \) is a neighbourhood of \( p \). Moreover, if \( g \in U \), then, by (1), \( 0 < \varphi(f) = \varphi(\xi^{-1}(g)) = a \cdot \varphi(\xi(h(g))) \), whence \( \varphi(\xi(h(g))) > 0 \), and thus \( h(g) > 0 \) and \( h(g) < 0 \) for \( g \in V \). Thus \( h \) is continuous, and in the same manner one can show that \( \varphi \) is also continuous. Finally, \( h \) is a homeomorphism of \( X_1 \) onto \( X_2 \), and the theorem follows.

§ 3.

In this paragraph we give a generalization of Theorems 1 and 2 to the case of linear spaces which satisfy a weaker condition than condition (b), namely the following one:

(10) We say that \( p \) corresponds to the functional \( \varphi \).
for each $f$ in $E$ and each $e > 0$ there is a $g$ in $E$ with $\|f - g\| < e$, where $\|f\|$ denotes, as usual, the norm of $f$.

This generalization is based on the following lemma ($E$ denotes the set of all functions which are limits of uniformly convergent sequences of members of $E$; $f \Rightarrow f$ means that the sequence $f_n$ is uniformly convergent to $f$. Moreover, we say that $E \subset C^r(X)$ separates points and closed sets of $X$ provided that for any $p_A \in A \subset C$ there is an $f \in E$ such that $\|f(p_A) - f(p)\| \geq 1$ for $p \in A$.

**Lemma.** Suppose that $E \subset C^r(X)$ ($E_1 \subset C^r(X_1)$, $E_2 \subset C^r(X_2)$) is a linear space satisfying conditions (a) and (b). Then

1. $E$ is a linear space satisfying conditions (a), (b) and (c);
2. if $E$ separates points and closed sets of $X$, then $E$ distinguishes points and closed sets of $X$;
3. each pointwise continuous positive functional $\varepsilon$ defined on $E$ admits an extension to a pointwise continuous positive functional $\tilde{\varepsilon}$ defined on $E$;
4. each isomorphism $\tilde{\varepsilon}$ between $E_1$ and $E_2$ admits an extension to an isomorphism $\tilde{\varepsilon}$ between $E_1$ and $E_2$.

**Proof.** Part 1° is obvious.

Part 2°. Let $p_A \in A \subset C$. There is an $f \in E$ such that $\|f(p_A) - f(p)\| \geq 1$ for $p \in A$. Setting $\varepsilon(p) = f(p) - f(p_A)$, we have $\varepsilon \in E$, $\varepsilon(p_A) = 0$ and $\|\varepsilon(p)\| \geq 1$ for $p \in A$. Since $\varepsilon$ satisfies condition (b), $\varepsilon(p) = -\min \{1, \|\varepsilon_\delta\|\}$.

Part 3°. If $f_n \Rightarrow f$ in $E$, then, by the inequality $\|\varepsilon(f_n) - \varepsilon(f_m)\| \leq \|\varepsilon(f_n - f_m)\|$, we infer that the sequence $\varepsilon(f_n)$ is convergent;

$$\tilde{\varepsilon}(f) = \lim \varepsilon(f_n).$$

One can easily verify that formula (1) defines a linear functional $\tilde{\varepsilon}$ on $E$ which is an extension of $\varepsilon$. Of course, $\tilde{\varepsilon}$ is a positive functional. To prove that $\tilde{\varepsilon}$ is pointwise continuous, assume that $f_n \rightarrow 0$, whence $\tilde{\varepsilon}(f_n) \rightarrow 0$. Let $g_n$ be a member of $E$ with $\|f_n - g_n\| < 1/n$. Of course, $g_n \rightarrow 0$, whence $\varepsilon(f_n) = \varepsilon(g_n) \rightarrow 0$. But $\|\varepsilon(f_n) - \varepsilon(g_n)\| \leq \varepsilon(\|f_n - g_n\|^{-1})$ and it follows that $\varepsilon(f_n) \rightarrow 0$.

Part 4°. We have $\|f\| \leq \|\varepsilon^{-1}(g)\|$, whence $\|\varepsilon(f)\| \leq \|\varepsilon^{-1}(g)\| \cdot \|f\|$; but $\|\varepsilon(f)\| \leq \|\varepsilon^{-1}(g)\| \cdot \|f\|$, and thus

$$\|\varepsilon(f)\| = \|\varepsilon^{-1}(g)\| \cdot \|f\|$$

for each $f$ in $E$.

Analogously

$$\|\varepsilon^{-1}(g)\| = \|\varepsilon^{-1}(g)\| \cdot \|f\|$$

for each $g$ in $E$. It follows that a sequence $f_n$ of members of $E_1$ is uniformly convergent if and only if the sequence $\varepsilon(f_n)$ is uniformly convergent. Hence $\varepsilon$ can be extended to an isomorphism $\tilde{\varepsilon}$ between $E_1$ and $E_2$.

From Theorems 1 and 2 we obtain by the foregoing lemma (we recall that a continuous image of a Lindelöf space is again a Lindelöf space and a Lindelöf space is $Q$-closed in any of its compactifications).

**Theorem 1.** If $E \subset C^r(X)$ is a linear space satisfying conditions (a) and (b) and $X$ is a Lindelöf space, then each pointwise continuous positive functional $\varepsilon$ on $E$ can be written in the form (1°).

**Theorem 2.** Suppose that $E_1 \subset C^r(X_1)$ and $E_2 \subset C^r(X_2)$ are linear spaces satisfying conditions (a) and (b) and such that $E_1$ separates points and closed sets of $X_1$, and $X_1$ is a Lindelöf space ($i = 1, 2$). If the spaces $E_1$ and $E_2$ are isomorphic, then the spaces $X_1$ and $X_2$ are homeomorphic.

**Remark 2.** In section 1 we have shown that if spaces $E_1 \rightarrow C^r(X_1)$, $E_1 \subset C^r(X_1)$ satisfy conditions (a), (b) and (c) and $\xi$ is a linear one-to-one mapping of $E_1$ onto $E_2$, which satisfies conditions (I) and (V), then $\xi$ is an isomorphic. This is not true if the spaces $E_1$ and $E_2$ satisfy only conditions (a) and (b); moreover, in this case the existence of such a mapping of $E_1$ onto $E_2$ does not imply the existence of a homeomorphism between $X_1$ and $X_2$ even if they are Lindelöf spaces. Let us consider the following example:

Let $X_1 = [0, 1]$ and $X_2 = [0, 1]$. Let $E_1 \subset C^r(X_1)$ be the space of all continuous functions $f$ defined on $X_1$ for which there exists a positive number $\delta_1$ such that $f$ is constant in the interval $[1 - \delta_1, 1]$. Then $E_1$ and $E_2$ are linear spaces satisfying conditions (a) and (b) and $E_1$ distinguishes points and closed sets of $X_1$ ($i = 1, 2$). Moreover, the mapping $\xi(f) = f|X_2$ is a linear one-to-one mapping of $E_1$ onto $E_2$, which satisfies conditions (I) and (V); nevertheless the spaces $X_1$ and $X_2$ are not homeomorphic.

**Remark 3.** Theorems 1 and 2 can be applied to spaces of differentiable functions. Indeed, if $\mathbb{R}$ is a manifold of the class $C^r$ ($r = 1, 2, \ldots, \infty$) (or an analytic manifold), then the space $E$ of all bounded functions on $\mathbb{R}$ being of the class $C^r$ (or analytic functions) is a linear space which separates points and closed sets of $\mathbb{R}$ and satisfies conditions (a) and (b) ($\xi$) (the last follows from the fact that, for each member $f$ of $E$, $f$ can be uniformly approximated by means of polynomials with respect to $f$).
On quasi-modular spaces
by
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§ 1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a lattice ordered linear space in which there exists $a_i$ for every system of positive elements $(a_i; i \in A)$ of $R$.

H. Nakano has considered a kind of functional on $R$ which is called a modular (\textsuperscript{1}), and constructed the most important parts of the theory of modular spaces (i.e. spaces on which modulars are defined).

In this paper we shall consider a functional $q$ on $R$ which satisfies the following conditions, weaker than those of modulars:

\begin{itemize}
  \item[(p.1)] $0 \leq q(a) = q(-a) \leq +\infty$ for all $a \in R$;
  \item[(p.2)] $q(a + y) = q(a) + q(y)$ for every $a, y \in R$
\end{itemize}

with $|a| \cap |y| = 0$;

\begin{itemize}
  \item[(p.3)] for any system $(a_i; i \in A)$ such that $|a_i| \cap |a_j| = 0$ for $i \neq j, i, j \in A$ and $\sum_{i \in A} q(a_i) < +\infty$, there exists $a_0 \in E$ with $\sum_{i \in A} a_i = a_0$ and $\sum_{i \in A} q(a_i) = q(a_0)$;
  \item[(p.4)] $\lim_{a \to 0} q(a) < +\infty$ for all $a \in E$.
\end{itemize}

$E$ is called a quasi-modular space if the above $q$ is defined on $E$ and $q$ is called a quasi-modular. This quasi-modular is considered as a generalization of a Nakano’s monotone complete modular or of a concave modular [4 and 6].

Recently, J. Musielak and W. Orlicz considered the pseudo-modular on a linear space in [8]. If we add the order structure to linear spaces and additive conditions: $(p.3)$ and $(p.3)$ to those of a pseudo-modular, then a quasi-modular can be considered as a pseudo-modular in the case of semi-ordered linear spaces.

Some of the examples of a pseudo-modular established in [8] are regarded as those of a quasi-modular.

\textsuperscript{1} For the definition of a modular see § 2.